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OPTIMAL INEQUALITIES FOR SUBMANIFOLDS IN QUATERNION-SPACE-FORMS WITH SEMI-SYMMETRIC METRIC CONNECTION

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Abstract

We establish a version of B.-Y. Chen's inequality for totally real submanifolds of quaternion-space-forms with semi-symmetric metric connection. Also for quaternion CR-submanifolds of quaternion-space-forms with semi-symmetric metric connection we obtain an optimal inequality concerning the Ricci curvature.

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1 Preliminaries

Let $\tilde{M}$ be an m-dimensional Riemannian manifold with the Riemannian metric g, the linear connection $\tilde{\nabla}$ and the Riemannian connection $\dot{\tilde{\nabla}}$. For the vector fields $\tilde{X}, \tilde{Y}$ on $\tilde{M}$ the torsion tensor field $\tilde{T}$ of the linear connection $\tilde{\nabla}$ is defined by

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}\tilde{X} - [\tilde{X}, \tilde{Y}]. \tag{1}$$

A linear connection $\tilde{\nabla}$ is said to be *semi-symmetric connection* if the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ satisfies

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \Phi(\tilde{Y})\tilde{X} - \Phi(\tilde{X})\tilde{Y}, \tag{2}$$

where Φ is a 1-form on $\tilde{M}$. Further, if $\tilde{\nabla}$ satisfies the condition

$$\tilde{\nabla} g = 0,$$

then $\tilde{\nabla}$ is called a *semi-symmetric metric connection* [14]. K. Yano obtained in [14] a relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection $\dot{\tilde{\nabla}}$ which is given by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \dot{\tilde{\nabla}}_{\tilde{X}}\tilde{Y} + \Phi(\tilde{Y})\tilde{X} - g(\tilde{X}, \tilde{Y})P, \tag{3}$$

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where P is a vector field given by $g(P,\tilde{X})=\Phi(\tilde{X})$ for any vector field $\tilde{X}$ on $\tilde{M}$.
We denote by $\tilde{R}$ and $\dot{\tilde{R}}$ the curvature tensors associated to $\tilde{\nabla}$ and $\dot{\tilde{\nabla}}$, respectively.

Let M be an n-dimensional Riemannian submanifold of the Riemannian manifold $\tilde{M}$ with the induced semi-symmetric metric connection ∇ and the induced Riemannian connection $\dot{\nabla}$.
The Gauss formulae are

$$\tilde{\nabla}_X Y = \nabla_X^Y + h(X,Y)$$
$$\dot{\tilde{\nabla}}_X Y = \dot{\nabla}_X^Y + \dot{h}(X,Y),$$

for any X,Y vector fields on M, where h is a $(0,2)$ symmetric tensor on M and $\dot{h}$ is the second fundamental form associated to the Riemannian connection $\dot{\nabla}$ [12].
We denote by R and $\dot{R}$ the curvature tensors associated to ∇ and $\dot{\nabla}$, respectively. Let $\tilde{M}$ be an $4m$-dimensional Riemannian manifold with the Riemannian metric g. $\tilde{M}$ is called a quaternionic Kaehlerian manifold if there exists a 3-dimensional vector space V of tensors of type (1,1) with local basis of almost Hermitian structure I, J and K such that
(a) $IJ=-JI=K, JK=-KJ=I, KI=-IK=J, I^2=J^2=K^2=-1$,
(b) for any local cross-section φ of V, $\dot{\tilde{\nabla}}_{\tilde{X}}\varphi$ is also a cross-section of V, where $\tilde{X}$ is an arbitrary vector field on $\tilde{M}$ and $\dot{\tilde{\nabla}}$ the Riemannian connection on $\tilde{M}$.
The condition (b) is equivalent to the following condition:
(b') there exist the local 1-forms p, q and r such that

$$\begin{aligned} \dot{\tilde{\nabla}}_{\tilde{X}} I &= r(\tilde{X})J - q(\tilde{X})K \\ \dot{\tilde{\nabla}}_{\tilde{X}} J &= -r(\tilde{X})I + p(\tilde{X})K \\ \dot{\tilde{\nabla}}_{\tilde{X}} K &= q(\tilde{X})I - p(\tilde{X})J. \end{aligned} \tag{4}$$

Let $\tilde{X}$ be a unit vector on $\tilde{M}$. Then $\tilde{X}$, $I\tilde{X}$, $J\tilde{X}$ and $K\tilde{X}$ form an orthonormal frame on $\tilde{M}$, denoting by $Q(\tilde{X})$ the 4-plane spanned by them. For any two orthonormal vectors $\tilde{X}$, $\tilde{Y}$ on $\tilde{M}$, we denote by $\pi(\tilde{X},\tilde{Y})$ the 2-plane spanned by $\tilde{X}$ and $\tilde{Y}$. If Q($\tilde{X}$) and Q($\tilde{Y}$) are orthogonal, the plane $\pi(\tilde{X},\tilde{Y})$ is called a *totally real plane*. Any 2-plane in $Q(\tilde{X})$ is called a *quaternionic plane*. A sectional curvature of a quaternionic plane π is called the *quaternionic sectional curvature* of π. A quaternionic Kaehlerian manifold is *a quaternion-space-form* if its quaternionic sectional curvatures are equal to a constant 4c. We denote an $4m$-dimensional quaternion-space-form by $\tilde{M}(4c)$. A quaternionic Kaehlerian manifold $\tilde{M}$ is a quaternion-space-form if and only if its curvature tensor $\dot{\tilde{R}}$ has the following form [8]:

$$\begin{aligned} \dot{\tilde{R}}(\tilde{X},\tilde{Y})\tilde{Z} = {} & c\{g(\tilde{Y},\tilde{Z})\tilde{X} - g(\tilde{X},\tilde{Z})\tilde{Y} + g(I\tilde{Y},\tilde{Z})I\tilde{X} - g(I\tilde{X},\tilde{Z})I\tilde{Y} + 2g(\tilde{X},I\tilde{Y})I\tilde{Z} \\ & + g(J\tilde{Y},\tilde{Z})J\tilde{X} - g(J\tilde{X},\tilde{Z})J\tilde{Y} + 2g(\tilde{X},J\tilde{Y})J\tilde{Z} \\ & + g(K\tilde{Y},\tilde{Z})K\tilde{X} - g(K\tilde{X},\tilde{Z})K\tilde{Y} + 2g(\tilde{X},K\tilde{Y})K\tilde{Z}\} \end{aligned} \tag{5}$$

for the vectors $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ tangent to $\tilde{M}(4c)$.
Let (M,g) be an n-dimensional Riemannian submanifold of the quaternion-space-form

$\tilde{M}(4c)$. A submanifold M is called a *totally real submanifold* of $\tilde{M}(4c)$ if any 2-plane $\pi(X,Y)$ of M (spanned by any orthonormal vectors X,Y of M) is contained by a totally real plane of $\tilde{M}(4c)$. Also $Q(X)$ and $Q(Y)$ are orthogonal and $g(X,\varphi Y) = g(\psi X, Y) = 0$ for φ, ψ = I, J or K. Consequently, if M is a totally real submanifold of $\tilde{M}(4c)$, then $\varphi(TM) \subset T^{\perp}M$ for φ = I, J or K, where $T^{\perp}M$ is the normal bundle of M in $\tilde{M}(4c)$[5] .

By (5) results the following relation (see [12]):

$$\dot{\tilde{R}}(X,Y,Z,W) = c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}, \tag{6}$$

for any X, Y, Z, W vector fields on M.
If α is $(0,2)$-tensor such that:

$$\alpha(X,Y) = (\dot{\tilde{\nabla}}_X \Phi)Y - \Phi(X)\Phi(Y) + \frac{1}{2}\Phi(P)g(X,Y),$$

for any X, Y vector fields of M it occurs (see [7]):

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = \dot{\tilde{R}}(X,Y,Z,W) \; &- \; \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) \\ &- \; \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).\end{aligned} \tag{7}$$

By (6) and (7) we obtain:

$$\begin{aligned}\dot{\tilde{R}}(X,Y,Z,W) &= c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &- \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) \\ &- \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).\end{aligned} \tag{8}$$

The Gauss equation is

$$\dot{\tilde{R}}(X,Y,Z,W) = \dot{R}(X,Y,Z,W) + g(\dot{h}(X,Z),\dot{h}(Y,W)) - g(\dot{h}(X,W),\dot{h}(Y,Z)). \tag{9}$$

Let $\pi \subset T_pM$ and $\pi^{\perp} \subset T_p^{\perp}M$ be plane sections for any p in M and $K(\pi)$ the sectional curvature of M associated to induced semi-symmetric metric connection ∇.
In $\widetilde{M}(4c)$ we can choose a local orthonormal frame:

$$\begin{aligned}e_1,...,e_n,e_{n+1},...,e_m;\; e_{I(1)} = Ie_1,...,e_{I(m)} = Ie_m;\\ e_{J(1)} = Je_1,...,e_{J(m)} = Je_m;\; e_{K(1)} = Ke_1,...,e_{K(m)} = Ke_m = e_{4m},\end{aligned} \tag{10}$$

such that, restricting to M, $e_1,...,e_n$ are tangent to M.
We denote by τ the scalar curvature of M defined as $\tau(p) = \sum_{1\le i<j\le n} K(e_i \wedge e_j)$, by λ the trace of α. If we write $h^r_{ij} = g(h(e_i,e_j),e_r)$, we have $h^{\varphi(i)}_{jk} = h^{\varphi(j)}_{ik} = h^{\varphi(k)}_{ji}$ where $i,j,k \in \{1,...,n\}$, $r \in \{n+1,...,I(1),...,K(m)\}$, $\varphi \in \{I,J,K\}$. The squared length of h is

$$\|h\|^2 = \sum_{1\le i<j\le n} g(h(e_i,e_j),h(e_i,e_j)),$$

and the mean curvature vector of M associated to ∇ is $\vec{H} = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$, denoting by H the mean curvature of M associated to ∇. Similarly, the mean curvature vector of M associated to $\dot{\nabla}$ is $\dot{\vec{H}} = \frac{1}{n}\sum_{i=1}^{n} \dot{h}(e_i, e_i)$, denoting by $\dot{H}$ the mean curvature of M associated to $\dot{\nabla}$.

A submanifold M of a quaternion Kaehler manifold $\tilde{M}$ is called a *quaternion CR-submanifold* if there exist two orthogonal complementary distributions D_p and $D_p^{\perp}$ such that D_p is invariant under quaternion structure, that is, $\varphi(D_p) \subseteq D_p, i = 1, 2, 3, \forall p \in M$ and $D_p^{\perp}$ is totally real, that is, $\varphi_i(D_p^{\perp}) \subseteq T_pM, i = 1, 2, 3$, where we denoted $\varphi_1 = I, \varphi_2 = J$ and $\varphi_3 = K$. A submanifold M of a quaternion Kaehler manifold is a quaternion submanifold if dim $D^{\perp} = 0$. Let dim D $= 4s$ and dim $D^{\perp} = t$. For any X tangent to M, we put (see[11])

$$\varphi_i X = T_i X + F_i X, i = 1, 2, 3, \tag{11}$$

where T_iX (resp. F_iX) denotes tangential (resp. normal) component of φ_iX.

Recently A. Mihai and C. Ozgur established in [10] a Chen inequality for submanifolds of real space forms with a semi-symmetric metric connection. In the following we obtain a Chen inequality for totally real submanifolds in quaternion-space-forms with semi-symmetric metric connection and we estimate the Ricci curvature for quaternion CR-submanifolds of quaternion-space-forms with semi-symmetric metric connection, referring to [5] and [11] for basic results.

2 A Chen inequality for totally real submanifolds in quaternion-space-forms with semi-symmetric metric connection

We first recall an algebraic lemma (see [2]):

Lemma 1. *Let $a_1, ..., a_k, c$ be $k + 1$ $(k \geq 2)$ real numbers such that:*

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)\left(\sum_{i=1}^{k} a_i^2 + c\right). \tag{12}$$

Then $2a_1a_2 \geq c$, with equality holding if and only if $a_1 + a_2 = a_3 = ... = a_k$.

Now we can prove the following inequality.

Theorem 1. *Let M be an n-dimensional $(n \geq 2)$ totally real submanifold of an 4m-dimensional quaternion-space-form $\widetilde{M}(4c)$ with semi-symmetric metric connection $\tilde{\nabla}$. Then*

$$\tau(p) - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c - (n-2)\lambda - trace(\alpha|\pi^{\perp}) \tag{13}$$

Proof. The Gauss equation (see [12]) is:

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W)). \tag{14}$$

For $X = W = e_i$ and $Y = Z = e_j$, with $i \neq j \in \{1, ..., n\}$, (14) becomes:

$$\begin{aligned} \tilde{R}(e_i, e_j, e_j, e_i) &= c\{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_j, e_i)\} \\ &- \alpha(e_j, e_j)g(e_i, e_i) + \alpha(e_i, e_j)g(e_j, e_i) \\ &- \alpha(e_i, e_i)g(e_j, e_j) + \alpha(e_j, e_i)g(e_i, e_i) \\ &= c - \alpha(e_j, e_j) - \alpha(e_i, e_i). \end{aligned} \tag{15}$$

By (14) and (15) it results

$$c - \alpha(e_j, e_j) - \alpha(e_i, e_i) = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)). \tag{16}$$

We obtain by (16)

$$n(n-1)c - 2(n-1)\lambda = 2\tau - n^2 \|H\|^2 + \|h\|^2. \tag{17}$$

Denoting by

$$\epsilon = 2\tau - \frac{n^2(n-2)}{n-1} \|H\|^2 + 2(n-1)\lambda - (n+1)(n-2)c \tag{18}$$

and substituing (17) in (18) we obtain:

$$n^2 \|H\|^2 = (n-1)(\|h\|^2 + \epsilon - 2c). \tag{19}$$

Let $e_{n+1} = \frac{H}{\|H\|^2}$ be the unit vector in the direction H, then (19) can be rewritten as

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left\{\sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i\neq j} \left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - 2c + \epsilon\right\}. \tag{20}$$

Using the Chen's lemma, we get the following inequality:

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j} \left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - 2c + \epsilon. \tag{21}$$

Let $\pi \subset T_pM$ be a plane section, with p in M, spanned by the orthonormal vectors e_1 and e_2. Then by (14) the sectional curvature is given by

$$\begin{aligned} K(\pi) = c + &\sum_{r=n+1}^{4m} \left[h_{11}^r h_{22}^r - (h_{12}^r)^2\right] - \alpha(e_1, e_1) - \alpha(e_2, e_2) \\ &\geq c + \frac{1}{2}\sum_{i\neq j} \left(h_{ij}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{4m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - c + \frac{\epsilon}{2} \\ &+ \sum_{r=n+2}^{4m} h_{11}^r h_{11}^r - \sum_{r=n+1}^{4m} (h_{12}^r)^2 - \alpha(e_1, e_1) - \alpha(e_2, e_2) \\ &\geq \frac{\epsilon}{2} - \alpha(e_1, e_1) + \alpha(e_2, e_2), \end{aligned}$$

the inequality (13) beeing obtained, where $\alpha(e_1, e_1) + \alpha(e_2, e_2) = \lambda - trace(\alpha|\pi^{\perp})$. The inequality (13) is known as Chen inequality. $\square$

Remark 1. *If P is a tangent vector field on M (see [6]), then $H = \dot{H}, h = \dot{h}$. In these conditions the equality case of (13) holds at a point $p \in M$ if and only if , with respect to a suitable orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{4m}\}$ at p, the shape operators $A_r = A_{e_r}$ take the following forms:*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu, \tag{22}$$

and

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n + 2 \leq r \leq 4m. \tag{23}$$

From the proof of theorem the relations (22)and (23)occur having conditions:

$$\begin{aligned} h_{1j}^{n+1} = 0, h_{2j}^{n+1} = 0, j > 2, \\ h_{ij}^{n+1} = 0, i \neq j > 2, \\ h_{1j}^r = h_{2j}^r = h_{ij}^r = 0, r \in \{n+2, ..., 4m\}, i, j \in \{3, ..., n\}, \\ h_{11}^r + h_{22}^r = 0, r \geq n + 2, \\ h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ... = h_{nn}^{n+1}. \end{aligned}$$

Corollary 1. *Let M be an n-dimensional totally real submanifold of a quaternion-space-form $\tilde{M}(4c)$ with semi-symmetric metric connection $\tilde{\nabla}$. If the equality in (13) holds (in the conditions of remark above) and ξ is a normal vector at a point p of M, then the operator A_ξ has at most 3 eigenspaces and the dimension of one of the eigenspaces is at least $n - 2$. Moreover, one of the following cases occurs:*

1. *$A_\xi = 0$, when $a = b = 0$ or $h_{11}^r = h_{12}^r = 0$ for some r where $\xi = e_{n+1}$ or e_r.*
2. *A_ξ has a two-dimensional eigenspace with a non-zero real number α as the eigenvalue and an $(n-2)$-dimensional eigenspace with eigenvalue 2α. This occurs only when $a = b = \alpha, \mu = 2\alpha, \xi = e_{n+1}$.*
3. *A_ξ has an one-dimensional eigenspace with eigenvalue zero and an $(n-1)$-dimensional eigenspace with nonzero eigenvalue α. This occurs when $\alpha = \mu, b = 0, \xi = e_{n+1}$.*
4. *A_ξ has an eigenspace with the eigenvalue α, another eigenspace with the eigenvalue β, both of them are one-dimensional, and an $(n-2)$-dimensional eigenspace with the eigenvalue $\alpha + \beta$. This occurs when $\xi = e_{n+1}$, $\alpha \neq \beta$ or $\xi = e_r$ for some r with $\alpha = -\beta = \pm\sqrt{(h_{11}^r)^2 + (h_{12}^r)^2}$.*

3 On Ricci curvature of quaternion CR-submanifolds in quaternion space forms with semi-symmetric metric connection

Theorem 2. *Let M be an n-dimensional quaternion CR-submanifold of an $4m$-dimensional quaternion-space-form $\widetilde{M}(4c)$ with semi-symmetric metric connection $\tilde{\nabla}$. Then*
a) For each unit vector $X \in D_p^{\perp}$, we have

$$\|H\|^2 \geq \frac{4}{n^2}[Ric(X) - (n-1)c + (2n-3)\lambda - (n-2)\alpha(X,X)]. \tag{24}$$

b) For each unit vector $X \in D_p$, we have

$$\|H\|^2 \geq \frac{4}{n^2}[Ric(X) - (n+8)c + (2n-3)\lambda - (n-2)\alpha(X,X)]. \tag{25}$$

c) If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (24) and (25), respectively, if and only if $X \in D_p^{\perp} \cap N_p$ (respectively $X \in D_p \cap N_p$).

Proof. Let M be a quaternion CR-submanifold of a quaternion space form $\widetilde{M}(4c)$ with semi-symmetric metric connection $\tilde{\nabla}$. Then using equation (11) in Gauss equation we have for any vector fields X, Y, Z, W tangent to M

$$\begin{aligned} R(X,Y,Z,W) = c\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ +\sum_{i=1}^{3}[g(T_iY,Z)g(T_iX,W) - g(T_iX,Z)g(T_iY,W) + 2g(X,T_iY)g(T_iZ,W)]\} \\ +g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) \\ -\alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z) + A(X,Y,Z,W), \end{aligned}$$

where we denoted

$$\begin{aligned} A(X,Y,Z,W) = \sum_{i=1}^{3}[-\alpha(T_iX,W)g(T_iY,Z) - \alpha(T_iY,Z)g(T_iX,W) \\ +\alpha(T_iX,Z)g(T_iY,W) + \alpha(T_iY,W)g(T_iX,Z) \\ -2\alpha(T_iZ,W)g(X,T_iY) - 2\alpha(X,T_iY)g(T_iX,Z)]. \end{aligned}$$

Let $p \in M$ and an orthonormal basis $\{e_1, ..., e_n = X\}$ be in T_pM. Then the Ricci tensor $S(X,Y)$ is given by

$$S(X,Y)=\sum_{j=1}^{n}R(e_j,X,Y,e_j)=c\sum_{j=1}^{n}\{g(X,Y)g(e_j,e_j)-g(e_j,Y)g(X,e_j)$$
$$+\sum_{i=1}^{3}[g(T_iX,Y)g(T_ie_j,e_j)-g(T_ie_j,Y)g(T_iX,e_j)+2g(e_j,T_iX)g(T_iY,e_j)]\}$$
$$+\sum_{j=1}^{n}\{-\alpha(X,Y)g(e_j,e_j)+\alpha(e_j,Y)g(X,e_j)-\alpha(e_j,e_j)g(X,Y)+\alpha(X,e_j)g(e_j,Y)$$
$$+A(e_j,X,Y,e_j)+g(h(e_j,e_j),h(X,Y))-g(h(e_j,Y),h(X,e_j))\}$$
$$=c\{(n-1)g(X,Y)+3\sum_{i=1}^{3}g(T_iX,T_iY)\}+\sum_{j=1}^{n}\{g(h(e_j,e_j),h(X,Y))$$
$$-g(h(e_j,Y),h(X,e_j))+A(e_j,X,Y,e_j)+(n-2)\alpha(X,Y)+\lambda\}.$$

The scalar curvature ρ is given by

$$\rho=\sum_{l=1}^{n}S(e_l,e_l)=c[(n-1)n+12s+\sum_{l=1}^{n}\sum_{j=1}^{n}A(e_j,e_l,e_l,e_j)]+n^2\|H\|^2-\|h\|^2-2\lambda n+2\lambda.$$

Denoting by

$$\delta=\rho-n(n-1)c-12sc-\frac{n^2}{2}\|H\|^2+2\lambda n-2\lambda-\sum_{l=1}^{n}\sum_{j=1}^{n}A(e_j,e_l,e_l,e_j),$$

we obtain

$$n^2\|H\|^2=2(\delta+\|h\|^2). \tag{26}$$

With respect to the above orthogonal basis, equation (26) becomes

$$\left(\sum_{i=1}^{n}h_{ii}^{n+1}\right)^2=2\left\{\delta+\sum_{i=1}^{n}(h_{ii}^{n+1})^2+\sum_{i\neq j}(h_{ij}^{n+1})^2+\sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right\}. \tag{27}$$

Applying Lemma 1 in equation (27) we get

$$\sum_{1\le\alpha\neq\beta\le n-1}h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1}\ge\delta+2\sum_{i<j}(h_{ij}^{n+1})^2+\sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2,$$

or equivalently,

$$n(n-1)c+12sc+\frac{n^2}{2}\|H\|^2-2\lambda n+2\lambda+\sum_{l=1}^{n}\sum_{j=1}^{n}A(e_j,e_l,e_l,e_j) \tag{28}$$
$$\ge\rho-\sum_{1\le\alpha\neq\beta\le n-1}h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1}+2\sum_{i<j}(h_{ij}^{n+1})^2+\sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.$$

We characterise two different cases:
(a) $e_n \in D_p^{\perp}$;
(b) $e_n \in D_p$.
Using Gauss equation in case (a) we obtain

$$\rho - \sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 = 2S(e_n,e_n) + 12sc + \tag{29}$$
$$(n-1)(n-2)c + 2\sum_{i<n}(h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m}\left\{(h_{nn}^r)^2 + 2\sum_{i=1}^{n-1}(h_{in}^r)^2 + \left(\sum_{j=1}^{n-1}h_{jj}^r\right)^2\right\}.$$

By equations (28) and (29) we have

$$\frac{n^2}{4}\|H\|^2 + (n-1)c - 2\lambda n + 2\lambda + \alpha(e_n,e_n)(n-2) + \lambda$$
$$\geq S(e_n,e_n) + 2\sum_{i<n}(h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m}\left\{\sum_{i=1}^{n}(h_{in}^r)^2 + \left(\sum_{j=1}^{n-1}h_{jj}^r\right)^2\right\} \tag{30}$$

Then we get

$$S(e_n,e_n) \leq (n-1)c + \frac{n^2}{4}\|H\|^2 - (2n-3)\lambda + (n-2)\alpha(e_n,e_n).$$

Using Gauss equation in case (b) we find similarly

$$\begin{aligned}
&\rho - \sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\
= \;& 2S(e_n,e_n) + 12sc + (n-1)(n-2)c + 2\sum_{i<n}(h_{in}^{n+1})^2 \\
&+ \sum_{r=n+2}^{4m}\left\{(h_{nn}^r)^2\right\} + 2\sum_{i=1}^{n-1}(h_{in}^r)^2 + \left(\sum_{j=1}^{n-1}h_{jj}^r\right)^2 - \frac{3c}{2}\sum_{i=1}^{3}\|T_ie_n\|^2,
\end{aligned} \tag{31}$$

leading to the second inequality of theorem.
If $H(p) = 0$, then the equality holds in (24) and (25), respectively, if and only if

$$h_{1n}^r = ... = h_{n-1,n}^r = 0,$$
$$h_{nn}^r = \sum_{i-1}^{n-1} h_{ii}^r,$$

where $r \in \{n+1,...,4m\}$, then $h_{in}^r = 0$, $\forall i \in \{1,...,n\}$, i.e. $X \in D_p^{\perp} \cap N_p$ (respectively $X \in D_p \cap N_p$), N_p being the relative null space of M at a point $p \in M$ defined by

$$N_p = \{X \in T_pM | h(X,Y) = 0, \forall\, Y \in T_pM\}.$$

□

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