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ON THE CONTINUITY OF POINT TO SET MAPS WITH APPLICATIONS TO PARETO OPTIMIZATION

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Abstract

Pareto optimization problems have sets as solution both in the decisions' space and in the objectives' space. When the problem depends on a real vector parameter the notion of the stability of the solution must be defined and studied. We introduce in this paper the notions of limit of sequences of sets, open, closed and continous application. Then we determine some relations between the stability of the solutions in the decisions' space and in the objectives' space.

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1 Introduction

Zangwill's paper [7] marks the beginning of the research on the continuity of the solutions of an optimization problem in relation with the convergence of the optimization algorithms.

Many authors describe and analyze the properties of point to set maps related to the semicontinuity of a real function, considering the solution of a single objective optimization problem. A generalization of Pareto optimization is found in Tigan [6].

Another approach to study the properties of point to set maps is to take functions $F : R^n \to \mathcal{P}(R^n)$ and to consider the continuity of these functions in some special topology defined on $\mathcal{P}(R^n)$, see Berge[1].

More effective is the study of the properties of point to set maps through the properties of their graphics.

2 Continuity of set to point maps

First we define two basic concepts :*open application* and *closed application*. These notions are named in Tanino and Sawaragi [5] *lower semicontinous* and *upper semicontinous applications*.

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Definition 1. *Let* $A : Y \subset R^k \to \mathcal{P}(R^n)$*. A is closed in* $y_* \in Y$ *if for all convergent sequences* $y_n \to y_*$ *and* $x_n \to x_*$ *with* $x_n \in A(y_n)$ *we have* $x_* \in A(y_*)$*.*

Definition 2. *Let* $A : Y \subset R^k \to \mathcal{P}(R^n)$*. A is open in* $y_* \in Y$ *if for all convergent sequence* $y_n \to y_*$ *and for all* $x_* \in A(y_*)$ *there is a convergent sequence* $x_n \to x_*$ *and a natural* N *so that* $x_n \in A(y_n)$ *for all* $n > N$*.*

Definition 3. *Let* $A : Y \subset R^k \to \mathcal{P}(R^n)$*. A is continuous in* $y_* \in Y$ *if A is simultaneously closed and open in* y_**.*

Definition 4. *Let* $A : Y \subset R^k \to \mathcal{P}(R^n)$*. A is continuous (or open, or closed) on Y if A is continuous (or open or closed) in all* $y_* \in Y$*.*

Definition 5. *Let* $B_j \subset R^k,\ j \in N$ *be a sequence of sets. We define the inner limit of the sequence of sets* B_j *as the set*

$$\overline{B} = \limsup_{j\to\infty} B_j = \left\{ x \mid x = \lim_{i\to\infty} x_{j_i}, \quad x_{j_i} \in B_{j_i} \right\} \tag{1}$$

where B_{j_i} *is a subsequence of* B_j*.*

Definition 6. *Let* $B_j \subset R^k,\ j \in N$ *a sequence of sets. We define the outer limit of the sequence of sets* B_j *as the set*

$$\underline{B} = \liminf_{j\to\infty} = \{x \mid x = \lim x_j\} \tag{2}$$

where x_j *is a sequence with the property that there exists a natural N so that* $x_j \in B_j\ \forall j > N$*.*

Definition 7. *Let* $B_j \subset R^k,\ j \in N$*, a sequence of sets. If*

$$\liminf_{j\to\infty} B_j = \limsup_{j\to\infty} B_j = B$$

then we state that B is the limit of the sequence of sets B_j*.*

We emphasise some relations between the convergence of set sequences and the property of a point to set application to be continuous (or open or closed) obtained directly from these definitions.

Proposition 1. *The application A is closed in* y_* *if* $\underline{A(y_j)} \subset A(y_*)$ *for all sequence* y_j *convergent to* y_**.*

Proposition 2. *The application A is open in* y_* *if* $\overline{A(y_j)} \supset A(y_*)$ *for all sequence* y_j *convergent to* y_**.*

If the codomain of the function A consists in a single point we have:

Lemma 1. *(Meyer[4]) Consider a function* $A : Y \subset R^k \to \mathcal{P}(R^n)$ *with the property* $A(y) = \{x_y\}\ \forall y \in Y$.
i) if A *is open in* y_* *then* A *is closed in* y_* *and therefore* A *is continuous in* y_*
ii) if $A : Y \subset R^k \to \mathcal{P}(Z)$, Z *compact and* A *closed then* A *is open in* y_* *and therefore is continuous in* y_*.

Hogan proves in [3] that in some cases the condition of uniform compactness near y_* is sufficient.

Definition 8. *Let* $A : Y \subset R^k \to \mathcal{P}(R^n)$. *A is uniform compact near* y_* *if there exists a neighbourhood* $V(y_*) \subset Y$ *such that the set* $\bigcup\limits_{y \in V(y_*)} A(y)$ *is compact.*

We can prove:

Lemma 2. *Let* $A : Y \subset R^k \to \mathcal{P}(R^n)$. *If*
i) A is uniform compact near y_*
ii) A is closed in y_*
iii) $A(y_*) = x_*$
then A is open in y_*.

Theorem 1. *(Danzing,Folkman and Shapiro[2])*
Let $f : R^n \to R$, $D : Y \subset R^k \to \mathcal{P}(R^n)$ *and* $\Omega : Y \to \mathcal{P}(R^n)$ *be defined by*

$$\Omega(y) = \{x_* \in D(y) \mid \ f(x_*) = \min_{x \in D(y)} f(x)\} \overset{not}{=} m(f \mid D(y)) \tag{3}$$

If Ω *is a closed application in* y_* *for all continuous function* f *then for any convergent sequence* $y_j \to y_*$ *we have that* $\lim\limits_{j \to \infty} D(y_j)$ *is empty set or is equal to* $D(y_*)$.

We extend this theorem to the multiobjective optimization in the following shape. We introduce first the set of minimum Pareto points and in this sense we consider: $F : R^n \to R^p,\ F = \{f_1, f_2, ..., f_p\},\ D : Y \subset R^k \to \mathcal{P}(R^n)$ and $y \in Y$.

Definition 9. $x_* \in D(y)$ *is a Pareto minimum point of* F *on* $D(y)$ *if*
there is no $x \in D(y),\quad x \neq x_*$ *such that* $F(x) \leq F(x_*)$
(i.e. 1)$f_i(x) \leq f_i(x_*)$ *for any* $i \in \{1, ..., p\}$

2) there exists $j \in \{1, ..., p\}$ *so that* $f_j(x) \neq f_j(x_*)$*)*

Denote this notion by $mP(F \mid D)$.

Definition 10. $x_* \in D(y)$ *is weak Pareto minimum point of* F *on* $D(y)$ *if there is no* $x \in D(y),\ \ x \neq x_*$ *such that* $F(x) < F(x_*)$ *where* $F(x) < F(x_*)$ *means that* $f_i(x) < f_i(x_*)$ *for all* $i \in \{1, ..., p\}$.

We denote this set by $mPw(F \mid D)$.

Definition 11. $x_* \in D(y)$ *is a strict Pareto minimum point if there is no* $x \in D(y),\ x \neq x_*$ *such that* $F(x) \preceq F(x_*)$ *(i.e. there is no* $x \in D(y),\ x \neq x_*$ *such that* $f_j(x) \leq f_j(x_*)$ *for all* $i \in \{1, ...p\}$*).*

We denote the strict Pareto minimum point by $mPst(F \mid D)$

3 Connections between stability in the decisions space and in the objectives space

Theorem 2. *Let $Y \subset R^k$, $F : R^n \times Y \to R^p$, $D : Y \to \mathcal{P}(R^n)$ and $\Omega : Y \to \mathcal{P}(R^n)$ defined by $\Omega(y) = mP(F(\cdot, y) \mid D(y))$. If Ω is closed in y_* for all continuous functions F then for all convergent sequences $y_j \to y_*$ we have that $\lim\limits_{j\to\infty} D(y_j)$ is empty set or equal to $D(y_*)$*

Proof. If Ω is closed in y_* for all continous vector function F, then it is closed in y_* for all $F = (f, 0, ..., 0)$ and, in this case, $mP(F \mid D(y)) = m(f \mid D(y))$ and from Th. 2.1 we obtain the conclusion. □

In single objective optimization problem we have as a solution in the objectives' space a unique point; in multiple objectives optimization problem, the solution in the objectives' space is a set and that is an essential difference.
We consider further $Y \subset R^k$, $F : R^n \times Y \to R^p$, $F = \{f_1, f_2, ..., f_p\}$, f_j is continuous function for all j, $D : Y \to \mathcal{P}(R^n)$, $\Omega : Y \to \mathcal{P}(R^n)$ defined by $\Omega(y) = mP(F(\cdot, y) \mid D(y))$ and Φ defined by $\Phi(y) = \bigcup\limits_{x\in\Omega(y)} F(x, y)$.

Theorem 3. *If there exists a compact D_* and a neighbourhood $V(y_*)$ of y_* such that $D(y) \subset D_*\ \forall y \in V(y_*)$ and if Ω is a closed application in y_* then Φ is a closed application in y_*.*

Proof. Consider the convergent sequence' $y_j \to y_*$ and $z_j \in \Phi(y_j)$, $z_j \to z_*$. Then there exist $x_j \in \Omega(y_j)$ and a natural N such that $z_j = F(x_j, y_j)$ with $x_j \in D_*$ for all $j > N$. But D_* is compact, $D(y_j) \subset D_* \forall j > N$ then x_j contains a convergent subsequence $x_{j_i} \to x_*$. Ω is closed application in $y_* \Rightarrow x_* \in \Omega(y_*)$. F is continuous on every component then $F(x_j, y_j) \to F(x_*, y_*) = z_*$. But $x_* \in \Omega(y_*)$ then $z_* \in \Omega(y_*)$. □

The reciprocal theorem holds in less restrictive conditions:

Theorem 4. *If Φ is a closed application in y_* then Ω is a closed application in y_*.*

Proof. We consider the convergent sequence $y_j \to y_*$, $x_j \in \Omega(y_j)$, $x_j \to x_*$. Then $F(x_j, y_j) \to F(x_*, y_*) = z_*$ and $z_* \in \Phi(y_*)$ that means $x_* \in \Omega(y_*)$, therefore Ω is a closed application in y_*. □

We can establish the same type of connections between open applications.

Theorem 5. *If Ω is open application in y_* then Φ is open application in y_*.*

Proof. Let $y_m \to y_*$, and $z_* \in \Phi(y_*)$. Then there is $x_* \in \Omega(y_*)$ such that $F(x_*, y_*) = z_*$. But, since Ω is open in y_*, there is a convergent sequence $x_j \to x_*$ and a natural N such that for all $m > N$ we have $x_m \in \Omega(y_m)$. Because F is continuous it follows $F(x_m, y_m) = z_m \to z_*$ and for all $m > N$, $z_m \in \Phi(y_m)$. □

The reciprocal theorem holds in more restrictive conditions:

Theorem 6. *If:*
i) there exists a compact D_* *and a neighbourhood* $V(y_*)$ *of* y_* *such that* $D(y) \subset D_*$ *for all* $y \in V(y_*)$
ii) $mP(F(.,y_*)|D(y_*)) = mPSt(F(.,y_*)|D(y_*))$
iii) Φ *is a continuous application in* y_*
then Ω *is an open application in* y_**.*

Proof. Let $y_l \to y_*$, and $x_* \in \Omega(y_*)$. Denote by $z_* = F(x_*, y_*)$, that means $z_* \in \Phi(y_*)$. Because Φ is continuous in y_* it is open in y_*. Therefore there exists a convergent sequence $z_j \to z_*$ and a natural N_1, such that $z_j \in \Phi(y_j)$, for $j > N_1$. It follows that there exists $x_j \in \Omega(y_j)$ such that $z_j = F(x_j, y_j)$.
Then there exists N_2 such that for $l > N_2$ we have $x_l \in D_*$, but D_* is compact, therefore x_l contains a convergent subsequence $x_{l_i} \to x^0$;
Let us suppose $x^0 \neq x_*$.
F is continuous so $F(x_{l_i}, y_{l_i}) \to F(x^0, y_*)$ and because the limit is unique we have $F(x^0, y_*) = F(x_*, y_*)$ and $x^0 \in \Omega(y_*)$.
Therefore $x^0 \in mP(F(.,y_*)|D(y_*)) = mPSt(F(.,y_*)|D(y_*))$ and that is in contradiction to $F(x^0, y_*) = F(x_*, y_*)$. It means $x_l \to x_*$ and, for $l > \max(N_1, N_2)$ we have $x_l \in \Omega(y_l)$. □

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