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ON MAXIMAL CHEEGER SETS

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Abstract

In this paper we consider a constrained weighted total variation minimization problem, which is motivated by landslide modeling and may be viewed as a relaxation of a generalized Cheeger problem. We prove that level sets of minimizers are generalized Cheeger sets and obtain qualitative properties of the minimizers: they are all bounded and all achieve their essential supremum on a set of positive measure. We then propose a method to find the maximal Cheeger sets and give numerical computations.

1 Introduction

We consider a landslide model proposed by Ionescu and Lachand-Robert [11] which is the following :

- the ground is represented by Ω a nonempty open bounded subset of $\mathbb{R}^d$ with a Lipschitz boundary,
- the forces applied on the ground are represented by $f \in L^\infty(\Omega)$, $f \geq f_0$ for a positive constant f_0,
- the geomaterial properties of the ground are represented by $g \in C^0(\overline{\Omega})$, $g \geq g_0$ for a positive constant g_0,

and we have to study

$$\mu := \inf_{u \in BV_0} \mathcal{R}(u) \tag{1}$$

where

$$BV_0 := \{u \in BV(\mathbb{R}^d),\ u \equiv 0 \text{ on } \mathbb{R}^d \setminus \overline{\Omega}\}, \tag{2}$$

and for $u \in BV_0$ such that $\int_\Omega fu \neq 0$,

$$\mathcal{R}(u) := \frac{\displaystyle\int_{\mathbb{R}^d} g(x)\,\mathrm{d}|Du(x)|}{\left|\displaystyle\int_\Omega f(x)u(x)\,\mathrm{d}x\right|}. \tag{3}$$

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Whenever $\int_\Omega fu = 0$, we set $\mathcal{R}(u) = +\infty$. What has been proven in [11], is that when $\mu > 1$ there is no landslide.

When $g = f = 1$ (which is not always a relevant assumption in landslides modeling), it is well-known that the infimum in (1) coincides with the infimum of $\mathcal{R}$ over characteristic functions of sets of finite perimeter. In this case, (1) appears as a natural relaxation of:

$$\lambda(\Omega) := \inf_{A\subset\overline{\Omega},\ \chi_A\in BV} \frac{\|D\chi_A\|(\mathbb{R}^d)}{|A|} \tag{4}$$

where $|A|$ and $\|D\chi_A\|(\mathbb{R}^d)$ denote respectively the Lebesgue measure of A and the total variation of $D\chi_A$. Problem (4) is famous and known as Cheeger's problem [5], its value $\lambda(\Omega)$ is called the Cheeger constant of Ω and its minimizers are called Cheeger sets of Ω (see [9], [10] and the references therein). Note also that $\lambda(\Omega)$ is the first eigenvalue of the 1-Laplacian on Ω, see for instance [7], [8].

Let us remark that the space $BV(\mathbb{R}^d)$ is the natural one to search for a minimizer of (1). Indeed the infimum is usually not achieved in a Sobolev space like $W^{1,1}(\mathbb{R}^d)$. It is also clear that one always have $\mathcal{R}(|u|) \le \mathcal{R}(u)$ so that we can restrict the minimization problem to non-negative functions.

In what follows, every $u \in BV(\Omega)$ will be extended by 0 outside $\overline{\Omega}$, and thus will also be considered as an element of $BV(\mathbb{R}^d)$, still denoted u. We reformulate (1) as the convex minimization problem

$$\mu = \inf_{u\in BV_f} \int_{\mathbb{R}^d} g(x)\,\mathrm{d}\,|Du(x)| \tag{5}$$

where

$$BV_f := \Big\{u \in BV(\mathbb{R}^d),\ u \ge 0,\ u \equiv 0 \text{ on } \mathbb{R}^d \setminus \overline{\Omega},\ \int_\Omega fu = 1\Big\}. \tag{6}$$

By analogy with the case $g = f = 1$, it is natural to consider the generalized Cheeger problem:

$$\lambda := \inf_{A\in\mathcal{E}} \frac{\displaystyle\int_{\mathbb{R}^d} g(x)\,\mathrm{d}|D\chi_A(x)|}{\displaystyle\int_A f(x)\,\mathrm{d}x} = \inf_{A\in\mathcal{E}} \mathcal{R}(\chi_A) \tag{7}$$

where

$$\mathcal{E} := \{A \subset \overline{\Omega} \quad \text{with} \quad \int_A f(x)\,\mathrm{d}x > 0 \quad \text{and} \quad \chi_A \in BV(\mathbb{R}^d)\}. \tag{8}$$

Again (1) can be interpreted as a relaxed formulation of (7) as proven in Ionescu and Lachand-Robert [11] or in [2]. Existence of minimizers for both problems (1) and (7) is easily obtained by the direct method in the Calculus of Variations (see [11] or [2]). In this paper we give an overview of the results obtained by the author in collaboration with Giuseppe Buttazzo, Guillaume Carlier and Gabriel Peyré. The first section is devoted to the link between (1) and (7) and to various properties of the solutions. Section two gives a strategy to obtain the maximal Cheeger set and the last section presents some numerical computations of the maximal Cheeger set.

2 Existence results

All the results of this section have been proven in [2].

Theorem 1. *Let Ω, f and g satisfy the previous assumptions. Then*
1) the infimum of (5) is achieved in BV_f,
2) the infimum of (7) is achieved in $\mathcal{E}$.

This result follows from the invariance property of the problem, that is :

Proposition 1. *Let $H \in W^{1,\infty}(\mathbb{R},\mathbb{R}) \cap C^\infty(\mathbb{R},\mathbb{R})$ be such that $H(0) = 0$ and $H' > 0$ on $\mathbb{R}$. If u is a solution of (5) then so is $T_H(u)$ defined by*

$$T_H(u) := \frac{H \circ u}{\displaystyle\int_\Omega f(x) H(u(x))\,\mathrm{d}x}. \tag{9}$$

In fact this property may be slightly improved as follows :

Corollary 1. *Let u be a solution of (5) and $H \in W^{1,\infty}(\mathbb{R},\mathbb{R})$ be a nondecreasing function such that $H(0) = 0$. If $H \circ u \neq 0$ then $T_H(u)$ defined by (9) also solves (5).*

This allows us in particular to apply the invariance property to $H(v) = (v - t_0)_+$ and $H(v) = \min(v, t_0)$.

Idea of the proof of the theorem :
Let us denote by $X_t(.)$ the flow of the ordinary differential equation

$$\dot{v} = -H(v).$$

In other words, for all $v \in \mathbb{R}$, $X_t(v)$ is defined by:

$$\partial_t X_t(v) = -H(X_t(v)),\ X_0(v) = v. \tag{10}$$

For $t \geq 0$, define u_t by $u_t(x) = X_t(u(x))$, it is immediate to check that $u_t \in BV_0$ and $u_t \geq 0$. Let us also define

$$h(t) := \int_{\mathbb{R}^d} g(x)\,\mathrm{d}|Du_t(x)| - \mu \int_\Omega f(x) u_t(x) dx.$$

Since u_t belongs to BV_0 and $u_t \geq 0$, we have $h(t) \geq 0$ and since $u_0 = u$ solves (5), we have $h(0) = 0$. For all $t > 0$, this yields:

$$\frac{h(t) - h(0)}{t} \geq 0. \tag{11}$$

This leads to

$$0 \geq \int_{\mathbb{R}^d} g(x)\,\mathrm{d}|D(H \circ u)(x)| - \mu \int_\Omega f(x) H(u(x))\,\mathrm{d}x, \tag{12}$$

and $H \circ u$ is solves (5) too. For more details see [2].

The invariance property (9) allows us to prove two other results :

Theorem 2. *Let u be a solution of (5) and for every $t \geq 0$, define $E_t := \{x \in \mathbb{R}^d \ : \ u(x) > t\}$. For every $t \geq 0$ such that E_t has positive Lebesgue measure $\dfrac{1}{\int_{E_t} f}\, \chi_{E_t}$ solves (5). In particular, $\dfrac{1}{\int_{\{u>0\}} f}\, \chi_{\{u>0\}}$ solves (5).*

and

Proposition 2. *Let $u \in BV_0$, $u \geq 0$. If for every $t \geq 0$ such that $E_t := \{x \in \mathbb{R}^d \ : \ u(x) > t\}$ has positive Lebesgue measure, χ_{E_t} solves (1) then u solves (1).*

Thanks to these results we are able to prove the link between problems (5) and (7)

Corollary 2. *The values of problems (5) and (7) coincide:*

$$\mu = \inf_{u \in BV_0} \mathcal{R}(u) = \lambda = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A).$$

This was already proven in Ionescu and Lachand-Robert [11]. Here we have not used the coarea formula, see [2].

Corollary 3. *$A \in \mathcal{E}$ solves (7) if and only if there exists u solving (5) such that $A = \{u > 0\}$.*

Corollary 4. *Let $(A_n)_n$ be a sequence of solutions of (7) then $\bigcup_n A_n$ is also a solution of (7).*

The novelty of the last corollary is that most of the previous results about Cheeger problem have been obtained in a convex case. For example Kawhol and Lachand-Robert have given a total classification of the Cheeger sets for the convex case of $\mathbb{R}^2$, see [10]. Here we don't have any convex assumption.

The invariance property (9) allows us to prove qualitative properties of the solutions too :

Theorem 3. *Let u be a solution of (5). Then u belongs to $L^\infty(\Omega)$.*

Theorem 4. *Let u be a solution of (5), then the set $\{u = \|u\|_\infty\}$ has positive Lebesgue measure.*

Except under special additional assumptions (for instance when $f = g = 1$ and Ω is convex, see [4]), one cannot expect Cheeger sets to be unique and examples are known where they are actually infinitely many (see for instance [9, 10]). On the other hand, the family of Cheeger sets $\mathcal{C}$ is stable by countable union (see Theorem 3 of [2]). This implies that $\mathcal{C}$ possesses a maximal element in the sense of inclusion, the *maximal Cheeger set* of Ω. Is there any strategy to obtain the maximal Cheeger set? This is the subject of the next section.

3 Maximal Cheeger set

All the results of this section have been proven in [1].

The first approach is to consider the $p-Laplacian$ problem for $p>1$ and let p tends to 1, since this problem admits an unique solution $u_p>0$. In [1], we prove that, up to a subsequence, $(u_p)_p$ converges in $L^1(\Omega)$, as $p\to 1$, to a solution u of (5). Unfortunately the solution u has no particular propriety in term of maximal Cheeger set : it is neither a characteristic function of the maximal Cheeger set, up to a multiplicative constant, nor its support is the maximal Cheeger set. Some counter examples have been given in [1]. We have to find a different approach.

The idea relies on a concave penalization of the problem. We first write problem (5) as a maximization problem

$$\sup\left\{\int_\Omega fu\,dx\ :\ u\in BV_0(\Omega),\ \int_{\mathbb{R}^d} g\,d|Du|\le 1\right\}, \tag{13}$$

and we approximate this maximization problem by the strictly concave penalization

$$\sup\left\{\int_\Omega f\big(u-\varepsilon\Phi(u)\big)\,dx\ :\ \int_{\mathbb{R}^d} g\,d|Du|\le 1,\ u\in BV_0(\Omega)\right\} \tag{14}$$

where $\varepsilon>0$ is a perturbation parameter and Φ is a strictly convex nonnegative function that satisfies:

$$\Phi(0)=0,\qquad 0\le\Phi(t)<+\infty\quad\forall t\in\mathbb{R}^+. \tag{15}$$

We recall that, from Theorem 3, the set Q of solutions of (13) is in fact included in $L^\infty(\Omega)$. We obtain

Theorem 5. *Let u_ε be the solution of (14); then the following holds:*

- *$(u_\varepsilon)_\varepsilon$ converges in $L^1(\Omega)$, as $\varepsilon\to 0^+$, to the solution $\overline{u}$ of*

$$\inf\left\{\int_\Omega f\Phi(u)\,dx\ :\ u\in Q\right\}, \tag{16}$$

- *$\overline{u}=\alpha\chi_{C_0}$ for some $\alpha>0$ and $C_0\subset\overline{\Omega}$,*
- *C_0 is the maximal Cheeger set, i.e. $C_0\in\mathcal{C}$ and C_0 contains every other Cheeger set (up to a Lebesgue negligible set).*

4 Numerical computation

The aim of this section is to give numerical computation and examples of maximal Cheeger sets in dimension 2 and 3. All the results of this section have been proven in [3].

A natural choice for the perturbation Φ is of course

$$\Phi(t):=\frac{t^2}{2}$$

in which case, the perturbed problem (14) is easily seen to be equivalent to the projection problem

$$\inf\Big\{ \int_\Omega f\Big(u-\frac{1}{\varepsilon}\Big)^2 dx \ :\ \int_\Omega g\, d|Du| + \int_{\partial\Omega} g|u| d\mathcal{H}^{d-1} \le 1,\ u\in BV(\Omega)\Big\}. \tag{17}$$

The solution of the previous problem u_ε can of course be expressed as

$$u_\varepsilon = \Pi_K\Big(\frac{1}{\varepsilon}\Big) \tag{18}$$

where Π_K denotes the projection (for the weighted L^2 inner product $(u,v) := \int_\Omega fuv$) on the closed subset K of $L^2(\Omega)$ defined by

$$K := \Big\{u\in L^2(\Omega)\cap BV(\Omega)\ :\ \int_\Omega g\, d|Du| + \int_{\partial\Omega} g|u| d\mathcal{H}^{d-1} \le 1\Big\}. \tag{19}$$

If we further assume that $g\in C^1(\overline{\Omega})$ then it is well-known that K can be described by a set of linear constraints as follows

$$K = \Big\{u\in L^2(\Omega)\ :\ \int_\Omega \operatorname{div}(gp)u \le 1,\ \forall p\in C^1(\Omega,\mathbb{R}^d),\ \|p\|_\infty \le 1\Big\}. \tag{20}$$

From now on we suppose that $f, g\in C^1(\overline{\Omega})$.

In fact we are interested in projecting $u^0\in L^2(\Omega)$ onto K i.e.

$$\inf_{u\in K} F(u),\ F(u) = \int_\Omega f(u-u^0)^2. \tag{21}$$

For sake of simplicity, we work in the case $d=2$ and $\Omega = (0,1)^2$.

Given a step size $h = 1/N$, we then consider the following discretization of (21). First, let E_h be the set of matrices u with entries $u_{i,j}$, $i,j\in\{0,N\}^2$, by convention we extend u by setting $u_{i,j}=0$ when either i or j belongs to $\{-1,N+1\}$. For $u=(u_{i,j})_{ij}\in E_h$ we set

$$\partial_x^h u_{i,j} := \begin{cases} h^{-1}(u_{i+1,j}-u_{i,j}) & \text{if } -1\le i\le N,\ -1\le j\le N-1 \\ 0 & \text{if } -1\le i\le N,\ j=N. \end{cases}$$

$$\partial_y^h u_{i,j} := \begin{cases} h^{-1}(u_{i,j+1}-u_{i,j}) & \text{if } -1\le i\le N-1,\ -1\le j\le N \\ 0, & \text{if } -1\le j\le N,\ i=N. \end{cases}$$

We also set $\nabla^h u_{i,j} = (\partial_x^h u_{i,j}, \partial_y^h u_{i,j})$. Denoting f_{ij}^h and g_{ij}^h some discrete approximation of the weights f and g (e.g. $f_{i,j}^h = f(ih,jh)$, $g_{i,j}^h = g(ih,jh)$) and $u_{i,j}^0$ some discretization of u^0 (approximation by mean values say) we then discretize $G(u) = \int_\Omega g\, d|Du| + \int_{\partial\Omega} g|u| d\mathcal{H}^{d-1}$, by definining, for all $u\in E_h$:

$$G_h(u) := h^2 \sum_{i=-1}^{N}\sum_{j=-1}^{N} g_{i,j}^h |\nabla^h u_{i,j}|.$$

Defining K_h by $K_h := \{u \in E_h \ : \ G_h(u) \leq 1\}$ and

$$F_h(u) := h^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f_{i,j}^h (u_{i,j} - u_{i,j}^0)^2,$$

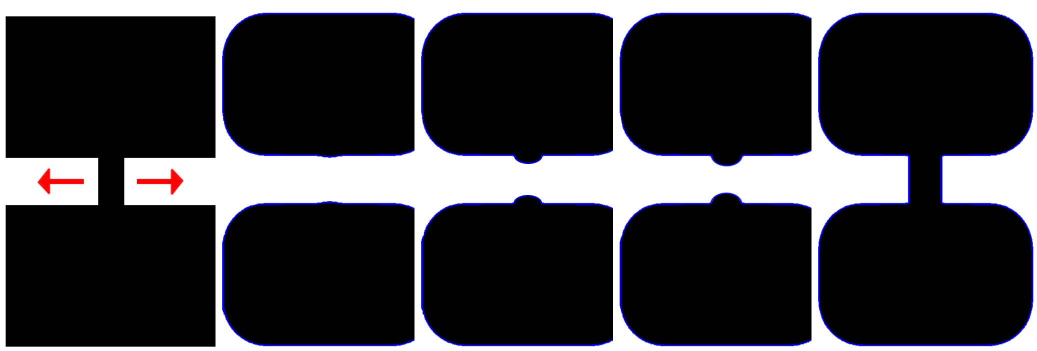
we then approximate (21) by

$$\inf_{u \in K_h} F_h(u) \tag{22}$$

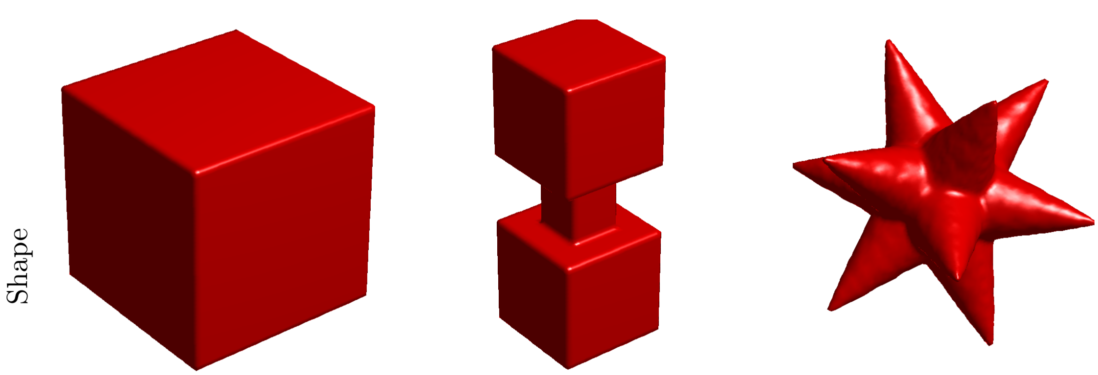
and denote by u^h the solution of (22). Denoting by C_{ij} the square $(ih,(i+1)h) \times (jh,(j+1)h)$, we define v_h as the piecewise constant function having value $u_{i,j}^h$ on C_{ij}. Denoting by $\mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ the space of bounded $\mathbb{R}^2$-valued measures on $\overline{\Omega}$, we then have the following convergence result.

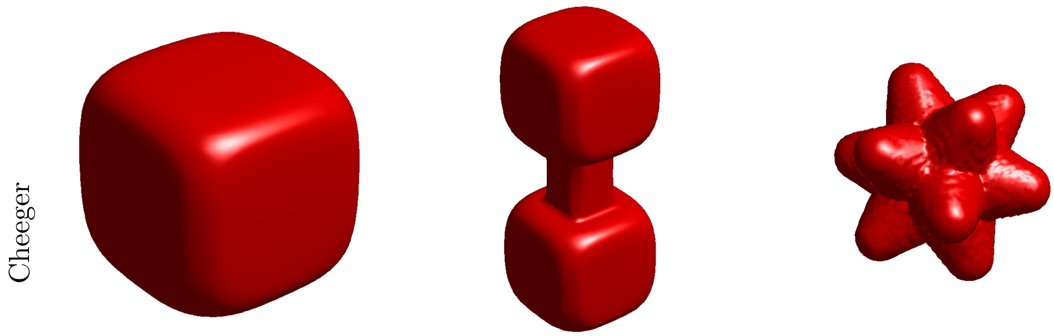
Theorem 6. *Let v_h be defined as above, then v_h converges to $\Pi_K(u_0)$ strongly in $L^2(\Omega)$ and ∇v_h converges weakly $\star$ to $\nabla \Pi_K(u_0)$ in $\mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ as $h \to 0$.*

The projection (22) of the discretized problem is computed numerically using the iterative algorithm of Combettes and Pesquet [6], see [3] for more details. Finally let us show some numerical results due to Gabriel Peyré.

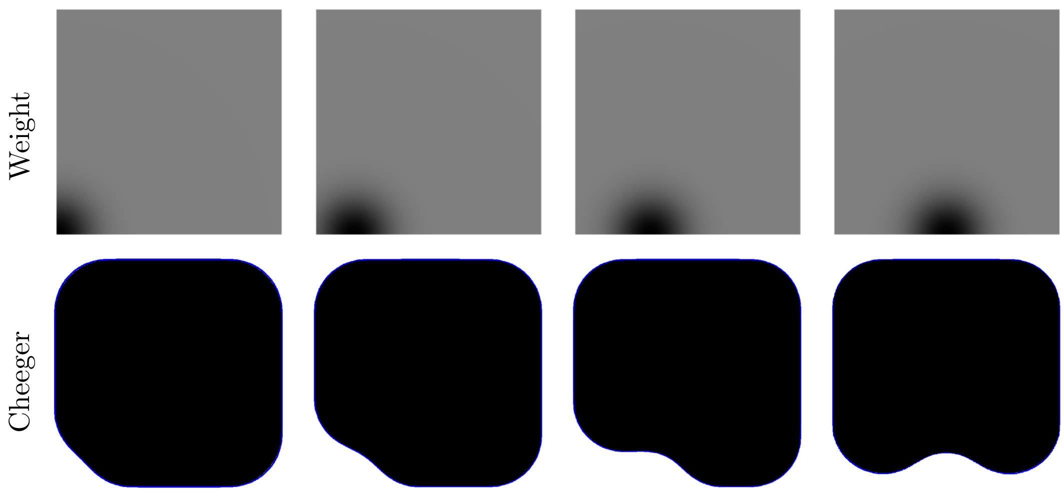


The original shape is composed of two rectangles linked with a tube of increasing width. The corresponding Cheeger sets are displayed on the right.

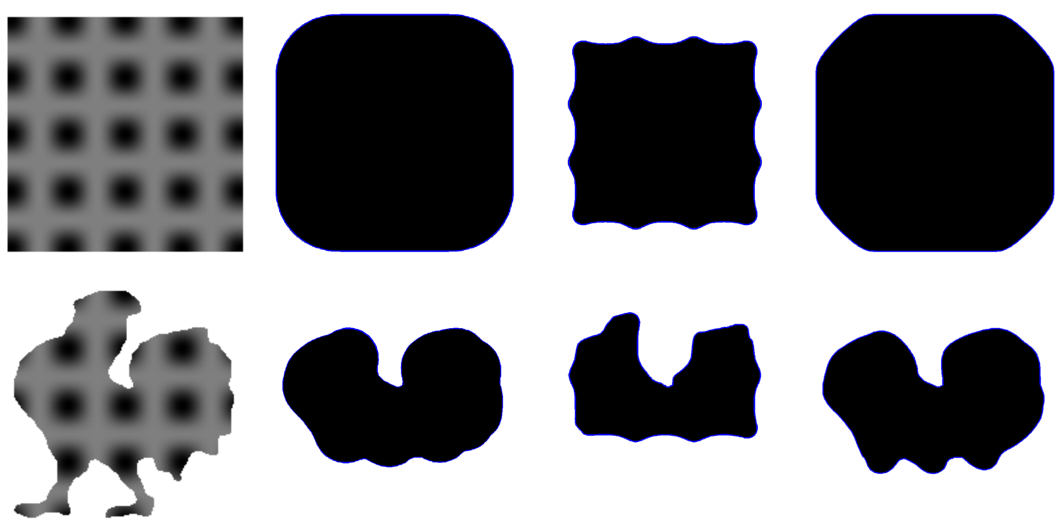




Cheeger sets in 3D with constant weights $f = g = 1$.



Cheeger sets in a 2D square with $f = 1$ and several non-constant weights g.



Weight w | $f = g = 1$ | $g = w, f = 1$ | $g = 1, f = w$

Comparison of the Cheeger with constant weights and with varying weights g or f.

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