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THE CONTROL VARIATIONAL METHOD FOR CONTACT OF EULER-BERNOULLI BEAMS

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Abstract

We consider two static problems of contact between an elastic beam and an obstacle, the so called foundation. The contact is modeled with normal compliance and the Signorini unilateral conditions, respectively. We state the variational formulation of the problems, then we analyse them via the control variational method. As a result, we obtain existence, uniqueness and regularity results.

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1 Introduction

Situations where deformable bodies or components come into contact are common in industrial settings and in everyday life. Consequently, problems involving contact and its additional phenomena (friction, wear, adhesion) have received a great deal of attention in the engineering literature. The formalism of general detailed models for these processes and their analysis has been developed in the mathematical literature, too. A general survey of some results on the analysis of three dimensional contact problems, via the study of variational inequalities, can be found in [3, 7]. There, various existence and uniqueness results were obtained and the error analysis in the study of discrete schemes for the corresponding problems is provided.

The interest in contact problems involving beams lies in the fact that their mathematical analysis is considerably easier and more transparent, as some of the difficulties associated with two or three dimensions are absent. The regularity of the solutions is usually better and the use of trace theorems more convenient. Such problems may provide insight into the possible types of behaviour of the solutions and on occasions lead to decoupling of some of the equations, thus simplifying the analysis even more. Moreover, one may use such models as tests and benchmarks for computer schemes meant for simulation

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of complicated multidimensional contact problems. Models, analysis and simulations of contact problems for beams can be found in [2, 4, 5, 8] and the references therein.

In the present paper we deal with two mathematical problems which describe the process of equilibrium of an elastic beam in contact with an obstacle. We use the Euler-Bernoulli model for the beam and we model the contact with normal compliance and the Signorini conditions, respectively. In a variational formulation, the problems lead to a nonlinear variational equation and to an elliptic variational inequality, respectively. The novelty of the paper is that we analyse these models by using the control variational method introduced in [1, 9]. A comprehensive presentation for this new variational method may be found in the recent monograph [6]. The main new idea in this method consists to perform the minimization of the energy of the system via the optimal control theory, which represents an extension of the arguments of minimization via the calculus of variations, used in the classical variational method. This new general framework is very flexible and may offer several different solutions for the same problem, as shown in [10]. It is relevant both from the theoretical and the numerical point of view, as illustrated in [6]. The interest in using the control variational method in the analysis of the problems described in this paper arise in the fact that it replaces the solution of nonlinear differential equations of order four by the solution of linear equations of second order and, moveover, it provides regularity results.

The rest of the paper is structured as follows. In Section 2 we present the two problems of contact and prove their unique weak solvability. Our main results are presented in Section 3; there, we analyze the variational models via the control variational method; we provide existence, uniqueness and regularity results, and we investigate additional properties related to our approach.

2 The problems and their unique weak solvability

The physical setting and the process are as follows. An elastic beam of length $L > 0$ is clamped at its left end and the right end is free. The beam is acted upon by an applied force of (linear) density $f = f(x)$ which is directed downward, $f \leq 0$, where x is the spatial variable. Let $g = g(x) \leq 0$ denote the gap between the beam in its reference configuration $[0, L]$ and an obstacle S, situated on the Ox axis. The beam comes into contact with S only when the vertical displacement exceeds g. The physical setting is depicted in Fig. 2.1.

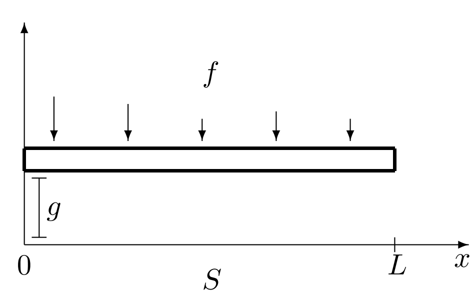

Fig. 2.1. The setting of the problem.

For $x \in [0, L]$, denote by $u = u(x)$ the vertical displacement of the beam. When the meaning is clear, we do not indicate explicitly the dependence of various variables on x. We use the Euler-Bernoulli model for the beam and we denote $A = EI$, where I is the beam's moment of inertia and E its Young modulus.

In the first problem we assume that S is deformable and, therefore, we model the contact with normal compliance. The classical formulation of the problem is the following.

Problem P_1. *Find a displacement field $u : [0, L] \to \mathbb{R}$ such that*

$$\frac{d^2}{dx^2}\Big(A\,\frac{d^2u}{dx^2}\Big) = f + p(g-u) \quad \text{in} \quad (0, L), \tag{1}$$

$$u(0) = \frac{du}{dx}(0) = 0, \tag{2}$$

$$\frac{d^2u}{dx^2}(L) = \frac{d^3u}{dx^3}(L) = 0. \tag{3}$$

We now provide explanations on the equations and conditions above. When $u > g$ there is no contact between the beam and the foundation and we have $\frac{d^2}{dx^2}(A\,\frac{d^2u}{dx^2}) = f$, which is the classical equilibrium equation of the beam. Therefore,

$$u > g \quad \Longrightarrow \quad \frac{d^2}{dx^2}\Big(A\,\frac{d^2u}{dx^2}\Big) = f. \tag{4}$$

When $u \le g$, there is contact between the beam and the foundation. In this case the foundation reacts with a normal force ξ directed upward, $\xi \ge 0$. The equilibrium equation now is $\frac{d^2}{dx^2}(A\,\frac{d^2u}{dx^2}) = f + \xi$. We assume that the reaction ξ depends on the penetration, i.e. $\xi = p(g - u)$ where p is a given nonnegative function. This assumption represents a version of the so-called normal compliance contact condition, see [7] and the reference therein. Thus,

$$u \le g \quad \Longrightarrow \quad \frac{d^2}{dx^2}\Big(A\,\frac{d^2u}{dx^2}\Big) = f + p(g-u). \tag{5}$$

Conditions (4) and (5) may be restated in the form (1), if we assume that $p(r) = 0$ for $r < 0$. Next, condition (2) is imposed since the beam is rigidly attached at its left and, finally, we use condition (3) since we assume that no moments act on the free end of the beam.

In the second problem we assume that the foundation is rigid and, therefore, we model the contact with the Signorini conditions. The classical formulation of the problem is the following.

Problem P_2. *Find a displacement field $u : [0, L] \to \mathbb{R}$ such that*

$$\frac{d^2}{dx^2}\Big(A\,\frac{d^2u}{dx^2}\Big) = f + \xi \quad \text{in} \quad (0, L), \tag{6}$$

$$u \geq g, \quad \xi \geq 0, \quad \xi(g - u) = 0 \quad \text{in} \quad (0, L), \tag{7}$$

$$u(0) = \frac{du}{dx}(0) = 0, \tag{8}$$

$$\frac{d^2u}{dx^2}(L) = \frac{d^3u}{dx^3}(L) = 0. \tag{9}$$

The meaning of the equations and conditions in Problem P_2 is similar to that in Problem P_1. The only difference arise from the fact that now we use the Signorini conditions (7). These conditions show that there is no penetration into the obstacle (since $u \geq g$), the reaction force ξ is directed upward (since $\xi \geq 0$), and vanishes when there is no contact (since $\xi = 0$ when $u > g$). More details in the use of the Signorini contact conditions can be found in [3, 7].

We turn now to derive a weak or variational formulation of Problems P_1 and P_2. To this end we assume in what follows that

$$A \in L^\infty(0, L), \quad \text{there exists } m > 0 \text{ such that } A(x) \geq m \ \text{ a.e. } x \in (0, L), \tag{10}$$

$$f \in L^2(0, L), \quad f(x) \leq 0 \ \text{ a.e. } x \in (0, L), \tag{11}$$

$$g \in L^2(0, L), \quad g(x) \leq 0 \ \text{ a.e. } x \in (0, L). \tag{12}$$

Also, the normal compliance function $p : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \leq L_p\,|r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(b) } (p(r_1) - p(r_2)) \cdot (r_1 - r_2) \geq 0 \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) } p(r) = 0 \quad \forall\, r < 0. \end{cases} \tag{13}$$

We remark that the assumptions (13) on $p(\cdot)$ are fairly general. The main severe restriction comes from condition (a) which, roughly speaking, requires that the function grows at most linearly for large values of the argument. From mechanical point of view, conditions (b) and (c) express the fact that the reaction force increases with the penetration and vanishes when there is no penetration into foundation. One standard example of function p which satisfies (13) is $p(r) = \mu r_+$, where $\mu > 0$ is the stiffness coefficient and r_+ denotes the positive part of r, i.e. $r_+ = max\,\{0, r\}$.

In what follows we use standard notation for L^p and Sobolev spaces and the subscripts x and xx will represent the first and the second derivatives with respect to x, respectively. We introduce the closed subspace of $H^2(0, L)$ given by

$$V = \{\ v \in H^2(0, L)\ :\ v(0) = v_x(0) = 0\ \}, \tag{14}$$

and we denote $(u,v)_V = (u,v)_{H^2(0,L)}$, whenever $u,\, v \in V$. In addition, we consider the bilinear form $a : V \times V \to \mathbb{R}$, the functional $j : V \times V \to \mathbb{R}$, and the set of admissible displacement fields K, defined by

$$a(u,v) = \int_0^L A\, u_{xx} v_{xx}\, dx \qquad \forall\, u,\, v \in V, \tag{15}$$

$$j(u,v) = -\int_0^L p(g-u)\, v\, dx \qquad \forall\, u,\, v \in V, \tag{16}$$

$$K = \{\ v \in V\ :\ v \geq g \quad \text{in}\ \ (0,L)\ \}. \tag{17}$$

We note that by conditions (10)–(13) the integrals in (15) and (16) are well-defined; moreover, by condition (12) it follows that K nonempty.

A standard computation, based on two integrations by parts and the boundary conditions (2), (3) leads to the following variational formulation of Problem P_1.

Problem P_1^V. *Find a displacement field u such that*

$$u \in V, \qquad a(u,v) + j(u,v) = (f,v)_{L^2(0,L)} \qquad \forall\, v \in V. \tag{18}$$

Next, we note that if u is a regular solution of Problem P_2 then $u \in K$ and, moreover, for all $v \in K$ we have

$$\xi(v-u) = \xi(v-g) + \xi(g-u) = \xi(v-g) \geq 0 \qquad \text{in}\ \ (0,L).$$

Therefore, using again integrations by parts and the boundary conditions (8), (9) we derive the following variational formulation of Problem P_2.

Problem P_2^V. *Find a displacement field u such that*

$$u \in K, \qquad a(u, v-u) \geq (f, v-u)_{L^2(0,L)} \qquad \forall\, v \in K. \tag{19}$$

We have the following existence and uniqueness results, which provide the unique weak solvability of the contact problems P_1 and P_2.

Theorem 1. *Assume that* (10)–(12) *hold. Then:*

1) There exists a unique solution $u^* \in V$ to the variational problem P_1^V, if (13) *holds.*

2) There exists a unique solution $\hat{u} \in K$ to the variational problem P_2^V.

Proof. 1) Let $A : V \to V$ be the operator given by $(Au,v)_V = a(u,v) + j(u,v)$ for all $v \in V$. We use assumptions (10) and (13) to see that A is a strongly monotone Lipschitz continuous operator on V. Moreover, by (11) it follows that there exists a unique element $\tilde{f} \in V$ such that $(\tilde{f}, v)_V = (f,v)_{L^2(0,L)}$ for all $v \in V$. The unique solvability of Problem P_1^V follows from the unique solvability of the equation $Au = \tilde{f}$, guaranteed by a standard result on nonlinear equations with monotone operators.

2) We use assumption (10) to see that the bilinear from $a(\cdot,\cdot)$ is continuous and V-elliptic. Also, it follows from (12) that K is a nonempty closed convex subset of V. The unique solvability of Problem P_2^V follows now from a standard result on elliptic variational inequality of the first kind. □

3 Analysis via the control variational method

In this section we perform the analysis of problems P_1^V and P_2^V by a method which is different from the method presented in the proof of Theorem 1. Everywhere below we assume that (10)–(12) hold and, when we deal with Problem P_1^V, we assume that (13) holds, too. We start with the analysis of Problem P_1^V and, to this end, we introduce the following optimal control problem:

$$min\Big\{\frac{1}{2}\int_0^L lh^2\,dx+\int_0^L\varphi(g-u)\,dx\Big\}, \tag{20}$$

$$u_{xx}=lz+lh \quad \text{in} \quad (0,L), \tag{21}$$

$$u(0)=u_x(0)=0. \tag{22}$$

Here and below $l=A^{-1}$ and note that, by condition (15), it follows that $l\in L^\infty(0,L)$. Also, $\varphi:\mathbb{R}\to\mathbb{R}$ is such that $\varphi'=p$ and $z\in H^2(0,L)$ is the solution of the problem

$$\frac{d^2z}{dx^2}=f \quad \text{in} \quad (0,L), \qquad z(L)=\frac{dz}{dx}(L)=0. \tag{23}$$

The solvability of the optimal problem (20)–(22) and its link with the variational problem P_1^V is given by the following result.

Theorem 2. *Assume that* (10)–(13) *hold. Then, problem* (20)–(22) *has a unique optimal pair* $[u^*,h^*]\in H^2(0,L)\times L^2(0,L)$ *and, moreover,* u^* *satisfies* (18).

Proof. Any pair $[u,h]\in H^2(0,L)\times L^2(0,L)$ that satisfies (21), (22) is admissible for the control problem. The assumptions on p show that φ is convex and therefore is bounded from below by some affine mapping. As the dependence $u\mapsto h$ in (21) is linear, it yields that the cost functional is coercive in $h\in L^2(0,L)$. This ensures the existence of an optimal pair $[u^*,h^*]$ and its uniqueness follows by the strict convexity of the cost functional (20).

Next, from (22) and the definition (14) of the space V it follows that $u^*\in V$. We consider now affine variations of the type $[u^*,h^*]+\lambda([w,k]-[u^*,h^*])$, where $\lambda\in\mathbb{R}$ and $[w,k]$ is an element of $H^2(0,L)\times L^2(0,L)$ which satisfies (21), (22). Since $[u^*,h^*]$ is the minimum point of (20), we obtain

$$\int_0^L\varphi(g-u^*)\,dx+\frac{1}{2}\int_0^L l(h^*)^2\,dx$$
$$\le\int_0^L\varphi(g-\lambda w-(1-\lambda)u^*)\,dx+\frac{1}{2}\int_0^L l(\lambda k+(1-\lambda)h^*)^2\,dx \quad \forall\,\lambda\in\mathbb{R}.$$

We divide the previous inequality by $\lambda>0$ and then pass to the limit as $\lambda\to 0$ to obtain

$$0\le\int_0^L lh^*(k-h^*)\,dx+j(u^*,w-u^*),$$

for all elements $[w,k]$ as above. Similar arguments with $\lambda < 0$ show that the converse inequality is valid, too; therefore, we conclude that

$$\int_0^L lh^*(k-h^*)\,dx + j(u^*, w-u^*) = 0. \tag{24}$$

Also, by (21) and the similar relation for $[w,k]$ we have

$$\begin{aligned}\int_0^L lh^*(k-h^*)\,dx &= \int_0^L (Au^*_{xx} - z)(w_{xx} - u^*_{xx})\,dx \\ &= \int_0^L (Au^*_{xx})(w_{xx} - u^*_{xx})\,dx - \int_0^L f(w-u^*)\,dx,\end{aligned}$$

due to the definition (23) of z. Using this equality in (24) and taking into account the definition of the bilinear form $a(\cdot,\cdot)$, we obtain (18), which concludes the proof. □

We now turn to the analysis of Problem P_2^V and, to this end, we introduce the optimal control problem

$$\min\Big\{\int_0^L lh^2 dx\Big\}, \tag{25}$$

subjected to (21), (22) and to the unilateral constraint

$$u \geq g \quad \text{in} \quad (0,L). \tag{26}$$

The solvability of this optimal problem and its link with the variational problem P_2^V is given by the following result.

Theorem 3. *Assume that* (10)–(12) *hold. Then, problem* (25), (21), (22), (26) *has a unique optimal pair* $[\hat{u},\hat{h}] \in K \times L^2(0,L)$ *and, moreover,* $\hat{u}$ *satisfies* (19).

Proof. Clearly, any element $u \in K$ generates via (21) the control $h \in L^2(0,L)$ such that the pair $[u,h]$ is admissible for the control problem (25), (21), (22), (26). The existence and the uniqueness of an optimal pair $[\hat{u},\hat{h}] \in K \times L^2(0,L)$ of this problem follows by using arguments similar to those presented in the proof of Theorem 2.

Let $[w,k] \in K \times L^2(0,L)$ be an admissible pair for the problem (25), (26) and consider convex variations of the form $[\hat{u},\hat{k}] + \lambda([w,k] - [\hat{u},\hat{h}])$ where $\lambda \in [0,1]$ and $[w,k]$ is an element of $K \times H^2(0,L)$ which satisfies (21), (22). Then, the optimality of $[\hat{u},\hat{h}]$ leads to the inequality

$$\int_0^L l(\hat{h})^2\,dx \leq \int_0^L l[\lambda k + (1-\lambda)\hat{h}]^2\,dx,$$

which implies that

$$0 \leq \int_0^L l\hat{h}(k-\hat{h})\,dx,$$

for any $k \in L^2(0,L)$ such that $[w,k]$ is admissible for (25), (26), with some $w \in K$. As in the proof of Theorem 2, we replace in the previous inequality $\hat{h}$ and k as they may be computed from the state equation (21). As a result we find that

$$0 \geq \int_0^L (A\hat{u}_{xx} - z)(\hat{u}_{xx} - w_{xx})\,dx \qquad \forall\, w \in K.$$

We now integrate by parts the term involving z to obtain (19), which concludes the proof. □

Note that by Theorems 2 and 3 we obtain the existence of the solutions to Problems P_1^V and P_2^V, respectively. We conclude that the control variational method presented above allows to recover the existence part in Theorem 1. The uniqueness part can be easily obtained by using arguments of monotonicity.

In what follows we provide more comments on the interest in using the control variational method and we illustrate them within the study of Problem P_1^V.

First, we note that the state equation (21) in the optimal control problems discussed above is linear and its form is extremely simple and easy to integrate. Therefore, the solution of (1)–(3) or, equivalently, the solution of (18), is reduced to the successive solution of such type of equations. Note also that, in contrast, both (1) and (18) are nonlinear ordinary differential equations of fourth order and, therefore, in principle, their integration is more difficult. We conclude that the interest on the control variational method presented above arises in the fact that it replaces the solution of nonlinear differential equations of order four by the solution of linear equations of second order.

A second interest in the optimal control method arises from the fact that it provides regularity results. To illustrate this, we turn again to the optimal control problem (20)−(22). We introduce the adjoint system and the adjoint state $r \in H^1(0,L)$ given by

$$r_{xx} = -p(g - u^*) \quad \text{in} \quad (0,L), \tag{27}$$

$$r(L) = r_x(L) = 0. \tag{28}$$

Performing integration by parts in (24) and using (27), (28), (21) and (22) we find that

$$\begin{aligned} 0 &= \int_0^L lh^*(k - h^*)\,dx - \int_0^L p(g-u)(w-u^*)\,dx \\ &= \int_0^L lh^*(k - h^*)\,dx + \int_0^L r_{xx}(w-u^*)\,dx \\ &= \int_0^L lh^*(k - h^*)\,dx + \int_0^L r(w_{xx}-u^*_{xx})\,dx \\ &= \int_0^L lh^*(k - h^*)\,dx + \int_0^L lr(k-h^*)\,dx. \end{aligned}$$

Since the above inequality is valid for any $k \in L^2(0,L)$ we obtain that

$$r + h^* = 0 \quad \text{in} \quad (0,L). \tag{29}$$

Relation (29) expresses the fact that the gradient of the cost functional (20) (as a function of h alone) is zero in the minimum point $h^* \in L^2(0,L)$. The left-hand side of (29) is exactly this gradient and it is used in the iterative procedures (the gradient methods) for the solution of (20)–(22). By Theorem 2 we see that this provides an alternative solution method for the original problem (1)–(3), involving just the equations (21) and (27), that may be integrated directly.

We also use relation (29) to note that h^* has the same regularity as r, i.e. $h^* \in H^2(0,L) \cap H_0^1(0,L)$. Here, we use the fact that $p(\cdot)$ is Lipschitz continuous and $g \in L^2(0,L)$. If l is smooth enough, by (21) we also obtain that $u^* \in H^4(0,L)$, which represents a regularity property for the solution of Problem P_1.

Note that such a regularity property seems not possible to be extended to the solution of the variational inequality (19) or, equivalently, to the solution of the optimal control problem (25), (21), (22), (26). The reason arise from the fact that in this case the adjoint equation (27) has the form

$$r_{xx} \in \partial I_K(u^*) \quad \text{in} \quad (0,L),$$

where I_K is the indicator function of the convex $K \subset V$. And, there may be a lack of regularity of the nonlinear term $\partial I_K(u^*)$ as, in general, I_K is not differentiable.

We end this paper with the remark that problem (20)–(22) may be reexpressed as the following mathematical programming problem

$$\min \Big\{ \frac{1}{2}\int_0^L lh^2\,dx + \int_0^L \varphi\Big(g - \int_0^x \int_0^s (lz+lh)(\zeta)d\zeta\,ds\Big)\,dx \Big\}$$

for any $h \in L^2(0,L)$. This formulation is useful in the numerical approach of the the contact problem P_1^V. And, finally, we note that arguments similar to those presend above can be used in the study of variational inequalities associated to partial differential equations of elliptic type, in arbitrary dimension.

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