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TRANSITION PROBABILITIES, TRANSITION FUNCTIONS AND AN ERGODIC DECOMPOSITION

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Dedicated to Şafak Alpay on the occasion of his 60th birthday

Abstract

Our goal in this paper is to discuss an ergodic decomposition of the state space of a transition probability or transition function. The decomposition allows us to have a better understanding of the structure of the set of invariant probabilities of the transition probability or transition function under consideration, and, as a by-product, it allows us to obtain various criteria for the existence of invariant probability measures. Also, the decomposition offers a "system of reference" for the invariant ergodic probability measures which is useful in various instances.

The present paper is a survey of results that I have obtained during the last ten years, and is an extended version of my talk at the 9-ème Colloque Franco-Roumain de Mathématiques Appliquées,28 août-2 septembre 2008, Braşov, Roumanie.

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Prelude

The father has given his son two dollar bills, and asked him to buy two pounds of sugar that cost one dollar and half a dozen eggs that also cost one dollar. A few moments later, the son returns and asks:

"Daddy, which dollar bill should I use for the sugar and which one for the eggs?"

1 Introduction

The joke in Prelude (an adaptation of one of the jokes about Bulă[2] that were popular in Romania in the sixties and seventies) points out an interesting aspect of reasoning in

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[2]Bulă is a folklore character, a ten years old boy who has been the hero of many Romanian jokes. The first vowel *u* in the name is pronounced like in the English word *book*; the last vowel *ă* is pronounced like the indefinite article *a* in English.

mathematics, and in science in general. If we have a finite or infinite set of objects, and we want to describe each in terms of certain characteristics, we first need a system of reference which will allow us to distinguish each object. A case in point is the following: suppose that we are given a transition probability or a transition function defined on a locally compact separable metric space, and suppose that we want to describe precisely the support of each invariant ergodic probability measure of the ! transition probability or transition function under consideration (for instance, we want to find a "formula" for these supports). This problem makes sense only if we have a system of reference for the measures whose supports we want to describe. In this paper we are going to discuss a decomposition of the state space of the transition probability or transition function in question which is just such a system of reference. Besides providing a system of reference for the invariant ergodic probability measures, the decomposition has various other features, the most important among these features being the fact that the decomposition allows us to obtain a better understanding of the structure of the set of invariant probabilities of the transition probability or transition function under study.

We call the decomposition discussed in this paper the Kryloff-Bogoliouboff-Beboutoff-Yosida decomposition (or the KBBY decomposition) because, among many earlier efforts to obtain the decomposition, the pioneering results of N. Kryloff and N. Bogoliouboff [20], M. Beboutoff [2], and K. Yosida [59] and [60] (see also Section 4 of Chapter 13 of Yosida's monograph [61]) are closest to the results discussed here.

The decomposition is valid under fairly general conditions, in the sense that any transition probability on a locally compact separable metric space defines a KBBY decomposition of the space and, under rather mild conditions, a transition function on a locally compact separable metric space defines a KBBY decomposition of the space, as well.

This work is an expanded version of my talk at the 9-ème Colloque Franco-Roumain de Mathématiques Appliquées, 28 août-2 septembre 2008, Braşov, Roumanie.[3] We will survey here results that have appeared in the monograph [65], the paper [68], and results that I plan to include in a small monograph that I am currently writing.

The paper is organized as follows: in the next section (Section 2), we discuss the KBBY decomposition for transition probabilities; in Section 3, we discuss transition functions and their KBBY decomposition; finally, in Section 4, we outline several directions for further research that stem from the results discussed in this paper.

2 Transition Probabilities

Let (X, d) be a locally compact separable metric space, and let $\mathcal{B}(X)$ be the σ-algebra of all Borel subsets of X (that is, $\mathcal{B}(X)$ is the σ-algebra generated by the open subsets of X).

As usual, a map $P : X \times \mathcal{B}(X) \to \mathbb{R}$ is called a transition probability (on X) if the following two conditions are satisfied:

[3]I would like to express my gratitude to Marius Iosifescu for inviting me to give a talk at the colloquium and to Eugen Păltănea for the invitation to write this paper.

(TP1) For every Borel subset A of X, the map $x \mapsto P(x, A)$ from X to $\mathbb{R}$ is Borel measurable.

(TP2) For every $x \in X$, the set function $\mu_x : \mathcal{B}(X) \to \mathbb{R}$ defined by $\mu_x(A) = P(x, A)$ for every $A \in \mathcal{B}(X)$ is a probability measure.

The transition probabilities are of fundamental importance in the study of Markov processes (for details on transition probabilities in the context of discrete-time Markov processes, see, for instance, the monographs by O. Hernández-Lerma and J. B. Lasserre [12], P. A. Meyer [35], S. P. Meyn and R. L. Tweedie [36], E. Nummelin [39], S. Orey [41], D. Revuz [45], and M. Rosenblatt [48]). This is so because, to any discrete-time Markov process, we can associate in a fairly standard manner a sequence of transition probabilities, and a large amount of information about the Markov process can be obtained by studying the corresponding sequence of transition probabilities (these transition probabilities are often called one-step transition probabilities). If the Markov process is also time-homogeneous, then all the corresponding one-step transition probabilities are equal; therefore, to such a process we associate a single transition probability. Interestingly enough (and well-known) is the fact that, given a transition probability, we can always construct a discrete-time Markov process homogeneous in time whose associated transition probability is the given one (for details, see Section 1.2 of Revuz [45]).

Transition probabilities are also of significant use in the study of discrete-time dynamical systems (see, for instance, Chapter 3 of U. Krengel's monograph [19], and [65]).

Let $B_b(X)$ be the Banach space of all real-valued bounded Borel measurable functions on X, where the norm on $B_b(X)$ is the uniform (sup) norm: $\|f\| = \sup_{x \in X} |f(x)|$ for every $f \in B_b(X)$, and let $\mathcal{M}(X)$ be the Banach space of all real-valued signed Borel measures on X, where the norm on $\mathcal{M}(X)$ is the total variation norm.

If $f \in B_b(X)$ and $\mu \in \mathcal{M}(X)$, we will use the notation $\langle f, \mu \rangle$ for $\int_X f(x)\,\mathrm{d}\mu(x)$.

Now, let P be a transition probability on (X, d).

Using P, we can define in a natural manner two linear operators $S : B_b(X) \to B_b(X)$ and $T : \mathcal{M}(X) \to \mathcal{M}(X)$ as follows:

$$Sf(x) = \int f(y)\,\mathrm{d}\mu_x(y) \tag{1}$$

for every $f \in B_b(X)$ and $x \in X$, where μ_x, $x \in X$, are the probability measures that appear in condition (TP2) in the definition of a transition probability; it is the custom to use the notation $P(x, \mathrm{d}y)$ for $\mathrm{d}\mu_x(y)$ in (1), so (1) becomes the more familiar equality

$$Sf(x) = \int f(y) P(x, \mathrm{d}y); \tag{2}$$

$$T\mu(A) = \int P(x, A)\,\mathrm{d}\mu(x) \tag{3}$$

for every $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}(X)$.

It is easy to see that S and T are well-defined (in the sense that Sf is indeed an element of $B_b(X)$ for every $f \in B_b(X)$ whenever Sf is defined by (1) or (2), and that

$T\mu$ belongs to $\mathcal{M}(X)$ whenever $T\mu$ is defined by (3). Also easy to see is that S and T are positive linear contractions (that is, S and T are linear bounded operators such that $\|S\| \leq 1$ and $\|T\| \leq 1$, and such that $Sf \geq 0$ and $T\mu \geq 0$ whenever $f \geq 0$ and $\mu \geq 0$), that T is a Markov operator (that is, $\|T\mu\| = \|\mu\|$ for every $\mu \in \mathcal{M}(X)$, $\mu \geq 0$; for additional details on Markov operators, see [65]), and that $S\mathbf{1}_X = \mathbf{1}_X$, where $\mathbf{1}_X$ is the (real-valued) constant 1 function on X. Finally, S and T are related by the equality

$$\langle Sf, \mu \rangle = \langle f, T\mu \rangle$$

for every $f \in B_b(X)$ and $\mu \in \mathcal{M}(X)$.

The ordered pair (S, T) is called the Markov pair defined by the transition probability P.

Let $C_b(X)$ be the Banach subspace of $B_b(X)$ of all real-valued bounded continuous functions on X.

Let P be a transition probability on (X, d), and let (S, T) be the Markov pair defined by P. We say that P is a Feller transition probability if $Sf \in C_b(X)$ whenever $f \in C_b(X)$. If P is a Feller transition probability, then (S, T) is called a Markov-Feller pair.

We will now discuss briefly a few examples of transition probabilities and their associated Markov pairs in order to illustrate how diverse these objects can be. We stress that the examples that follow are by no means the only important examples of transition probabilities. For many other examples, see the monograph by Meyn and Tweedie [36]. In the examples discussed below, and throughout the paper, we will use the following notation: if A is a subset of X, $\mathbf{1}_A$ stands for the real-valued function on X defined by $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \in X \setminus A$; given $x \in X$, δ_x stands for the Dirac probability measure concentrated at x (that is, δ_x is the probability measure on $(X, \mathcal{B}(X))$ defined by $\delta_x(\{x\}) = 1$).

Example 1 (Transition Probabilities Defined by Measurable Functions)**.** *Probably the simplest examples of transition probabilities and Markov pairs are those defined by measurable functions. In spite of their simple appearance, the study of these transition probabilities and Markov pairs is extremely interesting, often challenging and sophisticated. Under some additional conditions, the measurable functions and their iterates are studied under the name discrete-time dynamical systems (for additional details on discrete-time dynamical systems see, for example, the monographs by J. Aaronson [1], I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai [6], H. Furstenberg [11], A. Katok and B. Hasselblatt [18], U. Krengel [19], A. Lasota and M. C. Mackey [22], R. Mañé [32], W. de Melo and S. van Strien [34], K. Petersen [42], C. Robinson [47], D. J. Rudolph [49], Ya. G. Sinai [**?**] Si, P. Walters [58], and my book [65]).*

Given a locally compact separable metric space (X, d), let $w : X \to X$ be a measurable function (the measurability of w means, of course, that $w^{-1}(A) \in \mathcal{B}(X)$ for every $A \in \mathcal{B}(X)$).

Now, consider the map $P_w : X \times \mathcal{B}(X) \to \mathbb{R}$ defined by $P_w(x, A) = \mathbf{1}_A(w(x))$ for every $x \in X$ and $A \in \mathcal{B}(X)$. Since $P_w(x, A) = \delta_{w(x)}(A)$ for every $x \in X$ and $A \in \mathcal{B}(X)$, it follows that P_w is a transition probability.

It is easy to see that if $S_w : B_b(X) \to B_b(X)$ *is defined by* $S_w f(x) = f(w(x))$ *for every* $f \in B_b(X)$ *and* $x \in X$, *and if* $T_w : \mathcal{M}(X) \to \mathcal{M}(X)$ *is defined by* $T_w\mu(A) = \mu(w^{-1}(A))$ *for every* $\mu \in \mathcal{M}(X)$ *and* $A \in \mathcal{B}(X)$, *then* S_w *and* T_w *are well-defined (in the sense that* $S_w f$ *and* $T_w\mu$ *are indeed elements of* $B_b(X)$ *and* $\mathcal{M}(X)$ *whenever* $f \in B_b(X)$ *and* $\mu \in \mathcal{M}(X)$, *respectively) linear bounded operators, and that* (S_w, T_w) *is the Markov pair defined by* P_w. *We say that* P_w *and* (S_w, T_w) *are the transition probability and the Markov pair defined (or induced) by* w.

Note that P_w *is a Feller transition probability (or, equivalently, that* (S_w, T_w) *is a Markov-Feller pair) if and only if* w *is continuous.*

Note also that in the case of P_w, *the probability measures* μ_x, $x \in X$, *that appear in condition (TP2) of the definition of a transition probability, are all Dirac measures.* ■

The next example can be thought of as a natural extension of Example 1 to the case of several maps $w_1, w_2, \ldots, w_n$, $n \in \mathbb{N}$, $n \geq 2$.

Example 2 (Iterated Function Systems)**.** *As before, let* (X, d) *be a locally compact separable metric space.*

Let $n \in \mathbb{N}$, $n \geq 2$, *and let* $w_1, w_2, \ldots, w_n$, $p_1, p_2, \ldots, p_n$ *be* $2n$ *measurable functions,* $w_i : X \to X$, $p_i : X \to \mathbb{R}$ *for every* $i = 1, 2, \ldots, n$. *Set* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ *and* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. *Assume that* $p_i(x) \geq 0$ *for every* $i = 1, 2, \ldots, n$ *and every* $x \in X$, *that* $\sum_{i=1}^{n} p_i(x) = 1$ *for every* $x \in X$, *and that none of the functions* p_i, $i = 1, 2, \ldots, n$ *is identically zero. The ordered pair* $(\mathbf{w}, \mathbf{p})$ *is called a generalized iterated function system (or generalized i.f.s.) with probabilities. If the functions* w_i *and* p_i, $i = 1, 2, \ldots, n$ *are all continuous, then* $(\mathbf{w}, \mathbf{p})$ *is simply called an i.f.s. with probabilities (rather than a generalized i.f.s. with probabilities).*

The literature on i.f.s. with probabilities is impressively large (see, for instance, Section 12.8 of the monograph by A. Lasota and M. C. Mackey [22], the papers by P. M. Centore and E. R. Vrscay [4], J. Jaroszewska [17], A. Lasota and M. C. Mackey [21], A. Lasota and J. Myjak [23], [24], [25], [26], [27], and [28], A. Lasota and J. A. Yorke [29], J. Myjak and T. Szarek [37], M. Nicol, N. Sidorov, and D. Broomhead [38], Ö. Stenflo [51], [52], [53], [54], and [55], T. Szarek [56], E. R. Vrscay [57], our papers [62], [63], [64], and [67], and the references in the above-mentioned works). By contrast, the generalized i.f.s. with probabilities have not been considered in the literature so far; we believe that these generalized i.f.s. with probabilities should be studied because some of them, which we call simple generalized ! i.f.s. with probabilities (a simple generalized i.f.s. with probabilities is a generalized i.f.s. with probabilities which consists of simple functions (functions whose ranges are finite sets) only), are easier to handle computationally than the i.f.s. with probabilities that are currently employed, and we think that the simple generalized i.f.s. with probabilities could be used to solve some of the challenging problems that have emerged in image processing (for details, see [68]).

The generalized i.f.s. with probabilities are particular cases of OMIGT processes. The OMIGT processes (OMIGT stands for Onicescu, Mihoc, Iosifescu, Grigorescu, and Theodorescu) stem from a pioneering 1935 paper by O. Onicescu and G. Mihoc [40]; the interest in these processes, known as random systems with complete connections (r.s.c.c.), *has increased significantly when the theory of r.s.c.c. was made available to a large segment of the mathematical community by two monographs: one by M. Iosifescu and S. Grigorescu [15], and the other one by M. Iosifescu and R. Theodorescu [16]. The above historical remarks explain why we prefer to use the term OMIGT process rather than r.s.c.c.*

Now, let $(\mathbf{w},\mathbf{p})$, $\mathbf{w}=(w_1,w_2,\dots,w_n)$, $\mathbf{p}=(p_1,p_2,\dots,p_n)$, $n\in\mathbb{N}$, $n\geq 2$, *be a generalized i.f.s. with probabilities defined on* (X,d), *and let* $P_{(\mathbf{w},\mathbf{p})}: X\times\mathcal{B}(X)\to\mathbb{R}$ *be defined by* $P_{(\mathbf{w},\mathbf{p})}(x,A)=\sum_{i=1}^{n}p_i(x)\mathbf{1}_A(w_i(x))$ *for every* $x\in X$ *and* $A\in\mathcal{B}(X)$. *It is easy to see that* $P_{(\mathbf{w},\mathbf{p})}$ *is a transition probability (indeed, condition (TP1) is clearly satisfied; in order to see that condition (TP2) is also satisfied, note that* $P_{(\mathbf{w},\mathbf{p})}(x,A)=\sum_{i=1}^{n}p_i(x)\delta_{w_i(x)}(A)$ *for every* $x\in X$ *and* $A\in\mathcal{B}(X)$*). We say that* $P_{(\mathbf{w},\mathbf{p})}$ *is the transition probability defined by* $(\mathbf{w},\mathbf{p})$.

Let $S_{(\mathbf{w},\mathbf{p})}: B_b(X)\to B_b(X)$ *be defined by* $S_{(\mathbf{w},\mathbf{p})}f(x)=\sum_{i=1}^{n}p_i(x)f(w_i(x))$ *for every* $f\in B_b(X)$ *and* $x\in X$, *and let* $T_{(\mathbf{w},\mathbf{p})}:\mathcal{M}(X)\to\mathcal{M}(X)$ *be defined by* $T_{(\mathbf{w},\mathbf{p})}\mu(A)=\sum_{i=1}^{n}\int_{w_i^{-1}(A)}p_i(x)\,\mathrm{d}\mu(x)$ *for every* $\mu\in\mathcal{M}(X)$ *and* $A\in\mathcal{B}(X)$. *It is easy to see that* $(S_{(\mathbf{w},\mathbf{p})},T_{(\mathbf{w},\mathbf{p})})$ *is the Markov pair defined by* $P_{(\mathbf{w},\mathbf{p})}$. *We call* $(S_{(\mathbf{w},\mathbf{p})},T_{(\mathbf{w},\mathbf{p})})$ *the Markov pair defined by* $(\mathbf{w},\mathbf{p})$.

Note that if the functions w_i *and* p_i, $i=1,2,\dots,n$, *are continuous, then* $P_{(\mathbf{w},\mathbf{p})}$ *is a Feller transition probability (and, of course,* $(S_{(\mathbf{w},\mathbf{p})},T_{(\mathbf{w},\mathbf{p})})$ *is a Markov-Feller pair).*

Note also that, while in Example 1 the measures μ_x *described in condition (TP2) in the definition of a transition probability are Dirac measures, in the case of transition probabilities defined by generalized i.f.s. with probabilities, the measures* μ_x *are convex combinations of Dirac measures.* ■

In the two examples discussed so far, the transition probabilities that have been under consideration have the property that the supports of their measures μ_x described in condition (TP2) of the definition of a transition probability were finite subsets. In our next (and last) example of transition probabilities, we will see that, as expected, the supports of the probability measures μ_x can be fairly general closed subsets of the corresponding (locally compact separable) metric spaces on which these transition probabilities are defined.

Example 3 (Transition Probabilities Defined by Convolution Operators)**.** *The transition probabilities that we are going to dicuss here involve the use of the operation of convolution of two measures. For unexplained terminology and notation used in this example, see [66]. For additional details on convolutions of measures on groups, see H. Heyer's monograph [13]; for convolutions on semigroups, see the book by G. Högnäs and A. Mukherjea [14].*

Let (X,d) *be a locally compact separable metric semigroup, and let* $\mu\in\mathcal{M}(X)$. *The map* $T_\mu:\mathcal{M}(X)\to\mathcal{M}(X)$, $T_\mu\nu=\mu*\nu$ *for every* $\nu\in\mathcal{M}(X)$, *is called a convolution operator. If* μ *is a probability measure, then* T_μ *is a Markov operator.*

Let $\mu\in\mathcal{M}(X)$ *be a probability measure.*

Let $P_\mu : X \times \mathcal{B}(X) \to \mathbb{R}$ *be defined by* $P_\mu(x, A) = \mu * \delta_x(A)$ *for every* $x \in X$ *and* $A \in \mathcal{B}(X)$*. It is easy to see that* P_μ *is a transition probability. Since* $P_\mu(x, A) = T_\mu \delta_x(A)$ *for every* $x \in X$ *and* $A \in \mathcal{B}(X)$*, we say that* P_μ *is the transition probability defined by the convolution operator* T_μ*.*

Let $S_\mu : B_b(X) \to B_b(X)$ *be defined by* $S_\mu f(x) = \int f(yx)\,\mathrm{d}\mu(y)$ *for every* $f \in B_b(X)$ *and* $x \in X$*. Using Proposition 2.1 on p. 69 of Högnäs and Mukherjea [14], we obtain that the definition of* $S_\mu f$ *is correct in the sense that the integral defining* $S_\mu f(x)$ *exists for every* $x \in X$ *and the resulting real-valued function* $S_\mu f$ *on* X *is an element of* $B_b(X)$ *for every* $f \in B_b(X)$*. It is easy to see that* S_μ *is a positive linear contraction of* $B_b(X)$*, and it can be shown that* (S_μ, T_μ) *is the Markov pair defined by* P_μ*. Moreover, it can also be shown that* P_μ *is a Feller transition probability.*

The transition probability P_μ *and the Markov-Feller pair* (S_μ, T_μ) *are closely related to the study of random walks. For additional details on this relationship and for a study of random walks as stochastic processes in a setting similar to ours, see D. Revuz's monograph [45].*

Note that in the case of P_μ*, the probability measures* μ_x *that appear in condition (TP2) in the definition of a transition probability are of the form* $\mu * \delta_x$*, so* $\operatorname{supp}\mu_x = \overline{(\operatorname{supp}\mu)x}$*,* $x \in X$ *(here, and throughout the paper,* $\operatorname{supp}\nu$ *is the support of a measure* ν *and* $\overline{A}$ *is the closure of a subset* A *of* X*).* ■

We now return to the general setting described before the above examples. Thus, we assume given a locally compact separable metric space (X, d), a transition probability P on (X, d), and the Markov pair (S, T) defined by P.

An element μ of $\mathcal{M}(X)$ is said to be an invariant element for T (or for P, or for (S, T)) if $T\mu = \mu$.

Since the zero measure is always an invariant element for T, the interesting situation is the case in which T has also nonzero invariant elements.

There are two natural questions that appear in connection with the invariant elements of T:

(1) Does T have nonzero invariant elements?

and

(2) Assuming that T has nonzero invariant elements, what can be said about the structure of the set of all invariant elements of T?

In view of the fact that $\mathcal{M}(X)$ is a Banach lattice and since T is a Markov operator on $\mathcal{M}(X)$, we obtain that T has nonzero invariant elements if and only if T has invariant probability measures. Moreover, if T has invariant probabilities, then every invariant element of T in $\mathcal{M}(X)$ is a linear combination of invariant probability measures. Therefore, for most purposes, in order to understand the structure of the set of invariant elements of T, it is enough to understand the structure of the set of invariant probabilities of T. The KBBY decomposition that we are going to present below allows us to obtain a fairly complete understanding of the structure of the set of invariant probabilities, and, at the same time, the study of the decomposition yields various criteria for the existence of nonzero invariant elements of T.

Now, assume that T has invariant probabilities, let μ be such an invariant probability, and consider the following assertion:

(E) There exists a Borel measurable subset A of X such that $\mu(A) > 0$, $\mu(X \setminus A) > 0$, and such that the measures $\mu_1 : \mathcal{B}(X) \to \mathbb{R}$ and $\mu_2 : \mathcal{B}(X) \to \mathbb{R}$ defined by $\mu_1(B) = \mu(B \cap A)$ and $\mu_2(B) = \mu(B \cap (X \setminus A))$ for every $B \in \mathcal{B}(X)$ are both invariant for T.

The fact that assertion (E) holds true for μ means that we can write μ as a sum of two nonzero mutually singular T-invariant measures μ_1 and μ_2. By contrast, the fact that assertion (E) is false for μ means that we cannot "break" μ into such a sum of two nonzero mutually singular T-invariant measures.

A T-invariant probability measure for which assertion (E) is false is said to be ergodic. The ergodic T-invariant probabilities are, in a certain sense that will be made precise by the KBBY decomposition, the "building blocks" for all T-invariant probabilities, and, for T-invariance they play the same role as the role played by molecules for a substance. A molecule is the smallest part of a substance that preserves all the properties of that substance; similarly, an ergodic T-invariant probability measure is the "smallest" T-invariant probability in the sense of assertion (E) with respect to T-invariance.

Let $C_0(X)$ be the Banach space of all real-valued continuous functions on X that vanish at infinity, where the norm on $C_0(X)$ is the uniform (sup) norm (note that $C_0(X)$ can also be thought of as a Banach subspace of $B_b(X)$).

The KBBY decomposition of X defined by P is a splitting of X in terms of the convergence behaviour of the sequences of averages $\left(\frac{1}{n}\sum_{k=0}^{n-1} S^k f(x)\right)_{n\in\mathbb{N}}$, $f \in C_0(X)$, $x \in X$. The decomposition has various features, the most significant of them being the following two:

– it allows us to associate, in a fairly natural manner, a measurable subset of X to each ergodic T-invariant probability measure such that the measure is concentrated on the subset, and, for every two distinct ergodic T-invariant probabilities, the corresponding subsets are disjoint

and

– it allows us to express each T-invariant probability measure as a convex combination in integral form of ergodic T-invariant probability measures.

We will now describe the decomposition in detail.

Set

$$\mathcal{D} = \left\{ x \in X \,\middle|\, \begin{array}{l} \text{the sequence } \left(\frac{1}{n}\sum\limits_{k=0}^{n-1} S^k f(x)\right)_{n\in\mathbb{N}} \\ \text{converges to zero for every } f \in C_0(X) \end{array} \right\}$$

and $\Gamma_0 = X \setminus \mathcal{D}$.

The set $\mathcal{D}$ is called the dissipative part of X generated by P (or (S,T)). We say that P (or (S,T)) is dissipative if $\mathcal{D} = X$.

Now set

$$\Gamma_c = \left\{ x \in \Gamma_0 \;\middle|\; \begin{array}{l} \text{the sequence } \left(\frac{1}{n} \sum_{k=0}^{n-1} S^k f(x) \right)_{n\in\mathbb{N}} \\ \text{converges for every } f \in C_0(X) \end{array} \right\}.$$

The definition of Γ_c suggests to us that we consider, for every $x \in \Gamma_c$, a map $\varepsilon_x : C_0(X) \to \mathbb{R}$ defined by $\varepsilon_x(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k f(x)$ for every $f \in C_0(X)$. Clearly, the maps ε_x, $x \in \Gamma_c$, are linear and positive (the positivity of ε_x means that $\varepsilon_x(f) \geq 0$ whenever $f \geq 0$). Since ε_x, $x \in \Gamma_c$, are positive linear maps, it follows that these maps are also continuous; therefore, the maps belong to the topological dual $(C_0(X))^*$ of $C_0(X)$. But $(C_0(X))^*$ can be identified in a standard manner with $\mathcal{M}(X)$ (see, for instance, Theorem 7.3.5, pp. 220-223, of D. L. Cohn's book [5]). Thus, we may and do think of ε_x, $x \in \Gamma_c$, as usual (positive) Borel measures on X. Note that $0 < \varepsilon_x(X) \leq 1$ for every $x \in \Gamma_c$. We call ε_x, $x \in \Gamma_c$,! standard elementary measures (see [68] for the terminology). The elementary measures form a larger class of measures and are defined using Banach limits (see [65] and [68]). In order to exhibit the full flavor of the decomposition, in this paper we will not discuss elementary measures in full generality.

Set $\Gamma_{cp} = \{x \in \Gamma_c | \|\varepsilon_x\| = 1\}$. Thus, Γ_{cp} is the set of all $x \in \Gamma_c$ which have the property that ε_x is a standard elementary probability measure.

A natural question is whether or not all the standard elementary measures, or at least the standard elementary probability measures, are invariant. One is tempted to believe that the elementary measures are invariant because, in the case of a Feller transition probability, the elementary measures are indeed invariant (see Theorem 2.1.1 in [65]). However, in the general not necessarily Feller case, standard elementary measures, even if they are probability measures, may not be invariant (see Example 5.1 in [68]). Since, in general, standard elementary probability measures may or may not be invariant, it makes sense to set

$$\Gamma_{cpi} = \{x \in \Gamma_{cp} \,|\, \varepsilon_x \text{ is a } T\text{-invariant probability measure}\}$$

because, as shown in Example 5.1 of [68], $\Gamma_{cpi} \neq \Gamma_{cp}$ in general, even though $\Gamma_{cpi} = \Gamma_{cp}$ whenever P is a Feller transition probability.

A Borel subset A of X is said to be a set of maximal probability for T (or for P or for (S, T)) if either T does not have invariant probability measures, or else $\mu(A) = 1$ for every T-invariant probability measure μ.

The following theorem summarizes the results about the KBBY decomposition that can be stated at this time and that are needed to develop the decomposition further.

Theorem 4. *(a) The sets $\mathcal{D}$, Γ_0, Γ_c, Γ_{cp}, and Γ_{cpi} are Borel measurable subsets of X.*

(b) the sets Γ_0, Γ_c, Γ_{cp}, and Γ_{cpi} are sets of maximal probability for T.

For the proofs of the various assertions made in the above theorem, see [68].

As usual, if $\mu \in \mathcal{M}(X)$, we denote by $\operatorname{supp}\mu$ the support of μ in X.

Let $x \in X$. The subset $\mathcal{O}(x) = \bigcup_{n=1}^{\infty} \operatorname{supp}(T^n \delta_x)$ of X is called the orbit of x under the action of T (or P). The closure $\overline{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ in the metric topology of X is called the orbit-closure of x. The reason for the terminology stems from the fact that if P and T are induced by a measurable function $w : X \to X$, then $\mathcal{O}(X)$ and $\overline{\mathcal{O}(X)}$ are the usual orbit and orbit-closure of x defined by w that are studied in ergodic theory and the theory of dynamical systems.

Now, when we look at the standard elementary T-invariant probability measures ε_x, $x \in \Gamma_{cpi}$, we notice that the support of each such probability measure ε_x has a certain minimality property; namely, it can be shown that $\operatorname{supp} \varepsilon_x$ is a subset of the orbit-closure of x under P, $x \in \Gamma_{cpi}$. On the other hand, the ergodic invariant probability measures, by their definition have a certain minimality property, as we saw earlier in the paper, in the sense that an ergodic invariant probability measure cannot be "broken" into a sum of two nonzero mutually singular T-invariant measures. Thus, it is tempting to believe that all the measures ε_x, $x \in \Gamma_{cpi}$, are ergodic. However, a remarkably simple example, Example 2.2.4 on pp. 47-48 of [65], which was suggested by one of the anonymous referees of [65], can be used to show that, in general, some of the measures ε_x, $x \in ! \Gamma_{cpi}$, might fail to be ergodic. The example suggests that we should study further the set Γ_{cpi} if we want to identify the set of all $x \in \Gamma_{cpi}$ which have the property that ε_x is ergodic.

To this end, let us introduce first the following notation: if $f \in C_0(X)$, let $f^* : X \to \mathbb{R}$ be defined by

$$f^*(x) = \begin{cases} \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k f(x) & \text{if } \; x \in \Gamma_c \\ 0 & \text{if } \; x \notin \Gamma_c \end{cases} .$$

Now set

$$\Gamma_{cpie} = \left\{ x \in \Gamma_{cpi} \;\middle|\; \int_{\Gamma_{cpi}} (f^*(y) - f^*(x))^2 \, \mathrm{d}\varepsilon_x(y) = 0 \text{ for every } f \in C_0(X) \right\}.$$

An element x of Γ_{cpi} belongs to Γ_{cpie} if, for every $f \in C_0(X)$, the function f^* is constant on a measurable set $A_{x,f}$ (which depends on x and f) such that $\varepsilon_x(A_{x,f}) = 1$. However, since $C_0(X)$ is a separable Banach space, it can be shown that there exists a measurable set A_x, set which is independent of $f \in C_0(X)$ such that $\varepsilon_x(A_x) = 1$ and f^* is constant on A_x for every $f \in C_0(X)$. Now, the existence of such a set A_x implies that ε_x is ergodic. Moreover, it can be shown that any ergodic invariant probability measure is of the form ε_x for some $x \in \Gamma_{cpie}$. It can also be shown that Γ_{cpie} is a Borel measurable subset of X and that Γ_{cpie} is a set of maximal probability for P.

Using Theorem 1.2.6 of [65], the fact that Γ_{cpie} is a subset of Γ_{cpi}, and that Γ_{cpie} is a set of maximal probability, we obtain that, for every T-invariant probability measure μ, the following equality holds true:

$$\int_X f(x) \, \mathrm{d}\mu(x) = \int_{\Gamma_{cpie}} \left(\int_X f(y) \, \mathrm{d}\varepsilon_x(y) \right) \mathrm{d}\mu(x) \tag{4}$$

for every $f \in C_0(X)$. Using the notation $\langle f, \varepsilon_x \rangle = \int_X f(y)\, \mathrm{d}\varepsilon_x(y)$, $x \in \Gamma_{cpie}$, the equality (4) becomes

$$\int_X f(x)\, \mathrm{d}\mu(x) = \int_{\Gamma_{cpie}} \langle f, \varepsilon_x \rangle\, \mathrm{d}\mu(x) \tag{5}$$

for every $f \in C_0(X)$.

Note that the right-hand sides of (4) and (5) can be thought of as convex combinations in integral form of ergodic T-invariant probability measures. Since any element of $\mathcal{M}(X)$ can be thought of as a linear functional on $C_0(X)$, the equality (4) (or (5)) tells us that any T-invariant probability measure can be expressed as a convex combination in integral form of ergodic T-invariant probability measures, and this is a reason why we think of the ergodic invariant probability measures as being the "building blocks" for all the invariant probabilities.

Now, let us study further the structure of Γ_{cpie}. To this end, consider the relation $\sim$ defined on Γ_{cpie} as follows: $x \sim y$ if, by definition, $f^*(x) = f^*(y)$ for every $f \in C_0(X)$ where $x \in \Gamma_{cpie}$ and $y \in \Gamma_{cpie}$. It is easy to see that $\sim$ is an equivalence relation.

It can be shown that the equivalence classes $[x]$, $x \in \Gamma_{cpie}$, are measurable subsets of X, and that $\varepsilon_x([x]) = 1$ for every $x \in \Gamma_{cpie}$. Thus, for every $x \in \Gamma_{cpie}$, the set $[x]$ is a measurable subset of X of ε_x-measure 1 on which f^* is constant whenever $f \in C_0(X)$. For many purposes, the sets $[x]$, $x \in \Gamma_{cpie}$ can be thought of as a system of reference for the ergodic T-invariant probability measures.

Summing up, we can compare the KBBY decomposition of a space X defined by a transition probability P with a fruit. The ergodic invariant probability measures ε_x, $x \in \Gamma_{cpie}$, are the "seeds." Each "seed" ε_x is located at $[x]$. Thus, all the "seeds" are in the nucleus Γ_{cpie}. The nucleus is "surrounded" by several "protective" layers: $\mathcal{D}$, $\Gamma_0 \setminus \Gamma_c$, $\Gamma_c \setminus \Gamma_{cp}$, $\Gamma_{cp} \setminus \Gamma_{cpi}$, and $\Gamma_{cpi} \setminus \Gamma_{cpie}$. Some of the layers, or even all of them may be missing (there exist transition probabilities for which Γ_{cpie} is equal to the entire space X on which these transition probabilities are defined). On the other hand, it can happen that a transition probability does not have invariant probability measures; in this case, Γ_{cpi} and, of course, Γ_{cpie} are empty (actually, it can be shown that Γ_{cpi} is empty! if and only if Γ_{cpie} is empty); if a transition probability does not have invariant probability measures, then the space on which the transition probability is defined is the union of some of or all the sets $\mathcal{D}$, $\Gamma_0 \setminus \Gamma_c$, $\Gamma_c \setminus \Gamma_{cp}$, and Γ_{cp}, so, in this case, the KBBY decomposition consists of layers that protect nothing.

3 Transition Functions

A remarkable fact about the KBBY decomposition is that the decomposition is valid also for a rather large family of transition functions.

Let $\mathbb{T}$ stand for $\mathbb{R}$ or the interval $[0, +\infty)$ in $\mathbb{R}$, and, as in Section 2, let (X, d) be a locally compact separable metric space.

A family $(P_t)_{t\in\mathbb{T}}$ of transition probabilities on (X, d) is called a transition function if

$$P_{s+t}(x, A) = \int_X P_s(y, A)P_t(x, \mathrm{d}y) \tag{6}$$

for every $s \in \mathbb{T}$, $t \in \mathbb{T}$, $x \in X$, and $A \in \mathcal{B}(X)$, where, of course, $P_t(x, \mathrm{d}y)$ has the same meaning as $P(x, \mathrm{d}y)$ in formula (2).

As is well-known, transition functions play an important role in the study of continuous-time Markov processes (see, for instance, the monographs by E. B. Dynkin [9], S. N. Ethier and T. G. Kurtz [10], T. M. Liggett [30], M. B. Marcus and J. Rosen [33], and D. Revuz and M. Yor [46]). Transition functions play also a significant role in the study of a deep connection between potential theory and the theory of continuous-time Markov processes (for a recent nice exposition of this topic, see the monograph by L. Beznea and N. Boboc [3]).

The reader who is familiar with continuous-time Markov processes has no doubt recognized that the equalities (6) are the well-known Chapman-Kolmogorov equations. The same reader may wonder why do we allow the index set $\mathbb{T}$ to be also the real line rather than confining ourselves to study only transition functions of the form $(P_t)_{t\in[0,+\infty)}$ as is done in the above-mentioned literature. The reason for considering also transition functions of the form $(P_t)_{t\in\mathbb{R}}$ is that, given a measurable flow $(w_t)_{t\in\mathbb{R}}$ defined on (X, d), if we set $P_t(x, A) = \mathbf{1}_A(w_t(x))$ for every $t \in \mathbb{R}$, $A \in \mathcal{B}(X)$ and $x \in X$ (note the similarity with Example 1), then $(P_t)_{t\in\mathbb{R}}$ is a transition function in the sense of the definition stated above; thus, the results that we obtain for the transition functions as defined in this section can be used when dealing with flows, as well as when studying continuous-time Markov processes, se! miflows, or continuous one-parameter convolution semigroups of probability measures.

Now, let $(P_t)_{t\in\mathbb{T}}$ be a transition function defined on (X, d). Given $t \in \mathbb{T}$, P_t is a transition probability; therefore, as discussed in Section 2, P_t defines a Markov pair (S_t, T_t). The family of Markov pairs $((S_t, T_t))_{t\in\mathbb{T}}$ is said to be defined by, or associated with $(P_t)_{t\in\mathbb{T}}$. Since $(P_t)_{t\in\mathbb{T}}$ satisfies the Chapman-Kolmogorov equations (6), it follows that $(S_t)_{t\in\mathbb{T}}$ and $(T_t)_{t\in\mathbb{T}}$ are semigroups of operators if $\mathbb{T} = [0, +\infty)$, and groups of operators if $\mathbb{T} = \mathbb{R}$.

An element μ of $\mathcal{M}(X)$ is said to be invariant for $(T_t)_{t\in\mathbb{T}}$ or $(P_t)_{t\in\mathbb{T}}$ or $((S_t, T_t))_{t\in\mathbb{T}}$ if $T_t\mu = \mu$ for every $t \in \mathbb{T}$. As in the case of transition probabilities, we are interested to study the set of all invariant probability measures of $(P_t)_{t\in\mathbb{T}}$.

The definition of the ergodic invariant probability measures for a transition function is perfectly similar to the corresponding notion for transition probabilities; that is, an invariant probability measure μ for $(P_t)_{t\in\mathbb{T}}$ is ergodic if, by definition, μ cannot be written as a sum of two nonzero mutually singular invariant measures of $(P_t)_{t\in\mathbb{T}}$.

As in the case of transition probabilities, we say that a measurable subset A of X is a set of maximal probability for $(P_t)_{t\in\mathbb{T}}$ or for $(T_t)_{t\in\mathbb{T}}$ if either $(P_t)_{t\in\mathbb{T}}$ does not have invariant probability measures, or else $\mu(A) = 1$ for every invariant probability measure μ of $(P_t)_{t\in\mathbb{T}}$.

From now on in this section we will assume that the transition function $(P_t)_{t\in\mathbb{T}}$ under consideration satisfies the following two conditions:

(TF1) For every $A \in \mathcal{B}(X)$, the map $(t, x) \mapsto P_t(x, A)$, $(t, x) \in \mathbb{T} \times X$ is jointly measurable with respect to t and x; that is, the map is measurable with respect to the product σ-algebra $\mathcal{L}(\mathbb{T}) \otimes \mathcal{B}(X)$, where $\mathcal{L}(\mathbb{T})$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbb{T}$.

(TF2) For every $x \in X$ and $f \in C_b(X)$, the map $t \mapsto S_t f(x)$, $t \in \mathbb{T}$, is continuous.

Condition (TF1) is called the standard measurability assumption (s.m.a.); we call condition (TF2) the pointwise continuity of $(S_t)_{t\in\mathbb{T}}$ (or of $(P_t)_{t\in\mathbb{T}}$).

Note that the s.m.a. is indeed standard (as its name states) in the sense that the assumption is always made whenever one deals with transition functions to such an extent that it is sometimes made part of the definition of a transition function (see, for instance, p. 156 of Ethier and Kurtz's monograph [10]).

By contrast, the second condition (TF2), the pointwise continuity of $(S_t)_{t\in\mathbb{T}}$, is somewhat unusual, but fairly weak. It is satisfied by all transition functions defined by continuous semiflows, continuous flows, and many one-dimensional convolution semigroups of probability measures. In the case of transition functions defined by Markov processes, it often happens that the conditions imposed on these transition functions are significantly stronger than the pointwise continuity; for instance, if $(P_t)_{t\in\mathbb{T}}$ is a transition function defined by any of the interacting particle systems studied in Liggett's monographs [30] and [31], then $(P_t)_{t\in\mathbb{T}}$ is a Feller transition function (that is, $S_t f \in C_b(X)$ whenever $f \in C_b(X)$ and $t \in \mathbb{T}$, where $(S_t)_{t\in\mathbb{T}}$ is the semigroup of operators defined on $B_b(X)$), $\mathbb{T} = [0, +\infty)$, and the map $t \mapsto S_t f$, $t \in [0, +\infty)$, is continuous with res! pect to the topology of uniform convergence of $C_b(X)$ for every $f \in C_b(X)$.

It can be shown that if $(P_t)_{t\in\mathbb{T}}$ satisfies the s.m.a. and is pointwise continuous, then $(P_t)_{t\in\mathbb{T}}$ defines an ergodic decomposition of the space X that is perfectly similar to the KBBY decomposition defined by a transition probability, provided that we replace by integrals the sums that appear in the decomposition defined by a transition probability. For instance, the sets $\mathcal{D}$ and Γ_c that appear in the KBBY decomposition defined by a transition probability correspond to the sets

$$\mathcal{D}^{(\mathrm{TF})} = \left\{ x \in X \;\middle|\; \begin{array}{l} \lim\limits_{s\to+\infty} \frac{1}{s} \int\limits_0^s S_t f(x)\,\mathrm{d}t \text{ exists and is} \\ \text{equal to zero for every } f \in C_0(X) \end{array} \right\}$$

and

$$\Gamma_c^{(\mathrm{TF})} = \left\{ x \in \Gamma_0^{(\mathrm{TF})} \;\middle|\; \lim_{s\to+\infty} \frac{1}{s} \int\limits_0^s S_t f(x)\,\mathrm{d}t \text{ exists for every } f \in C_0(X) \right\},$$

where $\Gamma_0^{(\mathrm{TF})} = X \setminus \mathcal{D}^{(\mathrm{TF})}$; for every $x \in \Gamma_c^{(\mathrm{TF})}$, we can define the standard elementary measure $\varepsilon_x^{(\mathrm{TF})} : C_0(X) \to \mathbb{R}$, $\varepsilon_x^{(\mathrm{TF})}(f) = \lim\limits_{s\to+\infty} \frac{1}{s} \int\limits_0^s S_t f(x)\,\mathrm{d}t$ for every $f \in C_0(X)$; then we can set

$$\Gamma_{cp}^{(\mathrm{TF})} = \left\{ x \in \Gamma_c^{(\mathrm{TF})} \,|\, \|\varepsilon_x\| = 1 \right\},$$

$$\Gamma_{cpi}^{(\mathrm{TF})} = \left\{ x \in \Gamma_{cp}^{(\mathrm{TF})} \,|\varepsilon_x \text{ is an invariant probability measure for } (T_t)_{t\in\mathbb{T}} \right\},$$

and so on. All the notions and the results that appear in the study of the KBBY decomposition for transition probabilities have analogues for transition functions. However, the development of the KBBY decomposition for transition functions is significantly more sophisticated than the development for transition probabilities. As mentioned in Introduction, currently we are writing the final form of a small monograph essentially dedicated to the KBBY decomposition for transition functions.

We will conclude this section with a few words about the use of conditions (TF1) and (TF2). From our discussion so far, it is not difficult to see that we need the s.m.a. in order to make sure that the integrals $\int\limits_0^s S_t f(x)\,\mathrm{d}t$, $s \in \mathbb{R}$, $s > 0$, $x \in X$, $f \in C_0(X)$, do exist. By contrast, it is not at all clear from our outline of the decomposition here where do we use the pointwise continuity of $(S_t)_{t\in\mathbb{T}}$. During the colloquium in Braşov, Lucian Beznea asked me why do we need condition (TF2), and I believe that it would be of interest to elaborate here on my answer.

Condition (TF2) is used in several places when studying the KBBY decomposition defined by $(P_t)_{t\in\mathbb{T}}$. For instance, the condition is used to show that $\Gamma_{cpi}^{(\mathrm{TF})}$ is a measurable subset of X.

Let us outline briefly the main arguments used to prove the measurability of $\Gamma_{cpi}^{(\mathrm{TF})}$ in order to see why the pointwise continuity of $(S_t)_{t\in\mathbb{T}}$ is a necessary condition in our approach.

As in the case of the KBBY decomposition for transition probabilities, in order to prove that $\Gamma_{cpi}^{(\mathrm{TF})}$ is measurable, we have to prove first that $\mathcal{D}^{(\mathrm{TF})}$, $\Gamma_0^{(\mathrm{TF})}$, $\Gamma_c^{(\mathrm{TF})}$, and $\Gamma_{cp}^{(\mathrm{TF})}$ belong to $\mathcal{B}(X)$. As soon as we know that $\Gamma_{cp}^{(\mathrm{TF})}$ is measurable, taking into consideration that $\Gamma_{cpi}^{(\mathrm{TF})} \subseteq \Gamma_{cp}^{(\mathrm{TF})}$, we obtain that the proof of the measurability of $\Gamma_{cpi}^{(\mathrm{TF})}$ is completed if we prove that $\Gamma_{cp}^{(\mathrm{TF})} \setminus \Gamma_{cpi}^{(\mathrm{TF})}$ is measurable.

To this end, let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of $C_0(X)$ such that the range $\{g_n \,|\, n \in \mathbb{N}\}$ of $(g_n)_{n\in\mathbb{N}}$ is dense in $C_0(X)$ (there exists such a sequence $(g_n)_{n\in\mathbb{N}}$ because $C_0(X)$ is a separable Banach space). It can be shown that for every rational number q, $q \in \mathbb{T}$, and every $n \in \mathbb{N}$, the set $A_{q,n} = \left\{ x \in \Gamma_{cp}^{(\mathrm{TF})} \,|\langle S_q g_n, \varepsilon_x\rangle \neq \langle g_n, \varepsilon_x\rangle \right\}$ belongs to $\mathcal{B}(X)$. Finally, the proof of the measurability of $\Gamma_{cpi}^{(\mathrm{TF})}$ is completed by showing that

$$\Gamma_{cp}^{(\mathrm{TF})} \setminus \Gamma_{cpi}^{(\mathrm{TF})} = \bigcup_{\substack{q\in\mathbb{Q}\cap\mathbb{T} \\ n\in\mathbb{N}}} A_{q,n} \tag{7}$$

It is in the proof of the equality (7) that we use the pointwise continuity of $(S_t)_{t\in\mathbb{T}}$.

4 Future Research

We believe that, in sharp contrast with other ergodic decompositions, the KBBY decomposition is a dynamic one, in the sense that it will change in time as other types of invariant measures will start playing a role in the decomposition. For instance, it seems likely that the reversible probability measures defined on p. 91 of Liggett's monograph [30] will play such a role.

However, it seems to us that at this time the most pressing, important, and quite challenging research topic is to articulate the decomposition for the various cases of interest. By this, we mean to obtain results similar in spirit to Theorem 2.2.6 of [65] (in which we characterize the elements of Γ_{cp} in the case of symbolic flows) for the sets that appear in the KBBY decomposition in the cases of interest. We will discuss now a few such cases that come to mind. In all these situations, the idea of articulating the decomposition appears implicitly or explicitly.

1. The Simple Generalized I.F.S. with Probabilities. Recall (see Section 2) that these particular cases of OMIGT processes are fairly easy to use in computations but their transition probabilities are not Feller.

In many applications, one encounters the following situation: given a locally compact separable metric space (X, d), one would like to approximate a certain probability measure $\mu \in \mathcal{M}(X)$ by a sequence of probability measures $(\nu_n)_{n\in\mathbb{N}}$, $\nu_n \in \mathcal{M}(X)$ for every $n \in \mathbb{N}$, in the sense that one would like to find a sequence $(\nu_n)_{n\in\mathbb{N}}$ of probability measures that converges to μ in the weak* topology of $\mathcal{M}(X)$ (the convergence of $(\nu_n)_{n\in\mathbb{N}}$ to μ in the weak* topology means that $(\langle f, \nu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle f, \mu\rangle$ for every $f \in C_0(X)$). This is done by finding a sequence $((\mathbf{w}^{(n)}, \mathbf{p}^{(n)}))_{n\in\mathbb{N}}$ of i.f.s. with probabilities which has the property that each of the transition probabilities $P_{(\mathbf{w}^{(n)},\mathbf{p}^{(n)})}$ defined by $(\mathbf{w}^{(n)}, \mathbf{p}^{(n)})$ has only one invariant probability measure, namely ν_n, which is necessa! rily ergodic, and the KBBY decomposition of $P_{(\mathbf{w}^{(n)},\mathbf{p}^{(n)})}$ is simply $X = \Gamma_{cpie}$. Then one has to use the $P_{(\mathbf{w}^{(n)},\mathbf{p}^{(n)})}$ to approximate the ν_n, which in turn approximate μ.

The problem that has been encountered using this approach with usual i.f.s. with probabilities is that one has to solve linear systems that are ill-conditioned (that is, the absolute value of the determinant of the matrix that contains the coefficients of the unknowns is very close to zero).

By contrast, using generalized i.f.s. with probabilities in which the $w_i^{(n)}$ and the $p_i^{(n)}$ are simple functions is very handy because simple functions are computationally tractable and the use of such generalized i.f.s. with probabilities does not involve solving linear systems.

Thus, it is of interest to find conditions under which the transition probabilities of these simple generalized i.f.s. with probabilities (transition probabilities that are not Feller) have the KBBY decomposition $X = \Gamma_{cpie}$ and are uniquely ergodic (have only one invariant probability measure).

2. Transition Functions of Markov Processes. As we discussed in Section 3, the KBBY decomposition is valid for transition functions under very general conditions. In our approach, we have obtained the decomposition in terms of transition functions and the associated families of Markov pairs. However, when dealing with continuous-time Markov processes, in many cases of interest, the transition functions and the associated families of Markov pairs are not known explicitly (typical examples are the transition functions associated to interacting particle systems).

A situation of this kind can be described as follows: one is given a compact metric space (X, d) (to simplyfy matters, we assume that X is compact rather than locally compact and separable; since X is compact, it follows that $C_0(X) = C_b(X)$, so, as usual, we denote by $C(X)$ any of the Banach spaces $C_0(X)$ or $C_b(X)$) and a physical system that changes in time (for instance, an interacting particle system); the system is modelled as a continuous-time time-homogeneous Markov process with state space (X, d). The process is known to have a Feller transition function $(P_t)_{t\in[0,+\infty)}$, which in turn defines a family of Markov-Feller pairs $((S_t, T_t))_{t\in[0,+\infty)}$. It is also known that the restrictions of S_t to $C(X)$, which can be thought of as positive contractions of $C(X)$ and are also denoted by S_t, $t \in [0, +\infty)$, form a Markov semigroup (for the definition of a Markov semigroup, as well as for all other unexplained terminology on Markov p! rocesses, see Chapter 1 of Liggett's monograph [30]). By the Hille-Yosida theorem (see Theorem 2.9, p. 16 of Liggett [30]) there exists a unique Markov generator A defined by $(S_t)_{t\in[0,+\infty)}$ by the formula $Af = \lim\limits_{t\to 0} \dfrac{S_t f - f}{t}$ for every $f \in C(X)$ for which the limit exists in the norm topology of $C(X)$, and, conversely, every Markov generator A defines a unique Markov semigroup $(S_t)_{t\in[0,+\infty)}$ such that $\lim\limits_{t\to 0} \dfrac{S_t f - f}{t}$ exists for every f in the domain $\mathcal{D}(A)$ of A, and
$Af = \lim\limits_{t\to 0} \dfrac{S_t f - f}{t}$. Furthermore, it can be shown that a Markov.
semigroup $(S_t)_{t\in[0,+\infty)}$ defines a unique transition function $(P_t)_{t\in[0,+\infty)}$ with associated family of Markov-Feller pairs $((S_t, T_t))_{t\in[0,+\infty)}$ where T_t is the dual of S_t thought of as a positive contraction of $C(X)$, and S_t is also considered as a positive contraction of $B_b(X)$ as extended in the proof of Theorem 1.1.4 of [65], $t \in [0, +\infty)$. It can be shown that $(P_t)_{t\in[0,+\infty)}$ as defined by $(S_t)_{t\in[0,+\infty)}$ satisfies all the conditions necessary for the existence of the KBBY decomposition. Since $(S_t)_{t\in[0,+\infty)}$ defines a unique Markov generator A, it follows that we should be able to articulate the KBBY decomposition defined by $(P_t)_{t\in[0,+\infty)}$ on (X, d) in terms of the generator A without making explicit use of $(P_t)_{t\in[0,+\infty)}$ and of the family of Markov-Feller pairs $((S_t, T_t))_{t\in[0,+\infty)}$ defined by $(P_t)_{t\in[0,+\infty)}$. Since in many cases ! of interest we know explicitly only the generator A rather than $(P_t)_{t\in[0,+\infty)}$ and $((S_t, T_t))_{t\in[0,+\infty)}$, articulating the KBBY decomposition in terms of A is extremely useful. The problem of articulating the decomposition in terms of A seems to me rather challenging at this time, but I believe that the problem could be solved using some of the delicate but powerful potential theoretic methods discussed in the monograph of Beznea and Boboc [3].

3. Transition Functions of Flows on Spaces of Cosets. By a space of left (or right) cosets we mean the collection of all left (or all right) cosets of a not necessarily normal subgroup D in a group G. Under suitable conditions imposed on G and D, conditions which we assume to be satisfied here, the two collections of cosets (left and right) can be endowed with natural metrics which define locally compact separable topologies on the two collections of cosets. In contrast with the case of interacting particle systems, when we deal with flows or semiflows defined on a locally compact separable metric space, the transition functions and the corresponding families of Markov pairs can be described explicitly (in terms of the action of the flows or the semiflows, respectively). However, if the locally compact separable metric space is a space of cosets, then, in almost all cases of interest, the setting (that is, the structure of the cosets spaces and the action of th! e flows) is so sophisticated that articulating the KBBY decomposition (which exists because the corresponding transition functions satisfy all the conditions for the existence of the decomposition) is an extremely challenging problem. As far as I know, the only sophisticated flows for which the KBBY decomposition can be deduced from the results already obtained are the horocycle flow and the more general unipotent flows. For the horocycle flow, the decomposition can be deduced from the results of S. G. Dani [7] and S. G. Dani and J. Smillie [8]; for unipotent flows (in connected Lie groups), the decomposition can be obtained from the results of M. Ratner [43] and [44]. The impressive beauty of the results of [7], [8], [43], and [44] can never be overstated.

References

[1] Aaronson, J., *An Introduction to Infinite Ergodic Theory,* Mathematical Surveys and Monographs, Vol. 50, AMS, Providence, Rhode Island, 1997.

[2] Beboutoff, M., *Markoff chains with a compact state space,* Rec. Math. (Mat. Sbornik) N.S. **10 (52)** (1942), 213-238.

[3] Beznea, L. and Boboc, N., *Potential Theory and Right Processes,* Mathematics and Its Applications, Vol. 527 Kluwer, Dordrecht, Holland, 2004.

[4] Centore, P. M. and Vrscay, E. R., *Continuity of attractors and invariant measures for iterated function systems,* Canadian Math. Bull. **37** (1994), 315-329.

[5] Cohn, D. L. *Measure Theory,* Birkhäuser, Boston, 1980.

[6] Cornfeld, I. P., Fomin, S. V. and Sinai, Ya. G., *Ergodic Theory,* Springer, Berlin/New York, 1982.

[7] Dani, S. G., *Invariant measures of horospherical flows on noncompact homogeneous spaces,* Inventiones Math. **47** (1978), 101-138.

[8] Dani, S. G. and Smillie, J., *Uniform distribution of horocycle orbits for Fuchsian groups,* Duke Math. J. **51** (1984), 185-194.

[9] Dynkin, E. B., *Markov Processes,* 2 Vols., Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Vols. 121-122, Springer, Berlin/Göttingen/Heidelberg, 1965.

[10] Ethier, S. N. and Kurtz, T. G., *Markov Processes: Characterization and Convergence,* Wiley, Hoboken, New Jersey, 2005.

[11] Furstenberg, H., *Recurrence in Ergodic Theory and Combinatorial Number Theory,* Princeton Univ. Press, Princeton, New Jersey, 1981.

[12] Hernández-Lerma, O. and Lasserre, J. B., *Markov Chains and Invariant Probabilities,* Birkhäuser, Basel, 2003.

[13] Heyer, H., *Probability Measures on Locally Compact Groups,* Springer, Berlin/Heidelberg/New York, 1977.

[14] Högnäs, G. and Mukherjea, A., *Probability Measures on Semigroups: Convolution Products, Random Walks and Random Matrices,* Plenum Press, New York, 1995.

[15] Iosifescu, M. and Grigorescu, S., *Dependence with Complete Connections and Its Applications,* Cambridge Univ. Press, Cambridge, 1990.

[16] Iosifescu, and Theodorescu, R., *Random Processes and Learning,* Springer, New York, 1969.

[17] Jaroszewska, J., *Iterated function systems with continuous place dependent probabilities,* Univ. Iagellonicae Acta Math., Fasc. **40** (2002), 137-146.

[18] Katok A. and Hasselblatt, B., *Introduction to the Modern Theory of Dynamical Systems,* Encyclopedia of Mathematics and Its Applications, Vol. 54, Cambridge Univ. Press, Cambridge, England/New York/Melbourne, 1995.

[19] Krengel, U., *Ergodic Theorems,* de Gruyter, Berlin/New York, 1985.

[20] Kryloff, N. and Bogoliouboff, N., *La théorie générale de la mesure dans son application à l'étude des systèmes de la mécanique non linéaires,* Ann. Math., **38** (1937), 65-113.

[21] Lasota A. and Mackey, M. C., *Stochastic perturbation of dynamical systems: the weak convergence of measures,* J. Math. Anal. Appl. **138** (1989), 232-248.

[22] Lasota A. and Mackey, M. C., *Chaos, Fractals, and Noise,* Springer, New York, 1994.

[23] Lasota A. and Myjak, J., *Semifractals,* Bull. Polish Acad. Sci., Math. **44**(1996), 5-21.

[24] Lasota A. and Myjak, J., *Markov operators and fractals,* Bull. Polish Acad. Sci., Math. **45** (1997), 197-210.

[25] Lasota A. and Myjak, J., *Semifractals on Polish spaces,* Bull. Polish Acad. Sci., Math. **46** (1998), 179-196.

[26] Lasota A. and Myjak, J., *Fractals, semifractals, and Markov operators,* Int. J. Bifurcation and Chaos **9** (1999), 307-325.

[27] Lasota A. and Myjak, J., *Attractors of multifunctions,* Bull. Polish Acad. Sci., Math. **48** (2000), 319-334.

[28] Lasota A. and Myjak, J., *On a dimension of measures,* Bull. Polish Acad. Sci., Math. **50** (2002), 221-235.

[29] Lasota A. and Yorke, J. A., *Lower bound technique for Markov operators and iterated function systems,* Random & Computational Dynamics **2** (1994), 41-77.

[30] Liggett, T. M., *Interacting Particle Systems,* Grundlehren der Mathematischen Wissenschaften 276, Springer, New York, 1985.

[31] Liggett, T. M., *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes,* Springer, Berlin, 1999.

[32] Mañé, R., *Ergodic Theory and Differentiable Dynamics,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 8, Springer, Berlin/Heidelberg, 1987.

[33] Marcus, M. B. and Rosen, J., *Markov Processes, Gaussian Processes, and Local Times,* Cambridge Studies in Advanced Mathematics 100, Cambridge Univ. Press, New York, 2006.

[34] de Melo, W and van Strien, S., *One-Dimensional Dynamics,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 25, Springer, Berlin/Heidelberg, 1993.

[35] Meyer, P. A., *Probabilités et Potentiel,* Hermann, Paris, 1966.

[36] Meyn, S. P. and Tweedie, R. L., *Markov Chains and Stochastic Stability,* Springer, London, 1993.

[37] Myjak, J. and Szarek, T., *On Hausdorff dimension of invariant measures arising from non-contractive iterated function systems,* Annali di Matematica **181** (2002), 223-237.

[38] Nicol, M., Sidorov, N. and Broomhead, D., *On the fine structure of stationary measures in systems which contract-on-average,* J. Theoret. Probab. **15** (2002), 715-730.

[39] Nummelin, E., *General Irreducible Markov Chains and Non-Negative Operators,* Cambridge Univ. Press, Cambridge, England, 1984.

[40] Onicescu, O. and Mihoc, G., *Sur les chaines de variables statistiques,* Bull. Sci. Math. **59** (1935), 174-192.

[41] Orey, S., *Lecture Notes on Limit Theorems for Markov Chain Transition Probabilities,* Van Nostrand Reinhold, London/New York/Toronto, 1971.

[42] Petersen, K., *Ergodic Theory,* Cambridge Univ. Press, Cambridge, England/New York/Melbourne, 1983.

[43] Ratner, M., *Distribution rigidity for unipotent actions on homogeneous spaces,* Bull. Amer. Math. Soc. **24** (1991), 321-325.

[44] Ratner, M., *Raghunathan's topological conjecture and distributions of unipotent flows,* Duke Math. J. **63** (1991), 235-280.

[45] Revuz, D., *Markov Chains,* North-Holland/American Elsevier, Amsterdam/New York, 1975.

[46] Revuz, D. and Yor, M., *Continuous Martingales and Brownian Motion,* Second Edition, Grundlehren der Mathematischen Wissenschaften 293, Springer, Berlin/Heidelberg, 1994.

[47] Robinson, C., *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos,* CRC Press, Boca Raton/Ann Arbor/London/Tokyo, 1995.

[48] Rosenblatt, M., *Markov Processes. Structure and Asymptotic Behavior,* Springer, Berlin/Heidelberg, 1971.

[49] Rudolph, D. J., *Fundamentals of Measurable Dynamics: Ergodic Theory on Lebesgue Spaces,* Oxford Univ. Press, New York, 1990.

[50] Sinai, Ya. G., *Topics in Ergodic Theory,* Princeton Univ. Press, Princeton, New Jersey, 1994.

[51] Stenflo, Ö., *A note on a theorem of Karlin,* Statist. & Probab. Letters **54** (2001), 183-187.

[52] Stenflo, Ö., *Ergodic theorems for time-dependent random iteration of functions,* Fractals and Beyond (Valletta, 1998), World Sci. Publishing, River Edge, New Jersey, 1998, 129-136.

[53] Stenflo, Ö., *Uniqueness of invariant measures for place-dependent random iterations of functions,* Fractals in Multimedia, Edited by M. F. Barnsley, D. Saupe, and E. R. Vrscay, Springer, New York, 2002, 13-32.

[54] Stenflo, Ö., *Markov chains in random environments and random iterated function systems,* Trans. Amer. Math. Soc. **353** (2001), 3547-3562.

[55] Stenflo, Ö., *Uniqueness in g-measures,* Nonlinearity **16** (2003), 403-410.

[56] Szarek, T., *Invariant measures for Markov operators with application to function systems,* Studia Math. **154** (2003), 207-222.

[57] Vrscay, E. R., *From fractal image compression to fractal-based methods in mathematics,* Fractals in Multimedia, Edited by M. F. Barnsley, D. Saupe, and E. R. Vrscay, Springer, New York, 2002, 65-106.

[58] Walters, P., *An Introduction to Ergodic Theory,* Springer, New York, 1982.

[59] Yosida, K., *Markoff processes with a stable distribution,* Proc. Imp. Acad. Tokyo **16** (1940), 43-48.

[60] Yosida, K., *Simple Markoff process with a locally compact phase space,* Math. Japonicæ **1** (1948), 99-103.

[61] Yosida, K., *Functional Analysis,* Third Edition, Springer, New York, 1971.

[62] Zaharopol, R., *Iterated function systems generated by strict contractions and place-dependent probabilities,* Bull. Polish Acad. Sci., Math. **48** (2000), 429-438.

[63] Zaharopol, R., *Fortet-Mourier norms associated with some iterated function systems,* Statist. & Probab. Letters **50** (2000), 149-154.

[64] Zaharopol, R., *Attractive probability measures and their supports,* Rev. Roumaine Math. Pures Appl. **49** (2004), 397-418.

[65] Zaharopol, R., *Invariant Probabilities of Markov-Feller Operators and Their Supports,* Birkhäuser, Basel, 2005.

[66] Zaharopol, R., *Invariant probabilities of convolution operators,* Rev. Roumaine Math. Pures Appl. **50** (2005), 387-405.

[67] Zaharopol, R., *Equicontinuity and the existence of attractive probability measures for some iterated function systems,* Rev. Roumaine Math. Pures Appl. **52**(2007), 259-286. *Erratum:* **52**(2007), Issue No. 6.

[68] R. Zaharopol, *An ergodic decomposition defined by transition probabilities,* Acta Appl. Math. **104** (2008), 47-81.