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T-STABILITY OF PICARD ITERATION IN METRIC SPACES

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Abstract

In this paper we extend a general result for the stability of Picard iteration. Several theorems in the literature are obtained as special cases. We give a correction to one result from [6].

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1 Indroduction

Let (X, d) be a complete metric space and T a selfmap of X. Let $x_{n+1} = f(T, x_n)$ be some iteration procedure. Suppose that $F(T)$, the fixed point set of T, is nonempty and that x_n converges to a point $q \in F(T)$. Let $\{y_n\} \subset X$, and define $\epsilon_n = d(y_{n+1}, f(T, y_n))$. If $\lim \epsilon_n = 0$ implies that $\lim y_n = q$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable. If $x_{n+1} = Tx_n$, then we say that Picard iteration is T-stable.

Qing and Rhoades [6] obtained sufficient conditions for that Picard iteration be T-stable for an arbitrary selfmap, and proved that Picard iteration is T-stable for many contractive selfmaps T.

Theorem 1. *Let (X, d) be a nonempty complete metric space, T a selfmap of X with $F(T) \neq \emptyset$. If there exist numbers $L \geq 0, 0 \leq h < 1$ such that*

$$d(Tx, q) \leq Ld(x, Tx) + hd(x, q) \tag{1}$$

for each $x \in X, q \in F(T)$, and, in addition, for every sequence $\{y_n\} \subset X$ with $\lim \epsilon_n = 0$, where $\epsilon_n = d(y_{n+1}, f(T, y_n))$, we have

$$\lim d(y_n, Ty_n) = 0, \tag{2}$$

then the Picard iteration is T-stable.

Corollary 1. *Let (X, d) be a nonempty complete metric space, T a selfmap of X satisfying: there exists $0 \leq h < 1$, such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leq h \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{3}$$

Then Picard iteration is T-stable.

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2 Main result

Our main result extends Theorem 1. We shall need the following lemma from [2].

Lemma 1. *Let $\{x_n\}, \{\epsilon_n\}$ be nonnegative sequences satisfying $x_{n+1} \le hx_n + \eta_n$ for all $n \in N, 0 \le h < 1, \lim \eta_n = 0$. Then $\lim x_n = 0$.*

Theorem 2. *Let (X, d) be a nonempty complete metric space, T a selfmap of X with $F(T) \neq \emptyset$. If there exist a real number $h, 0 \le h < 1$ and a function $\varphi : R^+ \to R^+$, $\varphi(0) = 0$ and $\lim_{t\to 0_+} \varphi(t) = 0$, such that*

$$d(Tx, q) \le hd(x, q) + \varphi(d(x, Tx)) \tag{4}$$

for each $x \in X, q \in F(T)$, and, in addition,

$$\lim d(y_n, Ty_n) = 0, \tag{5}$$

then Picard iteration is T-stable.

Proof. First we show that $F(T)$ is a singleton. Suppose p is another fixed point of T, then $d(p, q) = d(Tp, q) \le hd(p, q) + \varphi(d(p, Tp)) = hd(p, q)$. Since $0 \le h < 1$ we have $d(p, q) = 0$, i.e. $p = q$.
Let $\{y_n\} \subset X$, $\epsilon_n = d(y_{n+1}, Ty_n)$, and $\lim \epsilon_n = 0$. We need to show that $\lim y_n = q$. By (4) we have

$$d(y_{n+1}, q) \le d(y_{n+1}, Ty_n) + d(Ty_n, q) \le \epsilon_n + \varphi(d(y_n, Ty_n)) + hd(y_n, q).$$

Since $\lim_{t\to 0_+} \varphi(t) = 0$, by (5) and Lemma 1 with $\eta_n = \epsilon_n + \varphi(d(y_n, Ty_n))$ we get $\lim d(y_n, q) = 0$, so $\lim y_n = 0$. $\square$

To prove Corollary 1 Qing and Rhoades suppose that $\{y_n\}$ is bounded. The following example shows that it is possible for $\{y_n\}$ to be unbounded.

Example 1. Let $X = [0, \infty)$ and let d be the usual metric, $d(x, y) = |x - y|$. Let $T : X \to X$ such that $Tx = 1$ if $x \in \{0, 1\} \cup \cup_{n\ge 1}[1/(2n + 1, 1/(2n))$, $Tx = n$ if $x \in [1/(2n), 1/(2n - 1))$ for each $n \ge 1$, and $Tx = 1/n$ if $x \in (n - 1, n]$ for each $n \ge 2$.
Then $d(Tx, 1) \le d(x, Tx)$ for each $x \in [0, \infty)$. Indeed, if $Tx = 1$ the above relation is obvious. If $x \in [1/(2n), 1/(2n - 1))$, $n \ge 1$, then $Tx = n$ and

$$d(Tx, 1) = n - 1 \le n - 1/(2n - 1) < n - x = d(x, Tx).$$

If $x \in (n - 1, n]$, $n \ge 2$, then $Tx = 1/n$ and

$$d(Tx, 1) = 1 - 1/n < x - 1/n = d(x, Tx).$$

Hence

$$d(Tx, q) \le Ld(x, Tx) + hd(x, q)$$

for each $x \in X$, where $q = 1 \in F(T), L = 1, h = 0$.
It is easy to see that Picard iteration $x_{n+1} = Tx_n$ converges to 1 for every $x_0 \in X$. Let $y_{2n} = 2n, y_{2n+1} = 1/(4n+4), n \geq 1$. Then

$$d(y_{2n+1}, Ty_{2n}) = 1/(2n) - 1/(4n+4) = (n+2)/[4n(n+1)]$$

and

$$d(y_{2n+2}, Ty_{2n+1}) = 2n+2-2n-2 = 0,$$

so $d(y_{n+1}, Ty_n) \to 0$, but $\{y_n\}$ does not converge to $q = 1$. Clearly, the sequence $\{y_n\}$ is not bounded.
We give below a completion of the proof of Corollary 1.

Proof. From Theorem 11 of [3], T has a unique fixed point q. Define r_n to be the diameter of the following set $\{y_0, y_1, ..., y_n, Ty_0, Ty_1, ..., Ty_n\}$. Clearly, we have $r_n \leq r_{n+1}$. Since $\epsilon_n \to 0$, there exists $\epsilon > 0$ such that $\epsilon_n < \epsilon$ for every n.
If $r_n = d(Ty_i, Ty_j), 0 \leq i < j \leq n$, since

$$d(Ty_i, Ty_j) \leq h \max\{d(y_i, y_j), d(y_i, Ty_i), d(y_j, Ty_j), d(y_i, Ty_j), d(y_j, Ty_i)\} \leq hr_n, \tag{6}$$

we get $r_n \leq hr_n$, so $r_n = 0$.
If $r_n = d(Ty_i, y_j), 0 \leq i \leq n, 1 \leq j \leq n$, we have by (6)

$$d(Ty_i, y_j) \leq d(Ty_i, Ty_{j-1}) + d(Ty_{j-1}, y_j) \leq hr_n + \epsilon, \tag{7}$$

so $r_n \leq \epsilon/(1-h)$.
If $r_n = d(y_i, y_j), 1 \leq i < j \leq n$, then by (7) we have

$$d(y_i, y_j) \leq d(Ty_{j-1}, y_i) + d(Ty_{j-1}, y_j) \leq hr_n + \epsilon + \epsilon = hr_n + 2\epsilon, \tag{8}$$

so $r_n \leq 2\epsilon/(1-h)$.
If $r_n = d(y_0, y_i), 1 \leq i \leq n$, then by (7) we get

$$d(y_0, y_i) \leq d(y_0, Ty_0) + d(Ty_0, y_i) \leq d_0 + hr_n + \epsilon, \tag{9}$$

so $r_n \leq (d_0 + \epsilon)/(1-h)$, where $d_0 = d(y_0, Ty_0)$.
If $r_n = d(y_0, Ty_i), 0 \leq i \leq n$, then by (6) we get

$$d(y_0, Ty_i) \leq d(y_0, Ty_0) + d(Ty_0, Ty_i) \leq d_0 + hr_n, \tag{10}$$

so $r_n \leq d_0/(1-h)$.
Hence $r_n \leq (d_0 + 2\epsilon)/(1-h)$, so $\{r_n\}$ is bounded, and $\{y_n\}$ is bounded.
From this point the proof continues like in [6]. □

Theorem 3. *Let (X, d) be a nonempty complete metric space, T a selfmap of X satisfying*

$$d(Tx, Ty) \leq hd(x, y) + \varphi(d(x, Tx)) \tag{11}$$

for all $x, y \in X$, where $0 \leq h < 1$, and φ is a selfmap of R^+ such that $\varphi(0) = 0$ and $\lim_{t \to 0_+} \varphi(t) = 0$. Suppose that T has a fixed point p. Then T is Picard T-stable.

Proof. Let $p \in F(T)$. By (11) it follows $d(x_{n+1}, p) \leq hd(x_n, p)$ and so $d(x_n, p) \to 0$. Let $\{y_n\} \subset X$ and $\epsilon_n = d(y_{n+1}, Ty_n)$. Suppose $\lim \epsilon_n = 0$. Using triangle rule and (11) we get

$$d(y_{n+1}, p) \leq d(y_{n+1}, Ty_n) + d(Ty_n, p) \leq hd(y_n, p) + \epsilon_n. \tag{12}$$

Then, since $h \in [0, 1)$, it follows by Lemma 1 that $\lim y_n = p$. Moreover, we have

$$d(y_n, Ty_n) \leq d(y_n, p) + d(y_{n+1}, Ty_n) + d(y_{n+1}, p) \leq \epsilon_n + d(y_n, p) + d(y_{n+1}, p), \tag{13}$$

so $d(y_n, Ty_n) \to 0$. □

Corollary 2. *([5], Theorem 1; [6], Corollary 2) Let (X, d) be a nonempty complete metric space, T a selfmap of X satisfying*

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y) \tag{14}$$

for all $x, y \in X$, where $L \geq 0, 0 \leq a < 1$. Suppose that T has a fixed point p. Then T is Picard T-stable.

Proof. If we take $\varphi(t) = Lt$ and $h = a$ in Theorem 5 we get Corollary 6. □

Remark 1. *In the proof of Corollary 2, Qing and Rhoades [6] use the proof of Theorem 1 of [5] to show that $d(y_n, Ty_n) \to 0$. But for this it is necessary to prove that $y_n \to p$, so it is not correct to use this argument to show that Corollary 6 is a consequence of Theorem 1.*

As Qing and Rhoades [6] remark, many contractive conditions are special cases of (3), and so for each of these, Picard iteration is T-stable. For example, Theorems 1 and 2 of [1] and Theorem 1 of [4] are special cases of Corollary 2.

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