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ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A DIFFERENTIAL OPERATOR

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Abstract

In this paper we investigate a new subclass of analytic functions defined by Sălăgean differential operator. Some properties of functions belonging to this subclass are obtained.

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1 Indroduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U} \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Denote by S the class of functions $f \in \mathcal{A}$ which are univalent.

A function $f \in \mathcal{A}$ is said to be in the class S^* of starlike functions, if it satisfies the following inequality:

$$Re\frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{U}. \tag{2}$$

For a function $f \in \mathcal{A}$, Sălăgean differential operator D^n [8] is defined by

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z),$$
$$D^n f(z) = D(D^{n-1} f(z)), \quad n \in \{1, 2, \ldots\}.$$

If $f \in \mathcal{A}$ is given by (1), note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad n \in \{0, 1, 2, \ldots\}. \tag{3}$$

Making use of the generalized harmonic mean of the functions $D^n f(z)$ and $D^{n+1} f(z)$ we define the following class of functions.

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Definition 1. *Let α be a complex number. We say that a function $f \in \mathcal{A}$ belongs to the class $HS_n^*(\alpha)$ if the function F defined by*

$$\frac{1}{F(z)} = \frac{1-\alpha}{D^n f(z)} + \frac{\alpha}{D^{n+1} f(z)}, \quad z \in \mathbb{U} \tag{4}$$

is in the class S^*.

For $n = 0$, the class $HS_0^*(\alpha)$ reduces to the class $HS^*(\alpha)$ investigated by N. N. Pascu and D. Răducanu [6].

When $\alpha = 0$, the class $HS_n^*(0)$ reduces to the class of analytic n-starlike functions sudied by G. S. Sălăgean [8].

In this paper we find the relationship between the classes $HS_n^*(\alpha)$ and S^*. The Fekete-Szegö problem for the class $HS_n^*(\alpha)$ is also solved.

2 Relationship property

In order to prove the relationship between the classes $HS_n^*(\alpha)$ and S^* we need the following lemma.

Lemma 1 ([4]). *Let $p(z)$ be an analytic function in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 1$. If $0 < |z_0| < 1$ and*

$$Rep(z_0) = \min_{|z| \le |z_0|} Rep(z)$$

then

$$z_0 p'(z_0) \in \mathbb{R} \ \text{ and } \ z_0 p'(z_0) \le -\frac{|1-p(z_0)|^2}{2[1-\Re p(z_0)]}.$$

Theorem 1. *Let α be a complex number such that $\left|\alpha - \frac{1}{2}\right| \ge \frac{1}{2}$. Then $HS_n^*(\alpha) \subset S^*$.*

Proof. Assume that f belongs to the class $HS_n^*(\alpha)$. Simple calculations show that if f is in $HS_n^*(\alpha)$, then

$$Re\left[\frac{D^{n+1}f(z)}{D^n f(z)} + \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{(1-\alpha)D^{n+2}f(z) + \alpha D^{n+1}f(z)}{(1-\alpha)D^{n+1}f(z) + \alpha D^n f(z)}\right] > 0. \tag{5}$$

Consider the analytic function $p(z)$ in $\mathbb{U}$, given by

$$p(z) = \frac{D^{n+1}f(z)}{D^n f(z)}. \tag{6}$$

Then, the inequality (5) becomes

$$Re\left[p(z) + \frac{zp'(z)}{p(z)} - \frac{(1-\alpha)zp'(z)}{(1-\alpha)p(z) + \alpha}\right] > 0. \tag{7}$$

Suppose that there exists a point z_0 $(0 < |z_0| < 1)$ such that

$$Rep(z) > 0 \ \ (|z| < |z_0|) \ \text{ and } \ p(z_0) = i\rho, \tag{8}$$

where ρ is real and $\rho \neq 0$.Then, making use of Lemma 1., we get

$$z_0 p'(z_0) \leq -\frac{1+\rho^2}{2}. \tag{9}$$

By virtue of (7), (8) and (9) it follows that

$$R_0 := Re\left[p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} - \frac{(1-\alpha)z_0 p'(z_0)}{(1-\alpha)p(z_0)+\alpha}\right]$$

$$= Re\left[i\rho + \frac{z_0 p'(z_0)}{i\rho} - \frac{(1-\alpha)z_0 p'(z_0)}{(1-\alpha)i\rho+\alpha}\right].$$

Hence,

$$R_0 = \frac{z_0 p'(z_0)}{|(1-\alpha)i\rho+\alpha|^2} Re\left[|\alpha|^2 - \bar{\alpha}\right]. \tag{10}$$

Since $\left|\alpha - \frac{1}{2}\right| \geq \frac{1}{2}$ it follows that $Re\left[|\alpha|^2 - \bar{\alpha}\right] \geq 0$.

From (9) and (10) we get

$$R_0 \leq -\frac{1+\rho^2}{2|(1-\alpha)i\rho+\alpha|^2} Re\left[|\alpha|^2 - \bar{\alpha}\right] \leq 0,$$

which contradicts the assumption $f \in HS_n^*(\alpha)$. Therefore, we must have

$$Rep(z) = Re\frac{D^{n+1}f(z)}{D^n f(z)} > 0, \ \text{ for } \ z \in \mathbb{U}. \tag{11}$$

An important results in [8] states that the condition (11) implies

$$Re\frac{D^n f(z)}{D^{n-1} f(z)} > 0, \ \ z \in \mathbb{U}$$

which implies

$$Re\frac{D^{n-1} f(z)}{D^{n-2} f(z)} > 0, \ \ z \in \mathbb{U}$$

and so on. Finally, we obtain

$$Re\frac{D^1 f(z)}{D^0 f(z)} = Re\frac{zf'(z)}{f(z)} > 0, \ \ z \in \mathbb{U}$$

and thus, $f \in S^*$. □

3 The Fekete-Szegö problem

In 1933 M. Fekete and G. Szegö [1] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ for $f \in S$ and μ real number. For this reason, the determination of sharp upper bounds for the non-linear functional $|a_3 - \mu a_2^2|$ for any compact family $\mathbb{F}$ of functions $f \in \mathcal{A}$ is popularly known as the Fekete-Szegö problem for $\mathbb{F}$. For different subclasses of S, the Fekete-Szegö problem has been investigated by many authors including [[2], [5], [7], [9], [10]], etc.

In this section we will solve the Fekete-Szegö problem for the class $HS_n^*(\alpha)$, when α is a positive real number.

The following lemmas will be needed in order to prove our results.

Lemma 2 ([3]). *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$\left|c_2 - vc_1^2\right| \leq \begin{cases} -4v + 2, & \text{if } v \leq 0 \\ 2, & \text{if } 0 \leq v \leq 1 \\ 4v - 2, & \text{if } v \geq 1. \end{cases}$$

When $0 < v < 1$, the above upper bounds can be improved as follows:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2\ , \ \ 0 < v \leq \frac{1}{2}$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2\ , \ \ \frac{1}{2} < v \leq 1.$$

The results are sharp.

Lemma 3 ([7]). *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$|c_2 - vc_1^2| \leq 2 \max\{1; |2v - 1|\}\,.$$

The result is sharp.

Theorem 2. *Let α be a positive real number and let μ be a real number. Consider*

$$\sigma_1 := \frac{2^{2n-1}(1 + 4\alpha - \alpha^2)}{3^n(1 + 2\alpha)}$$

$$\sigma_2 := \frac{2^{2n}(1 + 3\alpha)}{3^n(1 + 2\alpha)}$$

$$\sigma_3 := \frac{2^{2n-2}(3 + 10\alpha - \alpha^2)}{3^n(1 + 2\alpha)}.$$

If the function f given by (1) belongs to the class $HS_n^*(\alpha)$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{1}{(1+\alpha)^2}\left[\dfrac{3 + 10\alpha - \alpha^2}{3^n(1 + 2\alpha)} - \dfrac{\mu}{2^{2n-2}}\right], & \text{if } \mu \leq \sigma_1 \\ \dfrac{1}{3^n(1 + 2\alpha)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{1}{(1+\alpha)^2}\left[\dfrac{\alpha^2 - 10\alpha - 3}{3^n(1 + 2\alpha)} + \dfrac{\mu}{2^{2n-2}}\right], & \text{if } \mu \geq \sigma_2. \end{cases}$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n-1}(\alpha^2 - 4\alpha - 1) + 3^n(1+2\alpha)\mu}{3^n(1+2\alpha)}|a_1|^2 \le \frac{1}{3^n(1+2\alpha)}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n}(1+3\alpha) - 3^n(1+2\alpha)\mu}{3^n(1+2\alpha)}|a_1|^2 \le \frac{1}{3^n(1+2\alpha)}.$$

Proof. Suppose f given by (1) belongs to the class $HS_n^*(\alpha)$. Let $p_1(z) = 1 + c_1 z + c_2 z^2 + \dots$ be an analytic function with positive real part in $\mathbb{U}$. From (5) we get

$$\frac{D^{n+1}f(z)}{D^n f(z)} + \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{(1-\alpha)D^{n+2}f(z) + \alpha D^{n+1}f(z)}{(1-\alpha)D^{n+1}f(z) + \alpha D^n f(z)} =$$

$$= 1 + c_1 z + c_2 z^2 + \dots. \tag{12}$$

We have

$$\frac{D^{n+1}f(z)}{D^n f(z)} = 1 + 2^n a_2 z + (2 \cdot 3^n a_3 - 2^{2n} a_2^2) z^2 + \dots \tag{13}$$

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = 1 + 2^{n+1} a_2 z + (2 \cdot 3^{n+1} a_3 - 2^{2n+2} a_2^2) z^2 + \dots \tag{14}$$

$$\frac{(1-\alpha)D^{n+2}f(z) + \alpha D^{n+1}f(z)}{(1-\alpha)D^{n+1}f(z) + \alpha D^n f(z)} = 1 + 2^n(2-\alpha)a_2 z +$$

$$+ \left[2 \cdot 3^n(3-2\alpha)a_3 - 2^{2n}(2-\alpha)^2 a_2^2\right] z^2 + \dots. \tag{15}$$

Using (13), (14) and (15) in (12) we find

$$c_1 = 2^n(1+\alpha)a_2 \text{ and } c_2 = 2 \cdot 3^n a_3(1+2\alpha) + 2^{2n}(\alpha^2 - 4\alpha - 1)a_2^2.$$

This gives

$$a_2 = \frac{c_1}{2^n(1+\alpha)} \text{ and } a_3 = \frac{1}{2 \cdot 3^n(1+2\alpha)}\left[c_2 - \frac{\alpha^2 - 4\alpha - 1}{(1+\alpha)^2}c_1^2\right]. \tag{16}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{1}{2 \cdot 3^n(1+2\alpha)}(c_2 - v c_1^2),$$

where

$$v := \frac{2^{2n-1}(\alpha^2 - 4\alpha - 1) + 3^n(1+2\alpha)\mu}{2^{2n-1}(1+\alpha)^2}. \tag{17}$$

The first part of our theorem now follows by an application of Lemma 2.

Assume $\sigma_1 \le \mu \le \sigma_3$. Then

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n-1}(\alpha^2 - 4\alpha - 1) + 3^n(1+2\alpha)\mu}{3^n(1+2\alpha)}|a_1|^2$$

$$= \left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)|a_1|^2$$
$$= \frac{1}{2\cdot 3^n(1+2\alpha)}|c_2 - vc_1^2| + \frac{3^n(1+2\alpha)\mu - 2^{2n-1}(1+4\alpha-\alpha^2)}{3^n(1+2\alpha)}\cdot\frac{|c_1|^2}{2^{2n}(1+\alpha)^2}$$
$$= \frac{1}{2\cdot 3^n(1+2\alpha)}[|c_2 - vc_1^2| + v|c_1|^2] \le \frac{1}{3^n(1+2\alpha)}.$$

Similarly, if $\sigma_3 \le \mu \le \sigma_2$, we have

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n}(1+3\alpha) - 3^n(1+2\alpha)\mu}{3^n(1+2\alpha)}|a_1|^2$$
$$= \left|a_3 - \mu a_2^2\right| + (\sigma_2 - \mu)|a_1|^2$$
$$= \frac{1}{2\cdot 3^n(1+2\alpha)}|c_2 - vc_1^2| + \frac{2^{2n}(1+3\alpha) - 3^n(1+2\alpha)\mu}{3^n(1+2\alpha)}\cdot\frac{|c_1|^2}{2^{2n}(1+\alpha)^2}$$
$$= \frac{1}{2\cdot 3^n(1+2\alpha)}[|c_2 - vc_1^2| + (1-v)|c_1|^2] \le \frac{1}{3^n(1+2\alpha)}.$$

Thus, we have completed the proof of the theorem. □

Making use of Lemma 3. and the equalities (16) and(17), we immediately obtain the following result.

Theorem 3. *Let α be a positive number and let μ a complex number. If the function f given by (1) belongs to the class $HS_n^*(\alpha)$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{3^n(1+2\alpha)}\max\left\{1; \frac{\left|2^{2n-1}(\alpha^2 - 10\alpha - 3) + 2\cdot 3^n(1+2\alpha)\mu\right|}{2^{2n-1}(1+\alpha)^2}\right\}.$$

References

[1] Fekete, M., Szegö, G., *Eine bemerkung über ungerade schlichte funktionen*, J. London Math. Soc. **8** (1933), 85-89.

[2] Keerthi. B. S., Stefen, B. A., Sivasubramanian S., *A coefficient inequality for certain subclasses of analytic functions related to complex order*, The Australian J.Math.Anal. Appl. **3**(2) (2006), 1-18.

[3] Ma, W., Minda, D., *A unified treatment of some special classes of univalent functions*, *Proc. of the Conference on Complex Analysis*, (Eds. Li, Z., Ren, F., Yang, L., Zhang, S.), Int. Press, 1994, 157-169.

[4] Miller, S. S., Mocanu, P. T., *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65** (1978), 289-305.

[5] Orhan, H., Gunes, E., *Fekete-Szegö inequality for certain subclass of analytic functions*, General Math. **14**, no. 1 (2006), 41-54.

[6] Pascu, N. N., Răducanu,D., *Generalized means and generalized convexity*, Seminar of Geometric Function Theory, Preprint **3** (1993), 95-98.

[7] Ravichandran, V., Polatoglu, Y., Bolcal, M., Sen, A., *Certain subclasses of starlike and convex functions of complex order*, Hact. J. Math. Stat. **34** (2005), 9-15.

[8] Sălăgean, G.,S., *Subclasses of univalent functions*, Lect. Notes in Math., Springer Verlag **1013** (1983), 362-372.

[9] Srivastava, H., M., Mishra, A.K., Das,M., K., *The Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Variable **44** (2001), 145-163.

[10] Tuneski, N., Darus,M., *Fekete-Szegö functional for non-Bazilevic functions*, Acta Math. Acad. Paed. Nyiregyhaziensis **18** (2002), 63-65.