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A NOTE ON THE GAUSSIAN CORRELATION CONJECTURE

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Abstract

We present an alternate proof of the well-known Gaussian correlation conjecture in the 1-dimensional case and we discuss the possibility of extending it to a proof in the general n-dimensional case.

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1 Introduction

The Gaussian correlation conjecture can be stated as follows:

Conjecture 1 (Gaussian correlation conjecture). *If μ denotes the standard normal distribution on $\mathbb{R}^n$ $(n \geq 1)$, then*

$$\mu\left(A \cap B\right) \geq \mu\left(A\right) \cdot \mu\left(B\right), \tag{1}$$

for any symmetric convex sets $A, B \subset \mathbb{R}^n$.

The above conjecture received a lot of attention since 1977, when L. D. Pitt ([5]) first proved the validity of the above conjecture in the 2-dimensional case. Other known results include the case when one of the sets is a strip ([4], [6]), when the two sets are centered ellipsoids ([7]), or when just one of the sets is an ellipsoid ([1]), the general case being still open in its full generality at the present moment. For more details and a history of the problem see for example [3].

In this paper we will present a simple proof of the above conjecture in the 1-dimensional case, and we will discuss the possibility of extending the proof to the general case.

We begin with some preliminary needed for the proof.

Recall that a n-dimensional Brownian motion starting at $x = \left(x^1, \ldots x^n\right) \in \mathbb{R}^n$ is a stochastic process $W_t = \left(W_t^1, \ldots, W_t^n\right)$, where $W_t^1, \ldots, W_t^n$ are independent 1-dimensional Brownian motions starting at $x^1, \ldots, x^n$.

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Remark 1. Since the transition density

$$p(t,x,y)=\frac{1}{(2\pi t)^{n/2}}e^{-\frac{||x-y||^2}{2t}} \tag{2}$$

of a n-dimensional Brownian motion $(W_t)_{t\geq 0}$ starting at $W_0=x$ is for $t=1$ and $x=0$ a standard normal distribution, we have

$$\mu(C)=\int_C \frac{1}{(2\pi)^{n/2}}e^{-\frac{||y||^2}{2}}dy=P^0(W_1\in C), \tag{3}$$

for any measurable set $C\subset\mathbb{R}^n$, where P^0 denotes the probability measure corresponding to a n-dimensional Brownian motion $(W_t)_{t\geq 0}$ starting at $W_0=0$.

Therefore, the inequality (1) in the Conjecture 1 can be rewritten equivalently in the form

$$P^0(W_1\in A\cap B)\geq P^0(W_1\in A)\,P^0(W_1\in B), \tag{4}$$

where $(W_t)_{t\geq 0}$ is a n-dimensional Brownian motion starting at the origin.

2 Main results

The main result is the following

Theorem 1. *The Gaussian correlation conjecture holds in the case $n=1$.*

Proof. Consider $A,B\subset\mathbb{R}$ arbitrary convex sets symmetric with respect to the origin and $W_t=\left(W_t^1,W_t^2\right)$ a 2-dimensional Brownian motion starting at the origin.

Since in particular A and B must be connected subsets of $\mathbb{R}$, it follows that there exists $a,b>0$ such that $A=(-a,a)$ and $B=(-b,b)$ (the proof is similar if either A or B are closed intervals). Moreover, without loss of generality we may assume that $0<a\leq b$.

It is easy to see that by the independence of W_t^1 and W_t^2, and since the rectangle $A\times B=(-a,a)\times(-b,b)$ is contained in the strip $A\times\mathbb{R}=(-a,a)\times\mathbb{R}$ (see Figure 1), we have:

$$\begin{aligned}
P^0\left(W_1^1\in A\right)P^0\left(W_1^2\in B\right) &= P^0\left(W_1^1\in A, W_1^2\in B\right)\\
&= P^0\left(W_1\in A\times B\right)\\
&\leq P^0\left(W_1\in A\times\mathbb{R}\right)\\
&= P^0\left(W_1^1\in A, W_1^2\in\mathbb{R}\right)\\
&= P^0\left(W_1^1\in A\right)P\left(W_1^2\in\mathbb{R}\right)\\
&= P^0\left(W_1^1\in A\right)\\
&= P^0\left(W_1^1\in A\cap B\right).
\end{aligned}$$

Since W_t^1 and W_t^2 are independent 1-dimensional Brownian motions starting at the origin, from the Remark 1 it follows that the above inequality can be written equivalently as

$$\mu(A)\,\mu(B)\leq\mu(A\times B),$$

concluding the proof of the theorem. $\square$

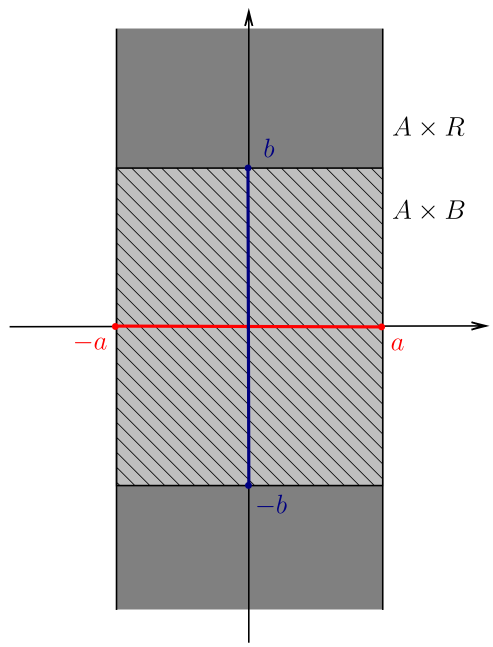

Figure 1: The sets $A \times B$ and $A \times \mathbb{R}$.

The fact that the Gaussian conjecture holds in the 1-dimensional case is well-known. In the 1-dimensional case the proof relies on the fact that if $A, B \subset \mathbb{R}$ symmetric convex sets, then either $A \cap B = A$ or $A \cap B = B$, hence the Gaussian conjecture reduces to $P(B) \leq 1$, respectively $P(A) \leq 1$, which is obviously true. Unfortunately, this argument cannot be extended in the case of the dimension $n > 1$.

The purpose of the above proof is to give an alternate proof, which might be suitable for generalization.

More precisely, following the above proof, the Gaussian conjecture can be reduced to showing

$$P^0\left(\widetilde{W}_1 \in A \cap B\right) \geq P^0\left(W_1 \in A \times B\right),$$

where $(W_t)_{t \geq 0}$ and $\left(\widetilde{W}_t\right)_{t \geq 0}$ are $2n$-dimensional, respectively n-dimensional Brownian motions starting at the origin.

If it can be shown that the projection of the set $A \times B \subset \mathbb{R}^{2n}$ on a n-dimensional hyperplane (i.e. on $\mathbb{R}^n$) is contained in the set $A \cap B$, then the above proof in the 1-dimensional case also applies, to give a proof of the Gaussian correlation conjecture in the general case.

This later equivalent problem is a difficult geometrical question, which we were unable to answer up to the present moment. However, we consider that the idea of the proof presented above is important, since it gives another possible line of approach for this well-known, still open problem.

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