Bulletin of the *Transilvania* University of Braşov • Vol 2(51) - 2009
Series III: Mathematics, Informatics, Physics, 199-206

THE VAISMAN CONNECTION ON THE VERTICAL BUNDLE OF A FINSLER MANIFOLD

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Abstract

The slit tangent manifold of a Finsler manifold is endowed with two foliations: the vertical foliation and the Liouville foliation, the last being a subfoliation of the first one, [1]. We give an adapted basis on the vertical bundle of such a manifold. In this paper we give the Vaisman connection on the vertical bundle with respect to the Liouville foliation and we compute its coefficients with respect to that adapted basis. We prove that the leaves of the vertical foliation are Reinhart spaces.

2000 *Mathematics Subject Classification:* 53C12, 58A20, 53C60.
Key words: Finsler manifold, Liouville foliation, Vaisman connection.

1 Preliminaries

We present two foliations on the slit tangent manifold TM^0 of a n-dimensional Finsler manifold (M, F), following [1]. In this paper the indices take the values $i, j, i_1, j_1, ... = \overline{1, n}$ and $a, b, a_1, b_1, ...= \overline{1, n-1}$.

Let (M, F) be a n-dimensional Finsler manifold and G the Sasaki-Finsler metric on its slit tangent manifold TM^0. The vertical bundle VTM^0 of TM^0 is the tangent (structural) bundle to the vertical foliation F_V determined by fibers $\pi : TM^0 \to M$. If $(x^i, y^i)_{i=\overline{1,n}}$ are local coordinates on TM^0, then VTM^0 is locally spanned by $\{\frac{\partial}{\partial y^i}\}_i$. A canonical transversal (also called horizontal) distribution is constructed in [1] as follows. We denote by $(g^{ij}(x, y))_{i,j}$ the inverse matrix of $g = (g_{ij}(x, y))_{i,j}$, where

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y), \tag{1.1}$$

where F is the fundamental function of the Finsler manifold. Obviously, we have the equalities $\frac{\partial g_{ij}}{\partial y^k}= \frac{\partial g_{ik}}{\partial y^j}= \frac{\partial g_{jk}}{\partial y^i}$.

Then locally define the functions

$$G^i = \frac{1}{4} g^{ik}\left(\frac{\partial^2 F^2}{\partial y^k \partial x^h} y^h - \frac{\partial F^2}{\partial x^k}\right), \quad G^j_i = \frac{\partial G^j}{\partial y^i}.$$

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There exists on TM^0 a n distribution HTM^0 locally spanned by the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}, \quad (\forall) i = \overline{1,n}. \tag{1.2}$$

The Riemannian metric G on TM^0 is satisfying

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g_{ij}, \quad G(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0, \quad (\forall) i, j. \tag{1.3}$$

The local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}_i$ is called adapted to the vertical foliation F_V and we have the decomposition

$$TTM^0 = HTM^0 \oplus VTM^0. \tag{1.4}$$

Now, let Z be the global defined vertical Liouville vector field on TM^0,

$$Z = y^i \frac{\partial}{\partial y^i}, \tag{1.5}$$

and L the space of line fields spanned by Z. We call this space *the Loiuville distridution* on TM^0. Yhe complementary orthogonal distributions to L in VTM^0 and TTM^0 are denoted by L' and $L^\perp$, respectively. It is proved, [1], that the both distributions L' and $L^\perp$ are integrable and we also have the decomposition

$$VTM^0 = L' \oplus L. \tag{1.6}$$

Moreover, we have, [1]:

Proposition 1.1. *a) The foliation determined by the distribution $L^\perp$ is just the foliation determined by the level hypersurfaces of the fundamental function F of the Finsler manifold.*

b) For every fixed point $x_0 \in M$, the leaves of the Liouville foliation $F_{L'}$ determined by the distribution L' on $T_{x_0}M$ are just the c-indicatrices of (M, F):

$$I_{x_0}M(c): \quad F(x_0, y) = c, \quad (\forall) y \in T_{x_0}M. \tag{1.7}$$

c) The foliation $F_{L'}$ is a subfoliation of the vertical foliation.

2 An adapted basis on VTM^0

As we have already seen in the previous section, the vertical bundle is locally spanned by $\{\frac{\partial}{\partial y^i}\}_{i=\overline{1,n}}$ and it admits decomposition (1.6). In this section we give another basis on VTM^0, adapted to $F_{L'}$.

There are some useful facts which follow from the homogenity of the fundamental function of the Finsler manifold (M, F). By the Euler theorem on positively homogeneous functions we have, [1],

$$F^2(x, y) = y^i y^j g_{ij}(x, y), \quad \frac{\partial F}{\partial y^k} = \frac{1}{F} y^i g_{ki}, \quad y^i \frac{\partial g_{ij}}{\partial y^k} = 0, \quad \forall k = \overline{1,n}. \tag{2.1}$$

Hence it results

$$G(Z,Z)=F^2. \tag{2.2}$$

We consider the following vertical vector fields:

$$X_k=\frac{\partial}{\partial y^k}-t_kZ, \quad k=\overline{1,n}, \tag{2.3}$$

where functions t_i are defined by the conditions

$$G(X_k,Z)=0, \forall k=\overline{1,n}. \tag{2.4}$$

The above conditions become

$$G(\frac{\partial}{\partial y^k}, y^i\frac{\partial}{\partial y^i})-t_kG(Z,Z)=0,$$

so, taking into account also (2.1) and (2.2), we obtain the local expression of functions t_k in a local chart $(U,(x^i,y^i))$:

$$t_k=\frac{1}{F^2}y^ig_{ki}=\frac{1}{F}\frac{\partial F}{\partial y^k}, \quad \forall k=\overline{1,n}. \tag{2.5}$$

If $(\tilde{U},(\tilde{x}^{i_1},\tilde{y}^{i_1}))$ is another local chart on TM^0, in $U\cap\tilde{U}\neq\oslash$ we have:

$$\tilde{t}_{k_1}=\frac{1}{F^2}\tilde{y}^{i_1}\tilde{g}_{i_1k_1}=\frac{1}{F^2}\frac{\partial\tilde{x}^{i_1}}{\partial x^i}y^i\frac{\partial x^k}{\partial\tilde{x}^{k_1}}\frac{\partial x^i}{\partial\tilde{x}^{i_1}}g_{ki}=\frac{\partial x^k}{\partial\tilde{x}^{k_1}}t_k,$$

so we obtain the following changing rule for the vector fields (2.3):

$$\tilde{X}_{k_1}=\frac{\partial x^k}{\partial\tilde{x}^{k_1}}X_k, \quad \forall k=\overline{1,n}. \tag{2.6}$$

By a straightforward computation, using (2.1), it results:

Proposition 2.1. *The functions $\{t_k\}_{k=\overline{1,n}}$ defined by (2.5) are satsfying:*

$$a) \quad y^it_i=1; \quad y^iX_i=0; \tag{2.7}$$

$$b) \quad \frac{\partial t_l}{\partial y^k}=-2t_kt_l+\frac{1}{F^2}g_{kl}, \quad Zt_k=-t_k, \quad \forall k,l=\overline{1,n}; \tag{2.8}$$

$$d) \quad y^j\frac{\partial t_j}{\partial y^i}=-t_i, \quad \forall i=\overline{1,n}, \quad y^i(Zt_i)=-1, \quad y^i(ZX_i)=0.; \tag{2.9}$$

Proposition 2.2. *There are the relations:*

$$[X_i,X_j]=t_iX_j-t_jX_i, \tag{2.10}$$

$$[X_i,Z]=X_i, \tag{2.11}$$

for all $i,j=\overline{1,n}$.

By conditions (2.4), $\{X_1, ..., X_n\}$ are n vector fields orthogonal to Z, so they belong to the $(n-1)$-dimensional distribution L'. It resuls that they are linear dependent and from (2.7)

$$X_n = -\frac{1}{y^n} y^a X_a, \tag{2.12}$$

since the local coordinate y^n is nonzero everywhere.

We could also prove that

Proposition 2.3. *The system of vector fields $\{X_1, X_2, ..., X_{n-1}, Z\}$ of vertical vector fields is a locally adapted basis to the Liouville foliation $F_{L'}$, on VTM^0.*

The entire proofs for propositions from this section are given in [4].

The Riemannian metric induced by G on VTM^0 has the matrix

$$h = \begin{pmatrix} (h_{ab})_{a,b} & 0_{n-1} \\ 0^t_{n-1} & F^2 \end{pmatrix}, \quad h_{ab} = G(X_a, X_b) = g_{ab} - F^2 t_a t_b, \tag{2.13}$$

with respect to the adapted basis $\{X_1, X_2, ..., X_{n-1}, Z\}$.

3 The Vaisman connection on VTM^0

Let (M, g) be a Riemannian foliated manifold, with D', D, the structural and transversal distributions, respectively. The *Vaisman connection* on M is a connection ∇^v on TM uniquely defined by the following conditions, [7]:

a) If $Y \in D'$ (respectively $\in D$), then $\nabla^v_X Y \in D'$ (respectively $\in D$), for every vector field X.

b) $v(T^v(X, Y)) = 0$, (respectively $h(T^v(X, Y)) = 0$) if at least one of the arguments is in D', respectively in D, where v, h are the projection morphisms of TM on D' and D, respectively.

c) The induced connections on D' and D are metric connections.

The foliated manifold (M, g) is called a *Reinhart space* if

$$(\nabla^v_X g)(Y, Y') = 0, \quad (\forall) X \in D', Y, Y' \in D. \tag{3.1}$$

We return now to the Finsler manifold (M, F, G). Let $v : VTM^0 \to L'$, $h : VTM^0 \to L$ be the projection morphisms. We search for a connection on VTM^0 with the same properties as the Vaisman connection:

a) If $Y \in L'$ (respectively $\in L$), then $\nabla^v_X Y \in L'$ (respectively $\in L$), for every vertical vector field X.

b) $v(T^v(X, Y)) = 0$, (respectively $h(T^v(X, Y)) = 0$) if at least one of the arguments is in L', respectively in L, where T^v is the torsion of the connection ∇^v.

c) For every $X, Y, Y' \in L'$ (respectively $\in L$),

$$(\nabla^v_X g)(Y, Y') = 0. \tag{3.2}$$

d) Moreover, we need to give $\nabla^v_X Y$ for every $X \in HTM^0$ and we put the above a), b) conditions also for X an horizontal vector field.

We request the above conditions on the adapted basis $\{X_1, ..., X_{n-1}, Z\}$ and let s^c_{ab}, s^c_a, s_a, s, be the local coefficients of ∇^v, which means:

$$\nabla^v_{X_b} X_a = s^c_{ab} X_c; \quad \nabla^v_Z X_a = s^c_a X_c; \quad \nabla^v_{X_a} Z = s_a Z; \quad \nabla^v_Z Z = sZ, \tag{3.3}$$

for all $a, b, c = \overline{1, n-1}$. Taking into account Proposition 2.2, we compute

$$T^v(X_a, X_b) = (s^c_{ab} - s^c_{ba}) X_c - t_a X_b + t_b X_a \in L', \tag{3.4}$$

and by condition b) it results

$$s^c_{ab} = s^c_{ba}, \quad (\forall) c \neq a, b, \quad s^b_{ba} = s^b_{ab} + t_a, \tag{3.5}$$

for all $a, b, c = \overline{1, n-1}$. We also have

$$T^v(X_a, Z) = s_a Z - s^b_a X_b - X_a = 0,$$

hence

$$s_a = 0; \quad s^b_a = 0, \quad (\forall) b \neq a; \quad s^a_a = -1, \quad (\forall) a = \overline{1, n-1}. \tag{3.6}$$

Since now we have

$$\nabla^v_{X_a} Z = 0; \quad \nabla^v_Z X_a = -X_a, \quad (\forall) a = \overline{1, n-1}. \tag{3.7}$$

By conditions c) and (2.2),

$$(\nabla^v_Z G)(Z, Z) = 0 \Rightarrow Z(F^2) = 2sF^2 \Rightarrow s = 1,$$

since $Z(F^2) = y^i \frac{\partial}{\partial y^i}(y^j y^k g_{jk}) = 2F^2$. We also have

$$(\nabla^v_{X_a} G)(X_b, X_c) = 0 \Rightarrow X_a(G(X_b, X_c)) - G(\nabla^v_{X_a} X_b, X_c) - G(\nabla^v_{X_a} X_c, X_b) = 0,$$

for all $a, b, c = \overline{1, n}$. We compute $X_a(G(X_b, X_c)) = \frac{\partial g_{bc}}{\partial y^a} - t_c h_{ba} + t_b h_{ca}$, where h_{ab} are given by (2.13). Using the same method while we determined the Christofell's symbols, we obtain:

$$2G(\nabla^v_{X_a} X_b, X_c) = \frac{\partial g_{bc}}{\partial y^a} - 2t_b h_{ca},$$

hence

$$2s^d_{ba} h_{dc} = \frac{\partial g_{bc}}{\partial y^a} - 2t_b h_{ca}. \tag{3.8}$$

Let $(h^{ab})_{a,b=\overline{1,n}}$ be the inverse of the matrix h from (2.13). Finally, we have

$$s^d_{ba} = \frac{1}{2} h^{dc} \frac{\partial g_{bc}}{\partial y^a} - t_b h^{dc} h_{ca} = \frac{1}{2} h^{dc} \frac{\partial g_{bc}}{\partial y^a} - t_b \delta^d_a,$$

which means

$$s^d_{ba} = \frac{1}{2}h^{dc}\frac{\partial g_{bc}}{\partial y^a}, \quad (\forall) d \neq a, \quad s^a_{ba} = \frac{1}{2}h^{ac}\frac{\partial g_{bc}}{\partial y^a} - t_b, \tag{3.9}$$

One can see that the conditions (3.5) are also satisfied.

The conditions d): Let β^b_{ai}, β_i be the coefficient functions

$$\nabla^v_{\frac{\delta}{\delta x^i}} X_a = \beta^b_{ai} X_b, \quad \nabla^v_{\frac{\delta}{\delta x^i}} Z = \beta_i Z,$$

the rest being zero.

We have $[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}] = \frac{\partial G^k_i}{\partial y^j}\frac{\partial}{\partial y^k}$, so we can compute

$$[\frac{\delta}{\delta x^i}, Z] = 0, \quad [\frac{\delta}{\delta x^i}, X_a] = \frac{\partial G^j_i}{\partial y^a}\frac{\partial}{\partial y^j} - \frac{\delta t_a}{\delta x^i} Z.$$

If we request similar conditions with b), we have

$$\beta_i = 0, \quad \beta^b_{ai} X_b = \frac{\partial G^j_i}{\partial y^a} X_j, \tag{3.10}$$

taking into account (2.12)

Proposition 3.1. *The local coefficients of the connection ∇^v on VTM^0 defined by conditions a), b), c), d) described above are given with respect to the adapted basis $\{X_1, ..., X_{n-1}, Z\}$ by the relations (3.6), (3.9), (3.10) and $\nabla^v_Z Z = Z$.*

Now, for every fixed point $x_0 \in M$, the leaf $T_{x_0}M$ of the vertical foliation F_V is also a Riemannian foliated manifold by L', see Proposition 1.1. The Riemannian metric is

$$h_{x_0} == \begin{pmatrix} (h_{ab}(x_0, y))_{a,b} & 0_{n-1} \\ 0^t_{n-1} & F^2(x_0, y) \end{pmatrix}.$$

The connection ∇^v is exactly the Vaisman connection on $(T_{x_0}M, h_{x_0}, L')$ and we have:

Proposition 3.2. *For every fixed point $x_0 \in M$, $(T_{x_0}M, h_{x_0}, L')$ is a Reinhart space.*

Proof: Indeed, we can compute

$$(\nabla^v_{X_a} G)(Z, Z) = X_a(F^2) - 2G(\nabla^v_{X_a} Z, Z) = X_a(F^2) = \frac{\partial F^2}{\partial y^a} - t_a Z(F^2) = 0,$$

for every $a = \overline{1, n-1}$, since $Z(F^2) = 2F^2$, and we have (2.1), (2.5).

In the end, we have to remark that the Levi-Civita connection ∇ of G on TM^0 also induces a connection ∇' on the vertical bundle by

$$\nabla'_X(VY) = V(\nabla_X(VY)), \quad V : TTM^0 \to VTM^0.$$

This connection has the well-known locally expression

$$\nabla'_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = C^k_{ij} \frac{\partial}{\partial y^k}, \quad C^k_{ij} = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h}.$$

We obtain

$$\nabla'_Z X_a = 0,$$

which is different by $\nabla^v_Z X_a = -X_a$. In conclusion we give a new connection on the vertical bundle of a Finsler manifold.

Acknowledgement: The present research was supported by the Romanian Academy Grant 4/2009.

References

[1] Bejancu, A., Farran, H.R., *Finsler Geomerty and Natural Foliations on the Tangent Bundle*, Reports on Mathematical Phisics, **58** (2006), 131-146.

[2] Bălan, V., Manea, A., *Leafwise 2-jet cohomology on foliated Finsler manifolds*, Balkan Society of Geometers Proceedings (BSGP) **16**, Geometry Balkan Press, Bucharest 2009, p. 28-41.

[3] Miron, R., Anastasiei, M., *Vector bundles, Lagrange spaces. Applications to the theory of relativity*, Balkan Press, Bucureşti, 1997.

[4] Manea, A., *Some new types of vertical 2-jets on the tangent bundle of a Finsler manifold*, submitted.

[5] Manea, A., *Mastrogiacomo leafwise cohomology on a foliated manifold*, Bull. Math. Soc. Sci. Romania, **46** (2003), 75-86.

[6] Tondeur, Ph., *Foliations on Riemannian Manifolds*, Universitext, Springer-Verlag, New-York, 1988.

[7] Vaisman, I., *Cohomology and differential forms*, Marcel Dekker Inc., New York, 1973.