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ON CERTAIN BOUNDARY VALUE PROBLEMS FOR SOME SECOND-ORDER DIFFERENTIAL INCLUSIONS

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Abstract

We establish Filippov type existence theorems for solutions of certain boundary value problems of some second-order differential inclusions.

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1 Introduction

This paper is concerned with differential inclusions of the form

$$\mathcal{D}x \in F(t,x), \tag{1.1}$$

where $\mathcal{D}$ is a differential operator and $F(.,.) : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map.

In the last years we observe a remarkable amount of interest in the study of existence of solutions of several boundary value problems associated to problem (1.1). Most of these existence results are obtained using fixed point techniques and are based on an integral form of the right inverse to the operator $\mathcal{D}$. This means that for every f the unique solution y of the equation $\mathcal{D}y = f$ can be written in the form $y = \mathcal{R}f$, where the operator $\mathcal{R}$ possesses nonnegative Green's function.

For a first order differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([8]) consists in proving the existence of a solution starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and the solution of the differential inclusion.

The aim of this paper is to show that Filippov's ideas can be suitably adapted in order to obtain the existence of solutions for the following problems

$$x'' - \lambda x' \in F(t,x), \quad a.e.\ (I), \quad x(0) = a_0, \quad x(1) = a_1, \tag{1.2}$$

$$x'' \in F(t,x), \quad a.e.\ (I), \quad x(0) - k_1 x'(0) = c_1, \quad x(1) + k_2 x'(1) = c_2, \tag{1.3}$$

where $I = [0,1]$, $F(.,.) : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map, $\lambda > 0$ and $a_i, c_i \in \mathbb{R}$, and $k_i \in \mathbb{R}_+$, $i = 1,2$.

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Existence results obtained using fixed point techniques for problem (1.2) may be found in [4,5] and for problem (1.3) may be found in [3,6,7]. We note that a similar results for another class of differential inclusions may be found in [2].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2 Preliminaries

Let (X, d) be a metric space. We recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $I = [0,1]$, by $C(I)$ we denote the Banach space of all continuous functions from I to $\mathbb{R}$ with the norm $||x(.)||_C = \sup_{t\in I} |x(t)|$, by AC^1 we denote the space of differentiable functions $x(.) : (0,1) \to \mathbb{R}$ whose first derivative $x'(.)$ is absolutely continuous and by L^1 we denote the Banach space of Lebesgue integrable functions $x(.) : [0,1] \to \mathbb{R}$ endowed with the norm $||u(.)||_1 = \int_0^1 |u(t)|dt$.

A function $x(.) \in AC^1$ is said to be a solution of (1.2) (resp., (1.3)) if there exists a function $v(.) \in L^1$ with $v(t) \in F(t, x(t))$, *a.e.* (I) such that $x''(t) - \lambda x' = v(t)$, *a.e.* (I) and $x(.)$ satisfies the corresponding boundary conditions.

Lemma 1. *If $v(.) : [0,1] \to \mathbb{R}$ is an integrable function then the problem*

$$x''(t) - \lambda x'(t) = v(t) \quad a.e.\ (I), \quad x(0) = a_0, \quad x(1) = a_1,$$

has a unique solution $x(.) \in AC^1$ given by

$$x(t) = P_a(t) + \int_0^1 G(t,s)v(s)ds,$$

where, if $a = (a_0, a_1) \in \mathbb{R}^2$, we denote

$$P_a(t) = \frac{1}{e^\lambda - 1}[(e^\lambda - e^{\lambda t})a_0 + (e^{\lambda t} - 1)a_1]$$

the unique solution of the problem

$$x'' - \lambda x' = 0 \quad x(0) = a_0, \quad x(1) = a_1,$$

and

$$G(t,s) = \frac{1}{e^{\lambda s}(1 - e^\lambda)} \begin{cases} (e^{\lambda t} - 1)(e^{\lambda s} - e^\lambda) & if \quad 0 \le t \le s \le 1 \\ (e^{\lambda s} - 1)(e^{\lambda t} - e^\lambda) & if \quad 0 \le s \le t \le 1 \end{cases}$$

is the Green function associated to the problem.

$$x'' - \lambda x' = 0 \quad x(0) = 0, \quad x(1) = 0.$$

The proof of Lemma 1 may be found in [4].

Note that if $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ we put $||a|| = |a_1| + |a_2|$ and

$$|P_a(t) - P_b(t)| \leq ||a - b||.$$

Denote $M := \sup_{t,s\in I} |G(t,s)|$.

Lemma 2. *If $v(.) : [0,1] \to \mathbb{R}$ is an integrable function then the problem*

$$\begin{array}{c} x''(t) = v(t) \quad a.e.\ (I) \\ x(0) - k_1 x'(0) = c_1, \\ x(1) + k_2 x'(1) = c_2, \end{array}$$

has a unique solution $x(.) \in AC^1$ given by

$$x(t) = Q_c(t) + \int_0^1 G_1(t,s)v(s)ds,$$

where if $c = (c_1, c_2) \in \mathbb{R}^2$ we denote

$$Q_c(t) = \frac{(1 - t + k_2)c_1 + (k_1 + t)c_2}{1 + k_1 + k_2}$$

and

$$G_1(t,s) = \frac{-1}{1 + k_1 + k_2} \begin{cases} (k_1 + t)(1 - s + k_2) & if \quad 0 \leq t < s \leq 1 \\ (k_1 + s)(1 - t + k_2) & if \quad 0 \leq s < t \leq 1 \end{cases}$$

is the Green function of the problem.

The proof of Lemma 1 may be found in [3].

Note that if $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ we put $||a|| = |a_1| + |a_2|$ and

$$|Q_a(t) - Q_b(t)| \leq ||a - b||.$$

On the other hand, it is well known that $\sup_{t,s\in I} |G_1(t,s)| = \frac{1+k_1+k_2}{4}$.

In what follows we impose the following conditions on F.

Hypothesis 1. (i) $F(.,.) : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ has nonempty closed values and for every $x \in \mathbb{R}$ $F(.,x)$ is measurable.

(ii) There exists $L(.) \in L^1$ such that for almost all $t \in I, F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x), F(t,y)) \leq L(t)|x - y| \quad \forall \ x, y \in \mathbb{R}.$$

3 The main result

We are now ready to prove the main result of this paper.

Theorem 1. *Assume that Hypothesis 1 is satisfied, assume that* $M||L||_1 < 1$ *and let* $y(.) \in AC^1$ *be such that there exists* $q(.) \in L^1$ *with* $d(y''(t) - \lambda y'(t), F(t, y(t))) \le q(t)$*, a.e.* (I)*. Denote* $\tilde{a}_0 = y(0)$*,* $\tilde{a}_1 = y(1)$ *and* $\tilde{a} = (\tilde{a}_0, \tilde{a}_1)$*.*

Then there exists $x(.)$ *a solution of (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \frac{1}{1 - M||L||_1}||a - \tilde{a}|| + \frac{M}{1 - M||L||_1}||q||_1. \tag{3.1}$$

Proof. The set-valued map $t \to F(t, y(t))$ is measurable with closed values and

$$F(t, y(t)) \cap \{y''(t) - \lambda y'(t) + q(t)[-1, 1]\} \ne \emptyset \quad a.e.\ (I).$$

It follows (e.g., Theorem 1.14.1 in [1]) that there exists a measurable selection $f_1(t) \in F(t, y(t))$ *a.e.* (I) such that

$$|f_1(t) - y''(t) + \lambda y'(t)| \le q(t) \quad a.e.\ (I) \tag{3.2}$$

Define $x_1(t) = P_a(t) + \int_0^1 G(t, s) f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \le ||a - \tilde{a}|| + M||q||_1.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I)$, $f_n(.) \in L^1$, $n \ge 1$ with the following properties

$$x_n(t) = P_a(t) + \int_0^1 G(t, s) f_n(s)ds, \quad t \in I, \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad a.e.\ (I),\ n \ge 1, \tag{3.4}$$

$$|f_{n+1}(t) - f_n(t)| \le L(t)|x_n(t) - x_{n-1}(t)| \quad a.e.\ (I), n \ge 1. \tag{3.5}$$

If this construction is realized then from (3.2)-(3.5) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \le \int_0^1 |G(t, t_1)|.|f_{n+1}(t_1) - f_n(t_1)|dt_1 \le$$

$$M \int_0^1 L(t_1)|x_n(t_1) - x_{n-1}(t_1)|dt_1 \le M \int_0^1 L(t_1) \int_0^1 |G(t_1, t_2)|.$$

$$|f_n(t_2) - f_{n-1}(t_2)|dt_2 \le M^2 \int_0^1 L(t_1) \int_0^1 L(t_2)|x_{n-1}(t_2) - x_{n-2}(t_2)|dt_2 dt_1$$

$$\le M^n \int_0^1 L(t_1) \int_0^1 L(t_2) ... \int_0^1 L(t_n)|x_1(t_n) - y(t_n)|dt_n ... dt_1 \le$$

$$\le (M||L||_1)^n (||a - \tilde{a}|| + M||q||_1).$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I)$, hence converging uniformly to some $x(.) \in C(I)$. Therefore, by (3.5), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbb{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.

Moreover, one has

$$\begin{aligned}&|x_n(t)-y(t)|\le|x_1(t)-y(t)|+\textstyle\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)|\le||a-\tilde{a}||\\&+M||q||_1+\textstyle\sum_{i=1}^{n-1}(||a-\tilde{a}||+M||q||_1)(M||L||_1)^i=\frac{||a-\tilde{a}||+M||q||_1}{1-M||L||_1}.\end{aligned}\tag{3.6}$$

On the other hand, from (3.2), (3.5) and (3.6) we obtain for almost all $t\in I$

$$\begin{aligned}&|f_n(t)-y''(t)+\lambda y'(t)|\le\textstyle\sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+|f_1(t)-y''(t)+\lambda y'(t)|\\&\le L(t)\frac{||a-\tilde{a}||+M||q||_1}{1-M||L||_1}+q(t).\end{aligned}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.)\in L^1$.

Using Lebesque's dominated convergence theorem and taking the limit in (3.3), (3.4) we deduce that $x(.)$ is a solution of (1.1). Finally, passing to the limit in (3.6) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.3)-(3.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N\ge 1$ we already constructed $x_n(.)\in C(I)$ and $f_n(.)\in L^1, n=1,2,...N$ satisfying (3.3),(3.5) for $n=1,2,...N$ and (3.4) for $n=1,2,...N-1$. The set-valued map $t\to F(t,x_N(t))$ is measurable. Moreover, the map $t\to L(t)|x_N(t)-x_{N-1}(t)|$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t\in I$

$$F(t,x_N(t))\cap\{f_N(t)+L(t)|x_N(t)-x_{N-1}(t)|[-1,1]\}\neq\emptyset.$$

Theorem 1.14.1 in [1] yields that there exist a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.))$ such that

$$|f_{N+1}(t)-f_N(t)|\le L(t)|x_N(t)-x_{N-1}(t)|\quad a.e.\ (I).$$

We define $x_{N+1}(.)$ as in (3.3) with $n=N+1$. Thus $f_{N+1}(.)$ satisfies (3.4) and (3.5) and the proof is complete.

Remark 1. According to Theorem 1 in [5], if the assumptions of Theorem 1 are satisfied then for any $\varepsilon>0$ there exists $x_\varepsilon(.)$ a solution of (1.2) satisfying for all $t\in I$

$$|x_\varepsilon(t)-y(t)|\le\frac{1}{1-M||L||_1}||a-\tilde{a}||+\frac{M}{1-M||L||_1}||q||_1+\varepsilon.\tag{3.7}$$

Obviously, the estimation in (3.1) is better than the one in (3.7).

We are concerned now with the boundary value problem (1.3).

Theorem 2. *Assume that Hypothesis 1 is satisfied,* $l:=\frac{1+k_1+k_2}{4}||L||_1<1$ *and let* $y(.)\in AC^1$ *be such that there exists* $q(.)\in L^1$ *with* $d(y''(t),F(t,y(t)))\le q(t)$, *a.e.* (I). *Denote* $\tilde{c}_0=y(0)-k_1y'(0)$, $\tilde{c}_1=y(1)+k_2y'(1)$ *and* $\tilde{c}=(\tilde{c}_1,\tilde{c}_2)$.

Then, there exists $x(.)$ *a solution of (1.3) satisfying for all* $t\in I$

$$|x(t)-y(t)|\le\frac{1}{1-l}||c-\tilde{c}||+\frac{1+k_1+k_2}{4(1-l)}||q||_1\qquad(3.8).$$

The proof of Theorem 2 is similar to the one of Theorem 1.

Remark 2. According to Theorem 3.1 in [6], if the assumptions of Theorem 2 are satisfied then for any $\varepsilon > 0$ there exists $x_\varepsilon(.)$ a solution of (1.3) satisfying for all $t \in I$

$$|x_\varepsilon(t) - y(t)| \leq \frac{1}{1-l}||c - \tilde{c}|| + \frac{1+k_1+k_2}{4(1-l)}||q||_1 + \varepsilon. \tag{3.9}$$

Obviously, the estimation in (3.8) is better than the one in (3.9).

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