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CONTINUOUS BRANCHING PROCESSES VIA POISSON RANDOM MEASURES

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Abstract

Continuous branching processes on $[0,\infty[$ are derived from Poisson random measures associated with Lévy processes

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1 The origin

From historical point of view, the first stochastic pattern in branching property is that of Galton-Watson chains, that is Markov processes with times $n \in \{0,1,\ldots\}$, states $x \in \{0,1,\ldots\}$ and one-step transition probabilities $Q(x;\mathrm{d}y)$ from time n to time $n+1$

$$Q(x;\cdot) := \nu^{*x} \text{ (no dependence on } n).$$

Here the probability ν on $\{0,1,\ldots\}$ is the offspring distribution (which is arbitrary but fixed, for each fixed chain) while the convolution power ν^{*x} is defined as $\nu * \ldots * \nu$ (x times). That is the shortest and most rigurous definition to Galton-Watson chains but it is often replaced by an inuitive description in terms of individuals which independently give birth to random numbers of ofsprings, each such random number obeying the same distribution ν from above.

From theoretical point of view, the most outstanding property of Galton-Watson chains is the following:

> If X and Y are independent Galton-Watson chains that have the same offspring distribution ν, then their sum $n \longmapsto X_n + Y_n$ is also a Galton-Watson chain with offspring distribution ν.

Roughly speaking, the modifications introduced by later branching patterns concern the time set $\{0,1,\ldots\}$ (which is often replaced by $[0,\infty[$) and the state space $\{0,1,\ldots\}$, which is often replaced by $[0,\infty[$ (or even by more complicated measure spaces). All of these patterns still obey intuitive descriptions in terms of particles which independently give birth to offsprings according to a given law. Other patterns also consider motions of these particles.

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2 Continuous Branching Processes

The continuous branching processes have the continuous time set $[0, \infty[$ and the state space $E := [0, \infty[$ (with a trap at $0 \in E$). Letting aside some technicalities, we may think of continuous branching processes as E-valued Feller processes which satisfy the following property:

> If X and Y are independent E-valued Markov processes with identical transition semigroup, then their sum process $X + Y$ is stil Markovian and has the same transition semigroup as X and Y.

Given a continuous branching process X, one may think intuitively of X_t as the random mass of a particle, at time t; clearly, we think of masses which evolve in time and vanish forever when hitting the state $0 \in E$. Under such an interpretation, no motion is assumed by this pattern.
Let us fix throughout $P_t(x; \mathrm{d}y)$ as to be the transition semigroup of a continuous branching process. Now clearly the above invariance under addition reads as the following convolution equation

$$P_t(x + y; \cdot) = P_t(x; \cdot) * P_t(y; \cdot) \text{ for all } t \geq 0 \text{ and } x, y \in E$$

Consequently we start with the assumption that the following setting is satisfied on a suitably large probability space.

1. $X = (X_t^x; x \in E, t \geq 0)$ is a doubly indexed family of E-valued random variables.
2. All the sample maps $t \longmapsto X_t^x$ are right continuous with left-hand limits and satisfy $X_0^x = x$.
3. All the sample maps $x \longmapsto X_t^x$ are non-decreasing and right continuous.
4. Whatever be $n \in \{2, 3, \ldots\}$ and $0 = x(0) \leq x(1) \leq \ldots \leq x(n) < \infty$, the processes $t \longmapsto (X_t^{x(i)} - X_t^{x(i-1)})$ (for $i = 1, 2, \ldots, n$) are independent continuous branching processe with transition semigroup P_t and a trap at $0 \in E$.

As for the relationship with the originating convolution power expression $Q(x; \cdot) = \nu^{*x}$ of transition probabilities, we still have

$$P_t(x; \cdot) = P_t(1; \cdot)^{*x} \text{ for } t \geq 0, x \in E = [0, \infty[\tag{1}$$

but the definition to the xth convolution power $(\cdot)^{*x}$ follows now from the theory of indefinite divisible distributions while the probability $P_t(1; \cdot)$ ceases to be arbitrary. In fact, the probabilities $P_t(1; \cdot)$ associated with all the continuous branching processes form a subclass of indefinite divisible laws which obey, via their Laplace transforms, some special differential equations.

Now fix arbitrarily $t > 0$ and think of the random maps $E \ni x \longmapsto X_t^x$ which correspond to a Lévy process. The Lévy procees $x \longmapsto X_t^x$ will have no Gaussian component since supported by a semibounded interval; the proof to this claim is not straightforward but we omit it since a general matter of indefinitely divisible laws. Therefore there exists a Poisson random measure $H_t(\mathrm{d}x \otimes \mathrm{d}y)$ on $E \times E$ such that

$$X_t^x - x = \int_{[0,x]\times E} v H_t(\mathrm{d}u \otimes \mathrm{d}v)$$

Furthermore, since the sample paths of the Lévy process $x \longmapsto X_t^x$ are non-decreasing (therefore with locally bounded variation) if follows that we have proved the following result:

Proposition 1 *The cadlag version of each sample map $x \longmapsto X_t^x - x$ can be expressed as a series*

$$X_t^x - x = \sum_{n\geq 1} \left(\sum_{1\leq k\leq n} \eta(t,k) \right) \chi_{[\tau(t,n),\infty[}(x) \tag{2}$$

with random variables $\eta(t,n) > 0$ and random variables $0 < \tau(t,1) < \tau(t,2) < \ldots < \tau(t,n) < \ldots$ which increase to ∞ with n. Finally, since $x \longmapsto X_t^x$ is a Lévy process, it follows that the random measures $H_t(\cdot; \mathrm{d}u\mathrm{d}x)$

$$H_t(\cdot; \mathrm{d}u\mathrm{d}y) := \sum_n \delta_{\eta(t,n)}(\mathrm{d}u) \otimes \delta_{\tau(t,n)}(\mathrm{d}y) \tag{3}$$

are Poisson random measures with respect to which the proceeses $x \longmapsto X_t^x$ express as integrals

$$X_t^x - x = \int_{[0,\infty[\times E} \chi_{[0,x]\times E}(u,y) y H_t(\cdot; \mathrm{d}u\mathrm{d}y) \tag{4}$$

As for the Poisson random measures $H_t(\cdot; \mathrm{d}u\mathrm{d}y)$, we have the following result.

Proposition 2 *The measure valued process $t \longmapsto H_t$ is a temporarily homogeneous Markov process.*

Proof. For every natural integer number $n \geq 2$, and arbitrary real numbers $x(0) \leq x(1) \leq \ldots \leq x(n)$ we know that the process

$$t \longmapsto (X_t^{x(0)}, X_t^{x(1)}, \ldots, X_t^{x(n)})$$

is Markovian since so is $t \longmapsto (X_t^{x(0)}, X_t^{x(1)} - X_t^{x(0)}, \ldots, X_t^{x(n)} - X_t^{x(n-1)})$ and the two processes follow from each other, by a one-to-one map. Since n and $x := (x(0), \ldots, x(n))$ are arbitrary, we extend the claim to the whole path valued procees $t \longmapsto (X_t^x; x)$ in the sense that

$$t \longmapsto \left([0,\infty[\ni u \longmapsto X_t^{x(u)} \right) \text{ is Markovian}$$

for every nondecreasing path $[0,\infty[\ni u \longmapsto x(u) \in E = [0,\infty[$ which has locally bounded variation. But our hypotheses ensure the fact that the path valued process $t \longmapsto \left(u \longmapsto X_t^{x(u)} \right)$ and the measure valued process $t \longmapsto H_t(\cdot; \mathrm{d}u\mathrm{d}x)$ follow from each other by the one to one integral map at (4). Therefore the measure valued process $t \longmapsto H_t(\cdot; \mathrm{d}u\mathrm{d}x)$ is Markovian too; finally its time homogenity also follows from the same property of $t \longmapsto \left(u \longmapsto X_t^{x(u)} \right)$. ∎

3 Continuous branching Brownian motion

Throughout this section, we will deal with a concrete example of continuous branching Markov process $t \longmapsto X_t$. Recall $(P_t)_{t\geq 0}$ denotes the transition semigroup of X.

Consider the function-to-function linear operator $\mathcal{G}$ which acts as

$$\mathcal{G}f(x) := xf''(x);\ x \in E = [0, \infty[\tag{5}$$

for all functions $f \in \mathrm{C}^2(E) := \mathrm{C}^2(\mathbb{R})_{|E}$. Finally assume that $\mathrm{C}^2(E)$ is included into the domain of the infinitessimal generator of $(P_t)_{t\geq 0}$ and that

$$\lim_{t\downarrow 0} \frac{P_t f - f}{t} = \mathcal{G}f \text{ in } \mathrm{C}(E) \text{ for all } f \in \mathrm{C}^2(E)$$

Let on the other hand $T_t : \mathrm{C}(E) \longrightarrow \mathrm{C}(E)$ be the contraction linear operators defined as

$$T_t f(x) := \int_E f(y) \exp\left(*; x\frac{1}{t}\mathrm{e}_{1/t}\right)(\mathrm{d}y);\ x \geq 0, t \geq 0$$

where e_λ

$$\mathrm{e}_\lambda(\mathrm{d}x) := \chi_{[0,\infty[}(x)\lambda e^{-\lambda x}\mathrm{d}x$$

is the exponential law while the probability $\exp(*;\mu)$ on $\mathbb{R}$ is the convolution exponential

$$\exp(*;\mu) := e^{-\mu(\mathbb{R})} \sum_{n\geq 0} \frac{1}{n!}\mu^{*n}$$

for every positive finite measure μ on $\mathbb{R}$. Now we have the following.result

Proposition 3

$$P_t = T_t \textit{ for all } t \geq 0. \tag{6}$$

Proof. With each $t \geq 0, \lambda \geq 0$ associate the number $\psi(t,\lambda)$

$$\psi(t,\lambda) := \frac{\lambda}{1 + t\lambda}$$

and let $f_\lambda : E \longrightarrow E$ be the function

$$f_\lambda(x) := e^{-\lambda x};\ x \in E = [0, \infty[.$$

Let Λ denote the linear space generated by all the f_λ functions from above, with $\lambda \geq 0$. Direct calculations show that

$$T_t f_\lambda = f_{\psi(t,\lambda)} \text{ for all } t \geq 0, \lambda \geq 0 \tag{7}$$

therefore

$$T_t \Lambda \subseteq \Lambda \text{ for all } t \geq 0. \tag{8}$$

But Λ is a dense linear subspace of $\mathrm{C}(E)$ therefore the equation $T_t f_\lambda = f_{\psi(t,\lambda)}$ and

$$\psi(s+t,\lambda) = \psi(t,\psi(s,\lambda))$$

toghether show that

$$T_t T_s = T_{s+t} \text{ on the whole linear space } \mathrm{C}(E)$$

therefore $(T_t)_{t\geq 0}$ is a semigroup, as the $(P_t)_{t\geq 0}$ is. Now it is the time to verify

$$\lim_{t\downarrow 0} \frac{T_t f - t}{t} = \mathcal{G} f \text{ in } \mathrm{C}(E) \text{ for all } f \in \Lambda$$

therefore Λ is a core for $\mathcal{G}$ and $(T_t)_{t\geq 0}$ is the unique Feller semigroup generated by $\mathcal{G}$ on $\mathrm{C}(E)$. That is $P_t = T_t$. ■

Let us now detail the time homogeneous Lévy process $x \longmapsto X_s^x$ for an arbitrarily fixed $s > 0$ (recall we think of x as the time parameter of $x \longmapsto X_s^x$) and the measure valued Markov process $t \longmapsto H_t(\cdot;\cdot)$. Since the Lévy measure of $x \longmapsto X_s^x$ is $\frac{1}{s}\mathrm{e}_{1/s}$ (see the expression of T_t at (3)) it follows that the measure intensity of each Poisson random measure $H_t(\cdot; \mathrm{d}u \otimes \mathrm{d}y)$ is equal to $\frac{1}{t}\ell(\mathrm{d}u) \otimes \mathrm{e}_{1/t}(\mathrm{d}y)$ where ℓ stands for the Lebesgue measure on $[0,\infty[$. Taking this into account, we make precise the distribution of the whole family of random variables $(\eta(t,k), \tau(t,k))_{k\geq 1}$ at (2) and (3). Concretely they can be expressed as

$$(\eta(t,n), \tau(t,n)) = t \times (\theta(t,n), \sigma(t,n)).$$

where all the random variables $\theta(t,n), \sigma(t,n)$ are independent (for running $n \geq 1$) and identically distributed with common distribution e_1.

Now, for fixed $x, y \in]0,\infty[$, the random measure $H_t(\delta_x \otimes \delta_y; \cdot)$ has the form

$$H_t(\delta_x \otimes \delta_y; \cdot) = \delta_{t\times\xi} \otimes \delta_y, \tag{9}$$

with ξ having the form

$$\xi = \sum_{n=1}^{\rho} \xi_n \tag{10}$$

where ξ_n are i.i.d. random variables with common distribution e_1 while the random variable ρ is independent of them and has Poisson distribution with expectation y/x.

3.1 Details on the correspondence $X \leftrightarrow H$

From the point of view of the Lévy process $x \longmapsto X_t^x$, the Dirac measure $\delta_x \otimes \delta_y$ on $[0,\infty[^2$ is not so addressing as it is the path

$$[0,\infty] \ni u \longmapsto x \cdot \chi_{[y,\infty[}(u).$$

whose right-hand values $x \cdot \chi_{[y,\infty[}(u)$ are integrals like the one at (4). Anyway we are interest in the path to path Markovian evolution

$$(x \mapsto X_s^x) \longmapsto \left(x \mapsto X_{s+t}^x\right),$$

therefore we think of (9)-(10) in terms of paths. In such terms, our result is now the following.

Proposition 4 *Let $s, t, x, y \in]0, \infty[$ be fixed numbers and assume that the sample path $X_s^\bullet$ has the form*

$$u \longmapsto X_s^u = x\chi_{[y,\infty[}(u)$$

Then the path $X_{s+t}^\bullet$ has the form

$$u \longmapsto X_{s+t}^u = \left(\sum_{n=1}^{\rho} \xi_n\right) \chi_{[y,\infty[}(u)$$

where ξ_n are i.i.d. random variables with common distribution e_1 while the random variable ρ is independent of them and has Poisson distribution with expectation y/x. Furthermore the branching property of X extends this representation by additivity, with independent summands for $X_{s+t}^\bullet$. Namely if $(x_i, y_i)_{1 \leq i \leq N}$ are strictly positive real numbers, with arbitrary natural number $N \geq 2$, and if $X_s^\bullet$ has the form

$$u \longmapsto X_s^u = \sum_{1 \leq i \leq N} x_i \chi_{[y_i,\infty[}(u)$$

then the path $X_{s+t}^\bullet$ has the form

$$u \longmapsto X_{s+t}^u = \sum_{1 \leq i \leq N} \left[\left(\sum_{n=1}^{\rho_i} (\xi_i)_n\right) \chi_{[y_i,\infty[}(u)\right]$$

with independent family of random variables ρ_i and $(\xi_i)_n$ such that all the $(\xi_i)_n$ are e_1-distributed while each ρ_i is Poisson distributed with expectation x_i/y_i.

Remark It is to notice that each sum

$$\left(\sum_{n=1}^{\rho_i} (\xi_i)_n\right) \chi_{[y_i,\infty[}(u)$$

from above is null on the event $(\rho_i = 0)$.

References

[1] Kawazu, K.,Watanabe, S., *Branching peocesses with immigration and related limit theorems* Theory of Probability and its Applications **26** (1971), 36-54.