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A NOTE ON LOCALLY CONFORMAL COMPLEX LAGRANGE SPACES

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Abstract

In this note we study locally conformal complex Lagrange spaces. This notion is introduced here by analogy with the real case [12], [13].

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1 Introduction and preliminaries notions

Complex Lagrange geometry [6], is the extension of complex Finsler geometry [1], [2], [4], to transversal "metrics" of the vertical foliation (the foliation by fibers) of holomorphic tangent bundle, which are defined as the complex Hessian of a non degenerate complex Lagrangian function. In the present paper, we consider the generalization of complex Lagrange geometry to an arbitrary tangent complex manifold, namely an affine foliated complex manifold endowed with a tangent structure. In the first section we briefly recall the tangent complex manifold notion [7]. In the second section we define the locally conformal complex Lagrange spaces, and using the complex Cartan tensors we study some local properties.

Let $\mathcal{M}$ be a foliated complex manifold [10], [11], with $2n$ dimension and n codimension, and $u=(z^i,\eta^i)$ complex coordinates in a local map (U,φ).

Definition 1. *$\mathcal{M}$ is said to be an affine foliated complex manifold if the local changes $(U_\alpha,\varphi_\alpha)\to(U_\beta,\varphi_\beta)$ are given by:*

$$z'^i = z'^i(z)\ ;\ \eta'^i = \frac{\partial z'^i}{\partial z^j}\eta^j + B'^i(z) \tag{1}$$

where z'^i and B'^i are holomorphic functions on z^j variables and $det(\frac{\partial z'^i}{\partial z^j})\neq 0$.

The leafs of this manifold, denoted by $\mathcal{V}$, are characterized by $z^i = const$.

Let J be the natural complex structure of the manifold and $T'\mathcal{M}$ and $T''\mathcal{M}=\overline{T'\mathcal{M}}$ be its holomorphic and antiholomorphic subbundles, respectively. By $T_{\mathbb{C}}\mathcal{M}=T'\mathcal{M}\oplus T''\mathcal{M}$

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we denote the complexified tangent bundle of the real tangent bundle $T_{\mathbb{R}}\mathcal{M}$. From (1) the following changes for the natural local frames on $T^{'}_{u}\mathcal{M}$ are true:

$$\frac{\partial}{\partial z^k} = \frac{\partial z^{'i}}{\partial z^k}\frac{\partial}{\partial z^{'i}} + (\frac{\partial^2 z^{'i}}{\partial z^j \partial z^k}\eta^j + \frac{\partial B^{'i}}{\partial z^k})\frac{\partial}{\partial \eta^{'i}} \ ; \ \frac{\partial}{\partial \eta^k} = \frac{\partial z^{'i}}{\partial z^k}\frac{\partial}{\partial \eta^{'i}} \quad (2)$$

By conjugation over all in (2) we obtain the change rules of the local frames on $T^{''}_{u}\mathcal{M}$, and then behaviour of the J complex structure is

$$J(\frac{\partial}{\partial z^k}) = i\frac{\partial}{\partial z^k} \, , \, J(\frac{\partial}{\partial \eta^k}) = i\frac{\partial}{\partial \eta^k} \, , \, J(\frac{\partial}{\partial \bar{z}^k}) = -i\frac{\partial}{\partial \bar{z}^k} \, , \, J(\frac{\partial}{\partial \overline{\eta}^k}) = -i\frac{\partial}{\partial \overline{\eta}^k} \quad (3)$$

Note that inside of (1) we can take into account the more general affine changes $\eta^{'i} = A^{'i}_j(z)\eta^j + B^{'i}(z)$, where $A^{'i}_j$ are also holomorphic functions. As well as in (1) we can consider the particular case $B^{'i} = 0$, when $\mathcal{M}$ is identified with the holomorphic tangent bundle of a complex manifold M, namely $\mathcal{M} = T^{'}M$. In this last situation we will say that $\mathcal{M}$ is of *holomorphic vector type*.

However, the general discussion $A^{'i}_j \neq \frac{\partial z^{'i}}{\partial z^j}$ limits the opportunity of other significant structures on $\mathcal{M}$ to considered. Such important for the geometry of $T_{\mathbb{C}}\mathcal{M}$ is the tangent structure defined by

$$S(\frac{\partial}{\partial z^k}) = \frac{\partial}{\partial \eta^k} \, , \, S(\frac{\partial}{\partial \eta^k}) = 0 \, , \, S(\frac{\partial}{\partial \bar{z}^k}) = \frac{\partial}{\partial \overline{\eta}^k} \, , \, S(\frac{\partial}{\partial \overline{\eta}^k}) = 0 \quad (4)$$

Proposition 1. *S is a global tangent structure acting on $T_{\mathbb{C}}\mathcal{M}$. Moreover, S is integrable.*

Proof. We easily check that $S^2 = 0$ and relation (4) is preserved by (2) changes. Since the Nijenhuis tensor

$$N_S(X,Y) = [S(X), S(Y)] - S([S(X), Y]) - S([X, S(Y)]) + S^2([X,Y]) = 0$$

for any $X, Y \in \Gamma(T_{\mathbb{C}}\mathcal{M})$, it follows that S is integrable. □

Proposition 2. *([7]) $(S, J, J \circ S = S \circ J)$ determines a commutative and integrable semiquaternionic structure on $T_{\mathbb{C}}\mathcal{M}$.*

Definition 2. *A pair $(\mathcal{M}, S)$ is called a tangent complex manifold.*

Let us consider the structural bundle of $\mathcal{V}$ given by $T^{'}\mathcal{V} = span\{\frac{\partial}{\partial \eta^i}\} \subset T^{'}\mathcal{M}$, or vertical distribution, which in view of (2) is an integrable and holomorphic one. According to [11], the transversal bundle of $\mathcal{V}$ is given by $N^{'}\mathcal{V} = T^{'}\mathcal{M}/T^{'}\mathcal{V}$ with holomorphic projection $p : T^{'}\mathcal{M} \to N^{'}\mathcal{V}$, with local bases defined by the equivalence classes $p(\frac{\partial}{\partial z^i}) = [\frac{\partial}{\partial z^i}]$ and with the transition functions $(\frac{\partial z^{'i}}{\partial z^j})$.

A normalization of the holomorphic foliation $\mathcal{V}$ is defined by a supplementary distribution $T^{'}\mathcal{H}$ of $T^{'}\mathcal{V}$ in $T^{'}\mathcal{M}$, locally spanned in $(U_\alpha, \varphi_\alpha)$ by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\}$, called a *complex nonlinear connection* on $\mathcal{M}$, briefly c.n.c.

However $T^{'}\mathcal{H}$ is smoothly isomorphic to $N^{'}\mathcal{V}$ which is holomorphic as $T^{'}\mathcal{V}$. Generally $T^{'}\mathcal{H}$ is not holomorphic subbundle of $T^{'}\mathcal{M}$. The existence of a holomorphic supplementary distribution $T^{'}\mathcal{H}$ is characterized in the general case of holomorphic foliations by the vanishing of a certain cohomological obstruction (for details see [11]).

By conjugation, we obtain the conjugated distributions $T^{''}\mathcal{V} = span\{\frac{\partial}{\partial\overline{\eta}^i}\}$ and $T^{''}\mathcal{H} = span\{\frac{\delta}{\delta\overline{z}^k} = \frac{\partial}{\partial\overline{z}^k} - N_{\overline{k}}^{\overline{j}}\frac{\partial}{\partial\overline{\eta}^j}\}$. The adapted coframes are given by $\{dz^k\}$, $\{\delta\eta^k = d\eta^k + N_j^k dz^j\}$, $\{d\overline{z}^k\}$ and $\{\delta\overline{\eta}^k = d\overline{\eta}^k + N_{\overline{j}}^{\overline{k}} d\overline{z}^j\}$ which span the dual bundles $T^{'*}\mathcal{H}$, $T^{'*}\mathcal{V}$, $T^{''*}\mathcal{H}$ and $T^{''*}\mathcal{V}$, respectively.

2 Locally conformal complex Lagrange spaces

In this section we generalized the notion of complex Lagrangians [6] for a general tangent complex manifold $(\mathcal{M}, S)$, and we refer to functions on $\mathcal{M}$ as *global complex Lagrangians* and to functions on open subsets of $\mathcal{M}$ as *local complex Lagrangians*.

Let $L_\alpha : \mathcal{M} \to \mathbb{R}$ be a Lagrangian function, defined on $U_\alpha \subset \mathcal{M}$, domain of local chart.

Definition 3. *We say that a family $\{\mathcal{M}, L_\alpha\}$ is a local complex Lagrange structure on $\mathcal{M}$, if there exists an atlas such that $g_{L_\alpha} = g_{i\overline{j}} = \partial^2 L_\alpha / \partial\eta^i \partial\overline{\eta}^j$ glue up to a global Hermitian metric on $\mathcal{M}$.*

If $\{\mathcal{M}, L_\alpha\}$ defines a local complex Lagrange structure on $\mathcal{M}$ then, by integration of $g_{i\overline{j}}$, we obtain a complex Lagrangian $L : \mathcal{M} \to \mathbb{R}$ such that $L_\alpha = L|_{U_\alpha} + l_\alpha$, where l_α is an affine real valued vertical form on $\mathcal{M}$, i.e. there exist $\overset{\alpha}{A}_i(z)$ and $\overset{\alpha}{B}(z) \in \mathbb{R}$ such that

$$L_\alpha = L|_{U_\alpha} + \overset{\alpha}{A}_i (\eta^i + \overline{\eta}^i) + \overset{\alpha}{B} \tag{5}$$

From (5), it follows that on the intersection $U_\alpha \cap U_\beta \neq \phi$ the local complex Lagrangians satisfy a compatibility relation of the form

$$L_\beta - L_\alpha = \overset{\alpha\beta}{A}_i (z)(\eta^i + \overline{\eta}^i) + \overset{\alpha\beta}{B}(z) \tag{6}$$

where $\overset{\alpha\beta}{A}_i(z)$ and $\overset{\alpha\beta}{B}(z)$ are real valued functions on $U_\alpha \cap U_\beta$.

On the intersection $U_\alpha \cap U_\beta$ we can define a cocycle $L_{\alpha\beta} := L_\beta - L_\alpha$, which is closed with respect to differential $(\delta L)_{\alpha\beta\gamma} := L_{\alpha\beta} - L_{\alpha\gamma} + L_{\beta\gamma} = 0$. Let $[L_\alpha] \in H^1(\mathcal{M}, \mathcal{A}_R^0(\mathcal{M}, \mathcal{V} \oplus \overline{\mathcal{V}}))$ be its cohomology class, where $\mathcal{A}_R^0(\mathcal{M}, \mathcal{V} \oplus \overline{\mathcal{V}})$ is the sheaf of germs of functions on $\mathcal{M}$ given by the right-hand side of (6). We have,

Proposition 3. $[L_\alpha] = 0$ *yields L is globally defined.*

In [7] are estabilished some conditions when $[L_\alpha] = 0$.

Definition 4. *A transversal metric $g \in \Gamma(N^{'*}\mathcal{V} \otimes N^{''*}\mathcal{V})$ of the vertical foliation $\mathcal{V}$ of a tangent complex manifold $(\mathcal{M}, S)$ is a symmetric 2 - covariant tensor field on $\mathcal{M}$, which is annihilated by imS.*

Similarly to [6], we define the complex Cartan tensors associated with the transversal metric $g_{i\bar{j}}$, by

$$C_{i\bar{j}k} = \frac{\partial g_{i\bar{j}}}{\partial \eta^k} \; ; \; C_{i\bar{j}\,\bar{k}} = \frac{\partial g_{i\bar{j}}}{\partial \overline{\eta}^k} \tag{7}$$

which generalized the Cartan tensor from [5] to complex Lagrange geometry.

Proposition 4. *The transversal metric g of the vertical foliation $\mathcal{V}$ of a tangent complex manifold $(\mathcal{M}, S)$ is a locally complex Lagrange structure if and only if the complex Cartan tensors satisfy the symmetry relations*

$$C_{i\bar{j}k} = C_{k\bar{j}i} \; ; \; C_{i\bar{j}\,\bar{k}} = C_{i\bar{k}\,\bar{j}} \tag{8}$$

Proof. If $g_{i\bar{j}} = \partial^2 L_\alpha / \partial\eta^i \partial\overline{\eta}^j$ is a locally complex Lagrange structure, then it is obiouvs that one has (8). Conversely, if we have (8), then by the same argument as in [1], [6] for the case $\mathcal{M} = T^{'}M$, there exists a local function L_α on U_α such that $g_{i\bar{j}} = \partial^2 L_\alpha / \partial\eta^i \partial\overline{\eta}^j$. □

Definition 5. *A locally conformal complex Lagrange structure on $\mathcal{M}$ is a maximal open covering $\mathcal{M} = \cup U_\alpha$ with local regular complex Lagrangians L_α such that, over the intersection $U_\alpha \cap U_\beta$, the local Lagrangians metrics satisfy a relation of the form,*

$$g_{L_\beta} = f_{\alpha\beta} g_{L_\alpha} \tag{9}$$

where $f_{\alpha\beta} > 0$ are foliated functions on $U_\alpha \cap U_\beta$.

Definition 6. *A tangent complex manifold $(\mathcal{M}, S)$ endoweed with this type of structure is called a locally conformal complex Lagrange space.*

Condition (9) is equivalent with the transition relations

$$L_\beta = f_{\alpha\beta} L_\alpha + \overset{\alpha\beta}{A_i} (\eta^i + \overline{\eta}^i) + \overset{\alpha\beta}{B} \tag{10}$$

where the last two terms are as in (6).

On the other hand, from (9) it results that $\{ln f_{\alpha\beta}\}$ is a Φ - valued 1 - cocycle, where Φ is the sheaf of germs of foliated functions on $\mathcal{M}$, and may be written as $ln f_{\alpha\beta} = \psi_\beta - \psi_\alpha$ where ψ_α is a smooth function on U_α (which may be assumed projectable (foliated) only if the cocycle is a coboundary).

Accordingly, the formula

$$g|_{U_\alpha} = e^{-\psi_\alpha} g_{L_\alpha} \tag{11}$$

defines a global transversal metric of the vertical foliation $\mathcal{V}$, which is locally conformal with local complex Lagrange metrics.

Proposition 5. *Let $(\mathcal{M}, S)$ be a tangent complex manifold. Then $\mathcal{M}$ is a locally conformal complex Lagrange space if and only if $\mathcal{M}$ has a global transversal metric g of the vertical foliation $\mathcal{V}$, which is locally conformal with local complex Lagrange metrics.*

Proof. We have only to prove that the existence of the metric g that satisfies (11) implies (9). Let g be a global transversal metric of the vertical foliation $\mathcal{V}$ of $(\mathcal{M}, S)$. Then, on the intersection $U_\alpha \cap U_\beta$ the relation $g|_{U_\alpha} = e^{-\psi_\alpha} g_{L_\alpha}$ leads to $g_{L_\beta} = e^{\psi_\beta - \psi_\alpha} g_{L_\alpha}$. The fact that $f_{\alpha\beta} = e^{\psi_\beta - \psi_\alpha} > 0$ are projectable follows from the Lagrangian character of the metrics g_{L_α} and g_{L_β}.

Indeed, with the usual local complex coordinates (z^i, η^i) we calculate in relation (9) the complex Cartan tensors of g_{L_α} and g_{L_β}:

$$\overset{\beta}{C}_{i\bar{j}k} = \frac{\partial (g_{L_\beta})_{i\bar{j}}}{\partial \eta^k} = \frac{\partial f_{\alpha\beta}}{\partial \eta^k} (g_{L_\alpha})_{i\bar{j}} + f_{\alpha\beta} \overset{\alpha}{C}_{i\bar{j}k}$$

$$\overset{\beta}{C}_{k\bar{j}i} = \frac{\partial (g_{L_\beta})_{k\bar{j}}}{\partial \eta^i} = \frac{\partial f_{\alpha\beta}}{\partial \eta^i} (g_{L_\alpha})_{k\bar{j}} + f_{\alpha\beta} \overset{\alpha}{C}_{k\bar{j}i}$$

By Proposition 4, we have $\overset{\beta}{C}_{i\bar{j}k} = \overset{\beta}{C}_{k\bar{j}i}$ and $\overset{\alpha}{C}_{i\bar{j}k} = \overset{\alpha}{C}_{k\bar{j}i}$. Then

$$\frac{\partial f_{\alpha\beta}}{\partial \eta^k} (g_{L_\alpha})_{i\bar{j}} = \frac{\partial f_{\alpha\beta}}{\partial \eta^i} (g_{L_\alpha})_{k\bar{j}} \tag{12}$$

and a contraction by $(g_{L_\alpha})^{\bar{j}i}$ in (12) leads to $\frac{\partial f_{\alpha\beta}}{\partial \eta^k} = 0$. Similarly, from $\overset{\beta}{C}_{i\bar{j}\,\bar{k}} = \overset{\beta}{C}_{i\bar{k}\,\bar{j}}$ and $\overset{\alpha}{C}_{i\bar{j}\,\bar{k}} = \overset{\alpha}{C}_{i\bar{k}\,\bar{j}}$ we obtain $\frac{\partial f_{\alpha\beta}}{\partial \bar{\eta}^k} = 0$. □

The cohomology class $\sigma = [lnf_{\alpha\beta}] \in H^1(\mathcal{M}, \Phi)$ will be called the complementary class of the metric g, and the locally conformal complex Lagrange metric g is a locally complex Lagrange metric iff $\sigma = 0$.

Indeed, if $\sigma = 0$, we may assume that the functions ψ_α are foliated and the complex Cartan tensors of $g = e^{-\psi_\alpha} g_{L_\alpha}$ satisfy the symmetry relations (8).

Let $T^{'}\mathcal{H}$ be a supplementary distribution of $T^{'}\mathcal{V}$ and $d^{'v}$ and $d^{''v}$ be the vertical differential operators (see[8]). Then, according to [9], the foliated differential d_f is related by $d_f = d^{'v} + d^{''v}$. Using the complex leafwise version of the de Rham theorem, the complementary class σ can be seen as the $(d^{'v} + d^{''v})$ - cohomology class of the global $(d^{'v} + d^{''v})$ - closed complementary form τ obtained by gluing up the local forms $\{(d^{'v} + d^{''v})\psi_\alpha\}$.

Finally, we give an example of the above discussion, represented by the Hopf manifold. Let Δ_λ be the cyclic group of the holomorphic transformation on $\mathbb{C} - \{0\}$ generated by $(z^1, ..., z^n) \to (\lambda z^1, ..., \lambda z^n)$ for $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$. The discrete group Δ_λ acts on $\mathbb{C} - \{0\}$ freely and properly. The quotient space $H^n := \mathbb{C} - \{0\}/\Delta_\lambda$ is a complex manifold, so-called Hopf manifold (see [9]).

The Hermitian metric $h = \frac{1}{|z|^2} dz^k \otimes d\bar{z}^k$ on $\mathbb{C} - \{0\}$ is invariant by the action of Δ_λ. Hence it defines a Hermitian metric h on H^n. The local functions $\sum_{i=1}^n \eta^i \bar{\eta}^i$ define a locally conformal complex Lagrange structure on $\mathcal{M} = T^{'} H^n$ and

$$g = \frac{1}{|z|^2} \sum_{i=1}^n \eta^i \bar{\eta}^i$$

is corresponding global metric [3], which, with previously used notation, corresponds to $\Psi_\alpha = ln|z|^2$. The corresponding complementary form is $\tau = 0$.

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