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ON THE TRANSFORMATIONS GROUP OF N-LINEAR CONNECTIONS ON THE DUAL BUNDLE OF 3-TANGENT BUNDLE

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Abstract

In the present paper we study the transformations for the coefficients of an $N-$linear connection on the dual bundle of 3-tangent bundle, $T^{*3}M$, by a transformation of nonlinear connections on $T^{*3}M$. We prove that the set $\mathcal{T}$ of these transformations together with the composition of mappings isn't a group, but we give some groups of transformations of $\mathcal{T}$, which keep invariant a part of the components of the local coefficients of an $N-$linear connection.

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1 Introduction

The notion of Hamilton spaces was introduced by R. Miron in [7] , [8] . The differential geometry of the dual bundle of $k-$osculator bundle was introduced and studied by R. Miron [13] , too.

In the present section the general setting from [13] is presented and subsequently only some needed notions are recalled.

Let M be a real $n-$dimensional C^{∞} -manifold and let $\left(T^{*3}M, \pi^{*3}, M\right)$ be the dual bundle of $3-$osculator bundle (or $3-$cotangent bundle), where the total space is:

$$T^{*3}M = T^{*2}M \times T^{*}M. \tag{1.1}$$

Let $\left(x^i, y^{(1)i}, y^{(2)i}, p_i\right), (i = 1, ..., n)$, be the local coordinates of a point $u = \left(x, y^{(1)}, y^{(2)}, p\right) \in T^{*3}M$ in a local chart on $T^{*3}M$.

The change of coordinates on the manifold $T^{*3}M$ is:

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$$\begin{cases} \tilde{x}^i = \tilde{x}^i\left(x^1, ..., x^n\right), \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j}\tilde{y}^{(1)j}, \\ 2\tilde{y}^{(2)i} = \frac{\partial \tilde{y}^{(1)i}}{\partial x^j}y^{(1)j} + 2\frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}}y^{(2)j}, \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i}p_j, (i, j = 1, 2, ..., n), \end{cases} \tag{1.2}$$

where the following relations hold:

$$\frac{\partial \tilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(\alpha+1)i}}{\partial y^{(1)j}} = \frac{\partial \tilde{y}^{(2)i}}{\partial y^{(2-\alpha)j}}, \left(\alpha = 0, 1; y^{(0)} = x\right). \tag{1.3}$$

$T^{*3}M$ is a real differential manifold of dimension $4n$.

With respect to (1.1) the natural basis of the vector space $T_u\left(T^{*3}M\right)$ at the point $u \in T^{*3}M$:

$$\left\{ \left.\frac{\partial}{\partial x^i}\right|_u, \left.\frac{\partial}{\partial y^{(1)i}}\right|_u, \left.\frac{\partial}{\partial y^{(2)i}}\right|_u, \left.\frac{\partial}{\partial p_i}\right|_u \right\} \tag{1.4}$$

is transformed as it follows by the Jacobi matrix of (1.2) changes.

We denote $\widetilde{T^{*3}M} = T^{*3}M \setminus \{0\}$. Let us consider the tangent bundle of the differentiable manifold $T^{*3}M$,
$\left(TT^{*3}M, d\pi^{*3}, T^{*3}M\right)$, where $d\pi^{*3}$ is the canonical projection and the vertical distribution $V : u \in T^{*3}M \to V(u) \in T_uT^{*3}M$, locally generated by the vector fields $\left\{\frac{\partial}{\partial y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}, \frac{\partial}{\partial p_i}\right\}$ at every point $u \in T^{*3}M$.

The following $\mathcal{F}\left(T^{*3}M\right)-$ linear mapping:

$$J : \chi\left(T^{*3}M\right) \to \chi\left(T^{*3}M\right),$$

defined by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0, J\left(\frac{\partial}{\partial p_i}\right) = 0, \tag{1.6}$$

at every point $u \in \widetilde{T^{*3}M}$ is a tangent structure on $T^{*3}M$.

We denote with N a nonlinear connection on the manifold $T^{*3}M$ with the coefficients:

$$\left(\underset{(1)}{N}{}^j{}_i\left(x, y^{(1)}, y^{(2)}, p\right), \underset{(2)}{N}{}^j{}_i\left(x, y^{(1)}, y^{(2)}, p\right), N_{ij}\left(x, y^{(1)}, y^{(2)}, p\right)\right), (i, j = 1, 2, ..., n).$$

The tangent space of $T^{*3}M$ in the point $u \in T^{*3}M$ is given by the direct sum of vector spaces:

$$T_u\left(T^{*3}M\right) = N_{0,u} \oplus N_{1,u} \oplus V_{2,u} \oplus W_{3,u}, \ \forall u \in T^{*3}M. \tag{1.5}$$

A local adapted basis to the direct decomposition (1.5) is given by:

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}}, \frac{\delta}{\delta p_i}\right\}, (i = 1, 2, ..., n), \tag{1.6}$$

where:

$$\begin{cases} \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N}{}^j{}_i \frac{\partial}{\partial y^{(2)j}} + N_{ij} \frac{\partial}{\partial p_j}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)j}} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(2)j}}, \\ \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}, \\ \frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i}. \end{cases} \tag{1.7}$$

Under a change of local coordinates on $T^{*3}M$, the vector fields of the adapted basis transform by the rule:

$$\begin{array}{c} \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}, \frac{\delta}{\delta y^{(1)i}} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{y}^{(1)j}}, \\ \frac{\delta}{\delta y^{(2)i}} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{y}^{(2)j}}, \frac{\delta}{\delta p_i} = \frac{\delta x^i}{\delta \tilde{x}^j} \frac{\delta}{\delta \tilde{p}_j}. \end{array} \tag{1.8}$$

The dual basis of the adapted basis (1.6) is given by:

$$\left\{\delta x^i, \delta y^{(1)i}, \delta y^{(2)i}, \delta p_i\right\}, \tag{1.9}$$

where:

$$\begin{cases} dx^i = \delta x^i, \\ dy^{(1)i} = \delta y^{(1)i} - \underset{(1)}{N}{}^i{}_j \delta x^j, \\ dy^{(2)i} = \delta y^{(2)i} - \underset{(1)}{N}{}^i{}_j \delta y^{(1)j} - \underset{(2)}{N}{}^i{}_j \delta x^j, \\ dp_i = \delta p_i + N_{ji} \delta x^j. \end{cases} \tag{1.10}$$

Let D be an $N-$linear connection on $T^{*3}M$, with the local coefficients in the adapted basis (1.6) :

$$D\Gamma(N) = \left(H^i{}_{jh}, \underset{(\alpha)}{C}{}^i{}_{jh}, C_i^{jh}\right), (\alpha = 1, 2). \tag{1.11}$$

An $N-$linear connection D is uniquely represented under the adapted basis in the following form:

$$\begin{array}{l} D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = H^s{}_{ij} \frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = H^s{}_{ij} \frac{\delta}{\delta y^{(\alpha)s}}, (\alpha = 1, 2), \\ D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta p_i} = -H^i{}_{sj} \frac{\delta}{\delta p_s}, \\ D_{\frac{\delta}{\delta y^{(\alpha)j}}} \frac{\delta}{\delta x^i} = \underset{(\alpha)}{C}{}^s{}_{ij} \frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta y^{(\alpha)j}}} \frac{\delta}{\delta y^{(\beta)i}} = \underset{(\alpha)}{C}{}^s{}_{ij} \frac{\delta}{\delta y^{(\beta)s}}, \\ D_{\frac{\delta}{\delta y^{(\alpha)j}}} \frac{\delta}{\delta p_i} = -\underset{(\alpha)}{C}{}^i{}_{sj} \frac{\delta}{\delta p_s}, (\alpha, \beta = 1, 2), \\ D_{\frac{\delta}{\delta p_j}} \frac{\delta}{\delta x^i} = C_i^{js} \frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta p_j}} \frac{\delta}{\delta y^{(\alpha)i}} = C_i^{js} \frac{\delta}{\delta y^{(\alpha)s}}, (\alpha = 1, 2), \\ D_{\frac{\delta}{\delta p_j}} \frac{\delta}{\delta p_i} = -C_s{}^{ij} \frac{\delta}{\delta p_s}. \end{array} \tag{1.12}$$

2 The set of the transformations of $N-$linear connections

In the following we shall give the transformations for the coefficients of an $N-$ linear connection on $T^{*3}M$, by a transformation of nonlinear connections and we shall prove that the set, $\mathcal{T}$, of all these transformations together with the mapping composition isn't a group. We shall find some groups which keep invariant a part of components of the local coefficients of an $N-$linear connection.

Let $\bar{N}$ be another nonlinear connection on $T^{*3}M$, with the local coefficients: $\left(\underset{(1)}{\bar{N}}{}^{j}{}_{i}\left(x,y^{(1)},y^{(2)},p\right),\underset{(2)}{\bar{N}}{}^{j}{}_{i}\left(x,y^{(1)},y^{(2)},p\right),N_{ij}\left(x,y^{(1)},y^{(2)},p\right)\right)(i,j=1,2,...,n)$

Then there exists the uniquely determined tensor fields $\underset{(\alpha)}{A}{}^{j}{}_{i}\in\tau_{1}^{1}\left(T^{*3}M\right),(\alpha=1,2)$ and $A_{ij}\in\tau_{2}^{0}\left(T^{*3}M\right)$, such that:

$$\begin{cases}\underset{(\alpha)}{\bar{N}}{}^{i}{}_{j}=\underset{(\alpha)}{N}{}^{i}{}_{j}-\underset{(\alpha)}{A}{}^{i}{}_{j},(\alpha=1,2),\\ \bar{N}_{ij}=N_{ij}-A_{ij},(i,j=1,2,...,n).\end{cases}\tag{2.1}$$

Conversely, if $\underset{(\alpha)}{N}{}^{i}{}_{j}$ and $\underset{(\alpha)}{A}{}^{i}{}_{j},(\alpha=1,2)$, respectively N_{ij} and A_{ij} are given, then $\underset{(\alpha)}{\bar{N}}{}^{i}{}_{j}$, $(\alpha=1,2)$, respectively $\bar{N}_{ij}$, given by (2.1) are the coefficients of a nonlinear connection.

Theorem 1 *Let N and $\bar{N}$ be two nonlinear connections on $T^{*3}M$, with local coefficients:*

$$\left(\underset{(1)}{N}{}^{j}{}_{i}\left(x,y^{(1)},y^{(2)},p\right),\underset{(2)}{N}{}^{j}{}_{i}\left(x,y^{(1)},y^{(2)},p\right),N_{ij}\left(x,y^{(1)},y^{(2)},p\right)\right),$$

$\left(\underset{(1)}{\bar{N}}{}^{j}{}_{i}\left(x,y^{(1)},y^{(2)},p\right),\underset{(2)}{\bar{N}}{}^{j}{}_{i}\left(x,y^{(1)},y^{(2)},p\right),N_{ij}\left(x,y^{(1)},y^{(2)},p\right)\right),\ (i,j=1,2,...,n)$ *respectively. If D is an $N-$linear connection on $T^{*3}M$, with local coefficients $D\Gamma(N)=\left(H^{i}{}_{jh},\underset{(1)}{C}{}^{i}{}_{jh},\underset{(2)}{C}{}^{i}{}_{jh},C_{i}^{jh}\right)$, then the transformation: $N\longrightarrow\bar{N}$, given by* (2.1) *of nonlinear connections implies for the coefficients* $D\Gamma\left(\bar{N}\right)=\left(\bar{H}^{i}{}_{jh},\underset{(1)}{\bar{C}}{}^{i}{}_{jh},\underset{(2)}{\bar{C}}{}^{i}{}_{jh},\bar{C}_{i}^{jh}\right)$ *of the $\bar{N}-$linear connection D the relations* (2.2), *that is the transformation $D\Gamma(N)\rightarrow D\Gamma\left(\bar{N}\right)$ is given by:*

$$\begin{cases}\bar{H}^{h}{}_{ij}=H^{h}{}_{ij}+\underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{C}{}^{h}{}_{il}+\underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}\underset{(2)}{C}{}^{h}{}_{ir}+\underset{(2)}{A}{}^{l}{}_{j}\underset{(2)}{C}{}^{h}{}_{il}-A_{jl}C_{i}^{lh},\\ \underset{(1)}{\bar{C}}{}^{h}{}_{ij}=\underset{(1)}{C}{}^{h}{}_{ij}+\underset{(1)}{A}{}^{l}{}_{j}\underset{(2)}{C}{}^{h}{}_{il},\\ \underset{(2)}{\bar{C}}{}^{h}{}_{ij}=\underset{(2)}{C}{}^{h}{}_{ij},\\ \bar{C}_{h}{}^{ij}=C_{h}{}^{ij},\\ \underset{(1)}{A}{}^{h}{}_{i|j}=0,\\ A_{ih|j}=0,(i,j,h=1,2,...,n),\end{cases}\tag{2.2}$$

where "$|$" denotes the $h-$covariant derivative with respect to $D\Gamma(N)$.

The proof results by a straghtforward computation, using (1.12) and (2.1)

Theorem 2 *Let N and $\bar{N}$ be tho nonlinear connections on $T^{*3}M$, with local coefficients:*
$\left(\underset{(1)}{N}{}^j{}_i\left(x,y^{(1)},y^{(2)},p\right),\underset{(2)}{N}{}^j{}_i\left(x,y^{(1)},y^{(2)},p\right),N_{ij}\left(x,y^{(1)},y^{(2)},p\right)\right)$,
$\left(\underset{(1)}{N}{}^j{}_i\left(x,y^{(1)},y^{(2)},p\right),\underset{(2)}{\bar{N}}{}^j{}_i\left(x,y^{(1)},y^{(2)},p\right),\bar{N}_{ij}\left(x,y^{(1)},y^{(2)},p\right)\right),\left(i,j=\overline{1,n}\right)$ *respectively.*

If $D\Gamma(N)=\left(H^i{}_{jh},\underset{(1)}{C}{}^i{}_{jh},\underset{(2)}{C}{}^i{}_{jh}C_i{}^{jh}\right)$ *and* $D\bar{\Gamma}\left(\bar{N}\right)=\left(\bar{H}^i{}_{jh},\underset{(1)}{\bar{C}}{}^i{}_{jh},\underset{(2)}{\bar{C}}{}^i{}_{jh},\bar{C}_i{}^{jh}\right)$, *are the local coefficients of two $N-$, respectively $\bar{N}$-linear connections, D, respectively $\bar{D}$ on the differentiable manifold $T^{*3}M$, then there exists only one system of tensor fields* $\left(\underset{(1)}{A}{}^i{}_j,\underset{(2)}{A}{}^i{}_j,A_{ij},B^i{}_{jh},\underset{(1)}{D}{}^i{}_{jh},\underset{(2)}{D}{}^i{}_{jh},D_i{}^{jh}\right)$ *such that:*

$$\begin{cases}\underset{(\alpha)}{\bar{N}}{}^i{}_j=\underset{(\alpha)}{N}{}^i{}_j-\underset{(\alpha)}{A}{}^i{}_j,(\alpha=1,2)\\ \bar{N}_{ij}=N_{ij}-A_{ij},\\ \bar{H}^i{}_{jh}=H^i{}_{jh}+\underset{(1)}{A}{}^l{}_h\underset{(1)}{C}{}^i{}_{jl}+\underset{(1)}{A}{}^l{}_h\underset{(1)}{N}{}^r{}_l\underset{(2)}{C}{}^i{}_{jr}+\underset{(2)}{A}{}^l{}_h\underset{(2)}{C}{}^i{}_{jl}-A_{hl}C_j{}^{li}-B^i{}_{jh},\\ \underset{(1)}{\bar{C}}{}^i{}_{jh}=\underset{(1)}{C}{}^i{}_{jh}+\underset{(1)}{A}{}^l{}_h\underset{(2)}{C}{}^i{}_{jl}-\underset{(1)}{D}{}^i{}_{jh},\\ \underset{(2)}{\bar{C}}{}^i{}_{jh}=\underset{(2)}{C}{}^i{}_{jh}-\underset{(2)}{D}{}^i{}_{jh},\\ \bar{C}_i{}^{jh}=C_i{}^{jh}-D_i{}^{jh}.\\ \underset{(1)}{A}{}^h{}_{i|j}=0,\\ A_{ih|j}=0,(i,j,h=1,2,...,n),\end{cases}\tag{2.3}$$

where "$|$" denotes the $h-$covariant derivative with respect to $D\Gamma(N)$.

Proof. The first equality (2.3) determines uniquely the tensor fields: $\underset{(\alpha)}{A}{}^i{}_j,(\alpha=1,2)$. The second equality (2.3) determines uniquely the tensor field A_{ij}. Since $\underset{(\alpha)}{C}{}^i{}_{jh},(\alpha=1,2)$ and $C_i{}^{jh}$ are $d-$tensor fields, the third equation (2.10) determines uniquely the tensor field $B^i{}_{jh}$. Similarly the fourth,... and the last equation (2.3) determines the tensor field $D_i{}^{jh}$ respectively.q.e.d.

We have immediately:

Theorem 3 *If* $D\Gamma(N)=\left(H^i{}_{jh},\underset{(1)}{C}{}^i{}_{jh},\underset{(2)}{C}{}^i{}_{jh},C_i{}^{jh}\right)$, *are the coefficients of an $N-$linear connection D on $T^{*3}M$ and*
$\left(\underset{(1)}{A}{}^i{}_j,\underset{(2)}{A}{}^i{}_j,A_{ij},B^i{}_{jh},\underset{(1)}{D}{}^i{}_{jh},\underset{(2)}{D}{}^i{}_{jh},D_i{}^{jh}\right)$, *is a system of tensor fields on $T^{*3}M$, then*
$D\bar{\Gamma}\left(\bar{N}\right)=\left(\bar{H}^i{}_{jh},\underset{(1)}{\bar{C}}{}^i{}_{jh},\underset{(2)}{\bar{C}}{}^i{}_{jh},\bar{C}_i{}^{jh}\right)$, *given by* (2.3) *are the coefficients of an $\bar{N}-$linear connection, $\bar{D}$, on $T^{*3}M$.*

Following the definition given by M. Matsumoto [4, 5] in the case of Finsler spaces, we have:

Definition 2.1 *i*) The system of tensor fields:
$\left(\underset{(1)}{A}{}^{i}{}_{j}, \underset{(2)}{A}{}^{i}{}_{j}, A_{ij}, B^{i}{}_{jh}, \underset{(1)}{D}{}^{i}{}_{jh}, \underset{(2)}{D}{}^{i}{}_{jh}, D_{i}{}^{jh}\right)$, is called the difference tensor fields of $D\Gamma\left(N\right)$ to $D\bar{\Gamma}\left(\bar{N}\right)$.

ii) The mapping: $D\Gamma\left(N\right) \longrightarrow D\bar{\Gamma}\left(\bar{N}\right)$ given by (2.3) is called a transformation of $N-$linear connection to $\bar{N}-$linear connection on $T^{*3}M$, and it is noted by:

$$t\left(\underset{(1)}{A}{}^{i}{}_{j}, \underset{(2)}{A}{}^{i}{}_{j}, A_{ij}, B^{i}{}_{jh}, \underset{(1)}{D}{}^{i}{}_{jh}, \underset{(2)}{D}{}^{i}{}_{jh}, D_{i}{}^{jh}\right).$$

Theorem 4 *The set $\mathcal{T}$ of the transformations of $N-$linear connections to $\bar{N}-$linear connections on $T^{*3}M$, together with the composition of mappings isn't a group.*

Proof. Let $t\left(\underset{(1)}{A}{}^{i}{}_{j}, \underset{(2)}{A}{}^{i}{}_{j}, A_{ij}, B^{i}{}_{jh}, \underset{(1)}{D}{}^{i}{}_{jh}, \underset{(2)}{D}{}^{i}{}_{jh}, D_{i}{}^{jh}\right) : D\Gamma\left(N\right) \longrightarrow D\bar{\Gamma}\left(\bar{N}\right)$ and $t\left(\underset{(1)}{\bar{A}}{}^{i}{}_{j}, \underset{(2)}{\bar{A}}{}^{i}{}_{j}, \bar{A}_{ij}, \bar{B}^{i}{}_{jh}, \underset{(1)}{\bar{D}}{}^{i}{}_{jh}, \underset{(2)}{\bar{D}}{}^{i}{}_{jh}, \bar{D}_{i}{}^{jh}\right) : D\bar{\Gamma}\left(\bar{N}\right) \longrightarrow D\bar{\bar{\Gamma}}\left(\bar{\bar{N}}\right)$, be two transformations from $\mathcal{T}$, given by (2.3).

From (2.3) we have:

$$\begin{aligned}
\underset{(\alpha)}{\bar{\bar{N}}}{}^{i}{}_{j} &= \underset{(\alpha)}{N}{}^{i}{}_{j} - \left(\underset{(\alpha)}{A}{}^{i}{}_{j} + \underset{(\alpha)}{\bar{A}}{}^{i}{}_{j}\right), \ (\alpha = 1, 2), \\
\bar{\bar{N}}_{ij} &= N_{ij} - \left(A_{ij} + \bar{\bar{A}}_{ij}\right).
\end{aligned}$$

We obtain for example:

$$\underset{(1)}{\bar{\bar{C}}}{}^{i}{}_{jh} = \underset{(1)}{C}{}^{i}{}_{jh} + \left(\underset{(1)}{A}{}^{l}{}_{h} + \underset{(1)}{\bar{A}}{}^{l}{}_{h}\right) \cdot \underset{(2)}{C}{}^{i}{}_{jl} - \left(\underset{(1)}{D}{}^{i}{}_{jh} + \underset{(1)}{\bar{D}}{}^{i}{}_{jh} + \underset{(2)}{D}{}^{i}{}_{jl} \underset{(1)}{\bar{A}}{}^{l}{}_{h}\right).$$

So $\underset{(1)}{\bar{\bar{C}}}{}^{i}{}_{jh}$ hasn't the form (2.10). It follows that the mapping of two transformations from $\mathcal{T}$ isn't a transformation from $\mathcal{T}$, that is $\mathcal{T}$, together with the composition of mappings isn't a group.q.e.d.

Remark 2.1. If we consider $\underset{(\alpha)}{A}{}^{i}{}_{j} = 0, (\alpha = 1, 2)$ and $A_{ij} = 0$ in (2.3) we obtain the set $\mathcal{T}_N$ of transformations of $N-$linear connections corresponding to the same nonlinear connection N :

$$\mathcal{T}_N = \left\{ t\left(0, 0, 0, B^{i}{}_{jh}, \underset{(1)}{D}{}^{i}{}_{jh}, \underset{(2)}{D}{}^{i}{}_{jh}, D_{i}{}^{jh}\right) \in \mathcal{T} \right\}.$$

We have:

Theorem 5 *The set $\mathcal{T}_N$ of the transformations of $N-$linear connections to $N-$linear connections on $T^{*3}M$, together with the composition of mappings is a group. This group, acts effectively and transitively on the set of $N-$linear connections.*

Proposition 6 *The sets:*$\mathcal{T}_{NH}, \mathcal{T}_{N\underset{(1)}{C}}, \mathcal{T}_{N\underset{(2)}{C}}, \mathcal{T}_{NC}, \mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}C}$ *are Abelian subgroups of* $\mathcal{T}_N$.

Proposition 7 *The group $\mathcal{T}_N$ preserves the nonlinear connection $N, \mathcal{T}_{NH}$ preserves the nonlinear connection N and the component ${H^i}_{jh}$ of the local coefficients $D\Gamma(N)$; $\mathcal{T}_{N\underset{(1)}{C}}$ preserves the nonlinear connection N and the component $\underset{(1)}{C}{}^i{}_{jh}$ of the local coefficients $D\Gamma(N)$, $\mathcal{T}_{N\underset{(2)}{C}}$ preserves the nonlinear connection N and the component $\underset{(2)}{C}{}^i{}_{jh}$ of the local coefficients $D\Gamma(N)$, $\mathcal{T}_{NC}$ preserves the nonlinear connection N and the component C_i^{jh} of the local coefficients $D\Gamma(N)$ and $\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}C}$ preserves the nonlinear connection N and the components $\underset{(1)}{C}{}^i{}_{jh}, \underset{(2)}{C}{}^i{}_{jh}, C_i^{jh}$ of the local coefficients $D\Gamma(N)$.*

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