

HOMOGENIZATION RESULTS FOR DYNAMICAL HEAT TRANSFER PROBLEMS IN HETEROGENEOUS BIOLOGICAL TISSUES

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Abstract

The effective behavior of the solution of a dynamical boundary-value problem modeling the bio-heat transfer in heterogeneous microvascular tissues is analyzed. We consider an ε -periodic structure Ω , consisting of two parts: a solid tissue part and small regions of blood of a certain temperature. In this domain, we consider a heat equation, with a dynamical condition imposed on the boundaries of the blood zones. The limit equation, as ε , the small parameter related to the characteristic size of the blood regions, tends to zero, is a new heat equation, with extra-terms coming from the influence of the nonhomogeneous dynamical boundary condition.

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1 Introduction and setting of the problem

The aim of this paper is to study the asymptotic behavior of the solution of a dynamical boundary-value problem modeling thermoregulation phenomena in the human microvascular system. The bio-heat transport in living tissues is a complex process involving multiple mechanisms, such as conduction, convection, radiation, metabolism, etc. Bio-heat transfer models have significant applications in many therapeutic practices, such as cancer hyperthermia, brain hypothermia resuscitation, disease diagnostics, cryosurgery, etc.

Let Ω be a bounded connected open subset of \mathbb{R}^n ($n \geq 2$), with $\partial\Omega$ of class C^2 and let $Y = [0, 1]^n$ be the representative cell in \mathbb{R}^n and F (the so-called *elementary obstacle*) an open subset of Y with boundary ∂F of class C^2 , such that $\bar{F} \subset Y$. We shall denote by $F(\varepsilon, \mathbf{k})$ the translated image of εF by the vector $\varepsilon \mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^n$:

$$F(\varepsilon, \mathbf{k}) = \varepsilon(\mathbf{k} + F).$$

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Also, we shall denote by F^ε the set of all the obstacles contained in Ω . So

$$F^\varepsilon = \bigcup_{\mathbf{k} \in \mathbb{Z}^n} \{F(\varepsilon, \mathbf{k}) \mid \overline{F}(\varepsilon, \mathbf{k}) \subset \Omega\}.$$

Let $\Omega^\varepsilon = \Omega \setminus \overline{F^\varepsilon}$. Hence, Ω^ε is a periodically perforated domain with obstacles of the same size as the period.

We shall use the following notations:

$$Y^* = Y \setminus \overline{F}, \quad \theta = \frac{|Y^*|}{|Y|}.$$

In fact, we consider that Ω is an ε -periodic structure, consisting of two parts: a solid tissue part Ω^ε of temperature u^ε and small regions of blood $\Omega \setminus \overline{\Omega^\varepsilon}$ of a certain temperature. ε represents a small parameter related to the characteristic size of the blood regions.

We shall assume that we are dealing with heterogeneous tissues. From a mathematical point of view, we shall consider the case of a general medium, having discontinuous properties, represented by a coercive periodic matrix with rapidly oscillating coefficients. Let $A \in L^\infty_{\#}(\Omega)^{n \times n}$ be a symmetric matrix whose entries are Y -periodic, bounded and measurable real functions. We use the symbol $\#$ to denote periodicity properties. Let us assume that for some $0 < \alpha < \beta$,

$$\alpha |\xi|^2 \leq A(y)\xi \cdot \xi \leq \beta |\xi|^2 \quad \forall \xi, y \in \mathbb{R}^n.$$

We shall denote by $A^\varepsilon(x)$ the value of $A(y)$ at the point $y = x/\varepsilon$, i.e.

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right).$$

The problem studied in this paper concerns the nonstationary heat transfer in the solid tissue part, in contact with the blood regions. We shall assume that we have some external thermal sources f inside Ω^ε and, due to the fact that this complicated microstructure is dynamically evolving, we shall impose a dynamical boundary condition on the boundaries of the blood zones.

If we denote by $(0, T)$ the time interval of interest, we shall analyze the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of the solution of the following problem:

$$\varepsilon \rho c_p \frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f(t, x), \quad \text{in } \Omega^\varepsilon \times (0, T), \quad (1)$$

$$A^\varepsilon \nabla u^\varepsilon \cdot \nu + \alpha \varepsilon \frac{\partial u^\varepsilon}{\partial t} = \varepsilon a(u_b^\varepsilon - u^\varepsilon), \quad \text{on } S^\varepsilon \times (0, T), \quad (2)$$

$$u^\varepsilon(0, x) = u^0(x), \quad \text{in } \Omega^\varepsilon, \quad (3)$$

$$u^\varepsilon(0, x) = v^0(x), \quad \text{on } S^\varepsilon, \quad (4)$$

$$u^\varepsilon = 0, \quad \text{on } \partial\Omega \times (0, T). \quad (5)$$

Here, ν is the exterior unit normal to Ω^ε , $f \in L^2(0, T; L^2(\Omega))$, $u^0 \in H_0^1(\Omega)$, $v^0 \in L^2(S^\varepsilon)$, $a > 0$, $c_p > 0$, $\rho > 0$, $\alpha > 0$, $u_b^\varepsilon \in H^1(\Omega)$ and S^ε is the boundary of the blood regions. Let us remark that on S^ε we assume that the temperature $v^0(x)$ is equal to the trace of $u^0(x)$.

The existence and uniqueness of a weak solution of problem (1)-(5) can be settled by using the theory of parabolic problems (see, for instance, [7] and [10]). We shall be interested in getting the asymptotic behavior, when $\varepsilon \rightarrow 0$, of this solution.

Our results constitute a generalization of some of the results obtained in [5], by considering nonstationary processes and dynamical conditions on the boundaries of the blood regions. Problems closed to this one have been considered by D. Cioranescu and P. Donato [1], D. Cioranescu, P. Donato and H.I. Ene [2], D. Cioranescu, P. Donato and R. Zaki [3], C. Conca, J.I. Díaz and C. Timofte [4] and H. Ene and D. Polisevski [6].

2 The main result

The main convergence result of this paper is given by the following theorem:

Theorem 1. *One can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε of the problem (1)-(5) such that $P^\varepsilon u^\varepsilon \rightharpoonup u$ weakly in $L^2(0, T; H_0^1(\Omega))$, where u is the unique solution of the following problem:*

$$\begin{cases} \alpha \frac{|\partial F|}{|Y^*|} \frac{\partial u}{\partial t} - \operatorname{div}(A^0 \nabla u) + a \frac{|\partial F|}{|Y^*|} (u - u_b) = f & x \in \Omega, t \in (0, T), \\ u = 0 & x \in \partial\Omega, t \in (0, T), \\ u(0, x) = u_0(x) & x \in \Omega. \end{cases} \tag{6}$$

Here, $A^0 = (a_{ij}^0)$ is the homogenized matrix, defined by:

$$a_{ij}^0 = \frac{1}{|Y^*|} \int_{Y^*} \left(a_{ij}(y) + a_{ik}(y) \frac{\partial \chi_j}{\partial y_k} \right) dy,$$

in terms of the functions χ_j , $j = 1, \dots, n$, solutions of the cell problems

$$\begin{cases} -\operatorname{div}_y A(y)(D_y \chi_j + \mathbf{e}_j) = 0 & \text{in } Y^*, \\ A(y)(D_y \chi_j + \mathbf{e}_j) \cdot \nu = 0 & \text{on } \partial F, \\ \chi_j \in H_{\#Y}^1(Y^*), \quad \int_{Y^*} \chi_j = 0, \end{cases}$$

where \mathbf{e}_i , $1 \leq i \leq n$, are the elements of the canonical basis in \mathbb{R}^n . The constant matrix A^0 is symmetric and positive-definite.

Thus, in the limit, when $\varepsilon \rightarrow 0$, we get a constant coefficient heat equation, with a Dirichlet boundary condition and with extra-terms coming from the well-balanced contribution of the dynamical part of our boundary condition on the surface of the blood regions.

3 Proof of the main result

Let us consider the variational formulation of problem (1)-(5):

$$\begin{aligned} & \varepsilon \rho c_p \int_0^T \int_{\Omega^\varepsilon} \dot{u}^\varepsilon \varphi dx dt + \int_0^T \int_{\Omega^\varepsilon} A^\varepsilon \nabla u^\varepsilon \cdot \nabla \varphi dx dt + \\ & \alpha \varepsilon \int_0^T \int_{S^\varepsilon} \dot{u}^\varepsilon \varphi dx dt + a \varepsilon \int_0^T \int_{S^\varepsilon} (u^\varepsilon - u_b^\varepsilon) \varphi dx dt = \int_0^T \int_{\Omega^\varepsilon} f \varphi dx dt, \end{aligned} \quad (7)$$

for any $\varphi \in C_0^\infty([0, T] \times \Omega^\varepsilon)$. Here, we have denoted by $\dot{\cdot}$ the partial derivative with respect to the time.

Following [7] or [10], we know that there exists a unique weak solution of (7). Taking it as a test function in (7) and using our assumptions on the data and Cauchy-Schwartz, Poincaré's and Young's inequalities, we can obtain suitable energy estimates, independent of ε , for our solution (see [4], [7] and [8]).

Denoting by $P^\varepsilon u^\varepsilon$ the classical extension of u^ε to Ω (see [1]), one can prove that $P^\varepsilon u^\varepsilon$ is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\frac{\partial P^\varepsilon u^\varepsilon}{\partial t}$ is bounded in $L^2(0, T; L^2(\Omega))$ (see, for details, [4], [8], [9] and [10]).

So, by passing to a subsequence, we have $P^\varepsilon u^\varepsilon \rightharpoonup u$ weakly in $L^2(0, T; H_0^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$ and $\frac{\partial P^\varepsilon u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ weakly in $L^2(0, T; L^2(\Omega))$.

It is well-known by now how to pass to the limit, with $\varepsilon \rightarrow 0$, in the terms of (7) defined on Ω^ε (see, for instance [4] and [8]). Also, recall that θ is the weak- \star limit in $L^\infty(\Omega)$ of χ^ε . Thus, we get:

$$\int_0^T \int_{\Omega^\varepsilon} \dot{u}^\varepsilon \varphi dx dt \rightarrow \int_0^T \int_{\Omega} \dot{u} \theta \varphi dx dt$$

and, therefore,

$$\varepsilon \rho c_p \int_0^T \int_{\Omega^\varepsilon} \dot{u}^\varepsilon \varphi dx dt \rightarrow 0. \quad (8)$$

Also,

$$\int_0^T \int_{\Omega^\varepsilon} A^\varepsilon \nabla u^\varepsilon \cdot \nabla \varphi dx dt \rightarrow \int_0^T \int_{\Omega} \theta A^0 \nabla u \cdot \nabla \varphi dx dt, \quad (9)$$

$$\int_0^T \int_{\Omega^\varepsilon} f \varphi dx dt \rightarrow \int_0^T \int_{\Omega} \theta f \varphi dx dt. \quad (10)$$

Let us see now how we can pass to the limit in the two terms on the boundary of the blood regions. To this end, let us remember a result of D. Cioranescu and P. Donato (see [1]). Introducing, for any $h \in L^{s'}(\partial F)$, $1 \leq s' \leq \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} h\left(\frac{x}{\varepsilon}\right) \varphi d\sigma \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$, it is proven in [1] that

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \quad (11)$$

where

$$\langle \mu_h, \varphi \rangle = \mu_h \int_{\Omega} \varphi dx,$$

with

$$\mu_h = \frac{1}{|Y|} \int_{\partial F} h(y) d\sigma.$$

If $h \in L^\infty(\partial F)$ or even if h is constant, using a result of D. Cioranescu, P. Donato and H. Ene (see [2]), we have

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega). \quad (12)$$

We denote by μ^ε the above introduced measure in the case in which $h = 1$.

Using the convergence (12) written for $h = 1$, we obtain

$$\varepsilon \int_{S^\varepsilon} u^\varepsilon \varphi dx = \langle \mu^\varepsilon, P^\varepsilon u^\varepsilon \varphi \rangle \rightarrow \frac{|\partial F|}{|Y|} \int_{\Omega} u \varphi dx.$$

Since $u_b^\varepsilon \in H^1(\Omega)$ and $\|u_b^\varepsilon\|_{H^1(\Omega)} \leq C$, then, up to a subsequence, we get

$$u_b^\varepsilon \rightharpoonup u_b \quad \text{weakly in } H^1(\Omega).$$

Hence, integrating in time and using Lebesgue's convergence theorem, it is not difficult to see that

$$a\varepsilon \int_0^T \int_{S^\varepsilon} (u_b^\varepsilon - u^\varepsilon) \varphi dx dt \rightarrow a \frac{|\partial F|}{|Y|} \int_0^T \int_{\Omega} (u_b - u) \varphi dx dt. \quad (13)$$

Also, since (see [8])

$$\frac{\partial \gamma(u^\varepsilon)}{\partial t} \in L^2(0, T; L^2(S^\varepsilon)),$$

where $\gamma : H^1(\Omega^\varepsilon) \rightarrow L^2(S^\varepsilon)$ is the trace operator with respect to S^ε , which is continuous, we have

$$\alpha\varepsilon \int_0^T \int_{S^\varepsilon} \dot{u}^\varepsilon \varphi dx dt \rightarrow \alpha \frac{|\partial F|}{|Y|} \int_0^T \int_{\Omega} \dot{u} \varphi dx dt. \quad (14)$$

Putting together (8)-(10) and (13)-(14), we can pass to the limit in all the terms in (7) and we obtain exactly the variational formulation of the limit problem (6). As u is uniquely determined, the whole sequence $P^\varepsilon u^\varepsilon$ converges to u and this completes the proof of Theorem 1. ■

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