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ON THE HOLOMORPHIC CURVATURE OF COMPLEX FINSLER HYPERSURFACES

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Abstract

Following the study of real hypersurfaces of Finsler spaces, in this paper we analyse the holomorphic hypersurfaces associated to a complex Finsler space (M, F) as holomorphic subspaces of complex codimension one. In this sense the induced complex Finsler metric, the induced nonlinear connection and, respectively, the linear connection and the equations of the holomorphic curvature are investigated. Moreover, based on the Gauss, Codazzi and Ricci equations we find the link between the holomorphic curvatures of the holomorphic hypersurface and the Finsler space (M, F), and the conditions under which the holomorphic hypersurface is totally geodesic, c-totally geodesic or generalized Einstein.

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1 Introduction

Many geometers have investigated subspaces of real or complex manifolds, the study of hypersurface being regarded as a particular case of them. Thus, a natural extension represents the study of complex Finsler subspaces. In order to analyze the geometry of hypersurfaces on a complex Finsler space, we use the basic ideas from real Finsler case from [8, 9] and we extend the study to the complex Finsler spaces by considering the complex Finsler hypersurfaces as holomorphic subspaces of complex codimension one. In this sense results from [10, 11, 12, 13, 15, 16] were used, particularized and extended.

Firstly, in Section 1, we recall some basic notions about complex Finsler geometry. By considering a holomorphic hypersurface of a complex Finsler space, we determine in the second Section the local frame along the hypersurface and the corresponding metric tensor of the induced Finsler metric. Also, by using the complex nonlinear connection of a complex Finsler, we determine the conditions

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under which a nonlinear connection of the hypersurface is induced by the Finsler space connection. In this sense we study the relation between their coefficients and also between the components of the associated adapted frames. If we take the induced nonlinear connection, we can consider three derivation rules which will determine three types of linear connection. Using the induced and the normal linear connection we can express the Gauss-Weingarten formulae of the holomorphic hypersurface, from which we deduce the Gauss, Ricci and Codazzi equations. Then, we will be able in Section 3 to give a geometric characterization of the holomorphic curvature of complex Finsler hypersurfaces and to obtain several of their properties.

Now, we make a short overview of the concepts and terminology used in complex Finsler geometry, (for more see [1, 10]). Let M be an n - dimensional complex manifold, with $z := (z^k),\ k = 1,..,n$, the complex coordinates on a local chart (U, φ). The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is in its turn a $2n$-dimensional complex manifold and the local coordinates in a local chart in $u \in T'M$ are $u := (z^k, \eta^k),\ k = 1,..,n$.

Definition 1. *A* complex Finsler space *is a pair* (M, F)*, with* $F : T'M \to \mathbb{R}^+,\ F = F(z, \eta)$ *a continuous function that satisfies the following conditions:*

i. F *is a smooth function on* $\widetilde{T'M} := T'M \setminus \{0\}$*;*

ii. $F(z, \eta) \geq 0$*, the equality holds if and only if* $\eta = 0$*;*

iii. $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, $\forall \lambda \in \mathbb{C}$*;*

iv. the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$ *is positive definite, where* $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ *is the fundamental metric tensor, with* $L := F^2$ *the complex Lagrangian associated to the complex Finsler function* F*.*

The positivity from the fourth condition is equivalent to the convexity of L and to the strongly pseudoconvex property of the complex indicatrix in a fixed point $I_z M = \{\eta \mid g_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1\}$, for any $z \in M$. Also, it ensures the existence of the inverse $(g^{\bar{j}i})$, with $g^{\bar{j}i} g_{i\bar{k}} = \delta^{\bar{j}}_{\bar{k}}$.

Moreover, condition iii. represents the fact that L is homogeneous with respect to the complex norm, $L(z, \lambda\eta) = \lambda\bar{\lambda} L(z, \eta),\ \forall \lambda \in \mathbb{C}$, and by applying Euler's formula we get that:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i \bar{\eta}^j. \tag{1}$$

The geometry of complex Finsler spaces consists of the study of geometric objects on the complex manifold $T'M$ endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$. A first step represents the analysis of the sections on the complexified tangent bundle $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$, where $T''_u(T'M) = \overline{T'_u(T'M)}$.

Let $V(T'M) = span\{\frac{\partial}{\partial \eta^k}\} \subset T'(T'M)$ be the vertical bundle and the complex non-linear connection, briefly (c.n.c.), is the supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of a (c.n.c.). Then, we will call the adapted frame of the (c.n.c.) the pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$, with the dual adapted base $\{\mathrm{d}z^k, \delta\eta^k := \mathrm{d}\eta^k + N_j^k \mathrm{d}z^j\}$.

Since $g_{i\bar{j}}$ is a nondegenerate $d-$complex tensor, by considering a fixed (c.n.c.) N we introduce a metric structure G on $T'M$, also named its N-Sasaki lift, as

$$G = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j. \tag{2}$$

One basic (c.n.c.) of a complex Finsler space is the Chern-Finsler (c.n.c.) ([1],[10]), with $N_j^k = g^{\bar{m}k} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l$, and determines the *Chern-Finsler linear connection*, locally given by the next set of coefficients ([10]) $L_{jk}^i = g^{\bar{l}i} \delta_k(g_{j\bar{l}})$, $C_{jk}^i = g^{\bar{l}i} \dot{\partial}_k(g_{j\bar{l}})$, $L_{\bar{j}k}^{\bar{i}} = 0$, $C_{\bar{j}k}^{\bar{i}} = 0$, where $D_{\delta_k}\delta_j = L_{jk}^i \delta_i$, $D_{\delta_k}\dot{\partial}_j = L_{jk}^i \dot{\partial}_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C_{jk}^i \dot{\partial}_i$, $D_{\dot{\partial}_k}\delta_j = C_{jk}^i \delta_i$ and $C_{jk}^i \eta^j = C_{jk}^i \eta^k = 0$ from (1). Since N_k^i is $(1,0)$-homogeneous, i.e. $(\dot{\partial}_k N_j^i)\eta^k = N_j^i$ and $(\dot{\partial}_{\bar{k}} N_j^i)\eta^{\bar{k}} = 0$ ([4]), it takes place $\eta^j L_{jk}^i = N_k^i$ and $L_{jk}^i = \dot{\partial}_j N_k^i$. Further we will use the following notation $\bar{\eta}^j =: \eta^{\bar{j}}$ to denote a conjugate object.

According to [10], for Chern-Finsler connection we have the following nonzero curvature coefficients

$$T_{jk}^i = L_{jk}^i - L_{kj}^i \; ; \quad Q_{jk}^i = C_{jk}^i \; ; \quad \Theta_{j\bar{k}}^i = \delta_{\bar{k}} N_j^i \; ; \quad \rho_{j\bar{k}}^i = \dot{\partial}_{\bar{k}} N_j^i. \tag{3}$$

where $hT(\delta_k, \delta_j) = T_{jk}^i \delta_i$, $hT(\dot{\partial}_k, \delta_j) = Q_{jk}^i \delta_i$, $vT(\delta_{\bar{k}}, \delta_j) = \Theta_{j\bar{k}}^i \dot{\partial}_i$, $vT(\dot{\partial}_{\bar{k}}, \delta_j) = \rho_{j\bar{k}}^i \dot{\partial}_i$, and the following nonzero curvature coefficients

$$\begin{aligned} R_{j\bar{k}h}^i &= -\,\delta_{\bar{k}} L_{jh}^i - \delta_{\bar{k}}(N_h^l) C_{jl}^i \; ; \qquad P_{j\bar{k}h}^i = -\delta_{\bar{k}} C_{jh}^i; \\ Q_{j\bar{k}h}^i &= -\,\dot{\partial}_{\bar{k}} L_{jh}^i - \dot{\partial}_{\bar{k}}(N_h^l) C_{jl}^i \; ; \qquad S_{j\bar{k}h}^i = -\dot{\partial}_{\bar{k}} C_{jh}^i. \end{aligned} \tag{4}$$

In [1]'s terminology, the complex Finsler space (M,F) is *strongly Kähler* iff $T_{jk}^i = 0$, *Kähler* iff $T_{jk}^i \eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}} T_{jk}^i \eta^j \bar{\eta}^l = 0$, where $T_{jk}^i := L_{jk}^i - L_{kj}^i$, but, cf. [7] strongly Kähler notion coincides with the Kähler notion.

2 The equations of holomorphic hypersurfaces

Let (M,F) be a complex Finsler space, $(z^k, \eta^k)_{k=\overline{1,n}}$ complex coordinates in a local chart, and a complex immersed hypersurface $\tilde{M}$ of M such that $\mathbf{i} : \tilde{M} \hookrightarrow M$ a holomorphic immersion locally given by $z^k = z^k(\xi^1, ..., \xi^{n-1})$, $k = 1, ..., n$. The complexified tangent map $\mathbf{i}_*^C : T'\tilde{M} \to T'M$ is defined by $\mathbf{i}_*^C(\xi,\theta) = (\mathbf{i}(\xi), \mathbf{i}_{*,\xi}^C \theta) = (z(\xi), \eta(\xi,\theta))$ and has the following local representation

$$z^k = z^k(\xi^1, .., \xi^{n-1}), \quad \eta^k = B_\alpha^k \theta^\alpha, \quad \text{where } B_\alpha^k(\xi) = \frac{\partial z^k}{\partial \xi^\alpha}. \tag{5}$$

From now on, the Latin indices $i, j, k, \ldots$ run from 1 to n and the Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to $n-1$.

The holomorphic immersion assumption implies that $B^k_{\bar{\alpha}} = \frac{\partial z^k}{\partial \bar{\xi}^\alpha} = 0$, $B^{\bar{k}}_{\alpha} = \frac{\partial \bar{z}^k}{\partial \xi^\alpha} = 0$, and in any point of the complexified tangent space $T_C(T'\tilde{M})$ the local frame $\{\frac{\partial}{\partial \xi^\alpha}, \frac{\partial}{\partial \theta^\alpha}\}$ is coupled to $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$ as follows

$$\frac{\partial}{\partial \xi^\alpha} = B^k_\alpha \frac{\partial}{\partial z^k} + B^k_{0\alpha} \frac{\partial}{\partial \eta^k}, \qquad \frac{\partial}{\partial \theta^\alpha} = B^k_\alpha \frac{\partial}{\partial \eta^k}, \tag{6}$$

where $B^k_{0\alpha} = \frac{\partial B^k_\alpha}{\partial \xi^\beta} \theta^\beta$. Its dual basis satisfies the conditions

$$dz^k = B^k_\alpha d\xi^\alpha, \quad d\eta^k = B^k_{0\alpha} d\xi^\alpha + B^k_\alpha d\theta^\alpha, \tag{7}$$

and their corresponding conjugates. Further, we use the notations $\partial_\alpha = \frac{\partial}{\partial \xi^\alpha}$, $\dot{\partial}_\alpha = \frac{\partial}{\partial \theta^\alpha}$, $\partial_{\bar{\alpha}} = \frac{\partial}{\partial \xi^{\bar{\alpha}}}$ and $\dot{\partial}_{\bar{\alpha}} = \frac{\partial}{\partial \theta^{\bar{\alpha}}}$.

In view of (5), the complex Finsler function F with metric tensor $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, induces a complex Finsler function $\tilde{F} : T'\tilde{M} \to \mathbb{R}^+$, $\tilde{F}(\xi, \theta) = F(z(\xi), \eta(\xi, \theta)) = F(z^k(\xi), \theta^\alpha B^k_\alpha)$, of associated complex Lagrangian $\tilde{L} := \tilde{F}^2$. By using (6), we can relate $g_{i\bar{j}}$ to the induced metric tensor $g_{\alpha\bar{\beta}} = \frac{\partial^2 \tilde{L}}{\partial \theta^\alpha \partial \bar{\theta}^\beta}$ as

$$g_{\alpha\bar{\beta}} = B^j_\alpha B^{\bar{k}}_{\bar{\beta}} g_{j\bar{k}}. \tag{8}$$

Then, $rank(g_{\alpha\bar{\beta}}) = n-1$ and it induces a metric structure on $\tilde{M}$ determined by $\tilde{L}$. Thus, we obtain an $(n-1)$-dimensional complex Finsler space $(\tilde{M}, \tilde{F})$, also called the *holomorphic hypersurface* of the complex Finsler space (M, F).

From (6) it is deduced that $V(T'\tilde{M}) = span\{\dot{\partial}_\alpha\}$ is a subdistribution of the vertical distribution $V(T'M)$. Thus, in any point (ξ^α) of $\tilde{M}$, we can define a *unit normal vector* $N = N^i(\xi, \theta)\frac{\partial}{\partial \eta^i}$ of $T_C(T'M)$ as

$$g_{i\bar{j}}(\tilde{u}) B^i_\alpha N^{\bar{j}} = 0, \quad g_{i\bar{j}}(\tilde{u}) N^i B^{\bar{j}}_{\bar{\alpha}} = 0, \quad g_{i\bar{j}}(\tilde{u}) N^i N^{\bar{j}} = 1. \tag{9}$$

This normal vector generates an orthogonal complement $VT'\tilde{M}^\perp$ in any point $\tilde{u}$, such that $VT'M = VT'\tilde{M} \oplus VT'\tilde{M}^\perp$.

Let us now consider in any $z(\xi) \in M$ the moving frame $\mathcal{R} = \{B^k_\alpha(\xi), N^k(\xi, \theta)\}$ along the complex Finsler hypersurface $(\tilde{M}, \tilde{F})$ and let $\mathcal{R}^{-1} = (B^\alpha_k \; N_k)^t$ be the inverse matrix associated to $\mathcal{R} = (B^k_\alpha \; N^k)$. Obviously, B^α_k and N_k are functions of z, θ and

$$B^i_\alpha B^\beta_i = \delta^\alpha_\beta, \quad B^i_\alpha N_i = 0, \quad N^i B^\alpha_i = 0, \quad N^i N_i = 1, \quad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j, \tag{10}$$

takes place. Moreover, $g_{k\bar{j}} N^{\bar{j}} = g_{i\bar{j}} N^i N^{\bar{j}} N_k = N_k$, $N^{\bar{j}} = g^{\bar{j}k} N_k$ occurs and thus $g^{\bar{j}k} N_k B^{\bar{\beta}}_{\bar{j}} = 0$. The last relation together with (9) and (10), leads to

$$g^{\bar{\beta}\alpha} = g^{\bar{j}i} B^\alpha_i B^{\bar{\beta}}_{\bar{j}}. \tag{11}$$

Therefore, using the inverse matrix $(g^{\bar{\beta}\alpha})$ of $(g_{\alpha\bar{\beta}})$, we get

$$B_i^\alpha = g^{\bar{\beta}\alpha} g_{i\bar{j}} B_{\bar{\beta}}^{\bar{j}}, \quad N_k = g_{k\bar{j}} N^{\bar{j}}, \quad N^{\bar{j}} = g^{\bar{j}k} N_k. \tag{12}$$

In addition, from (11) and the last relation from (12), it is deduced

$$g^{\bar{j}i} = B_\alpha^i B_{\bar{\beta}}^{\bar{j}} g^{\bar{\beta}\alpha} + N^i N^{\bar{j}}. \tag{11'}$$

In a similar manner, using (12), we determine an equivalent form for (8), as

$$g_{i\bar{j}} = \tilde{g}_{i\bar{j}} + N_i N_{\bar{j}}, \quad \text{unde} \quad \tilde{g}_{i\bar{j}} = B_i^\alpha B_{\bar{j}}^{\bar{\beta}} g_{\alpha\bar{\beta}}. \tag{8'}$$

Further, the next important step in our research is to obtain the induced complex nonlinear connection. A (c.n.c) $\tilde{N}$ on $T'\tilde{M}$ is said to be *induced* by the (c.n.c.) N on $T'M$ if $\delta\theta^\alpha = B_k^\alpha \delta\eta^k$. This condition implies [10]

$$\tilde{N}_\beta^\alpha = B_k^\alpha (B_{0\beta}^k + N_j^k B_\beta^j).$$

Then, the adapted bases are related by

$$dz^k = B_\alpha^k d\xi^\alpha; \qquad \delta\eta^k = B_\alpha^k \delta\theta^\alpha + N^k H_\alpha d\xi^\alpha; \tag{13}$$

$$\frac{\delta}{\delta\xi^\alpha} = B_\alpha^k \frac{\delta}{\delta z^k} + N^k H_\alpha \frac{\partial}{\partial\eta^k}; \qquad \frac{\partial}{\partial\theta^\alpha} = B_\alpha^k \frac{\partial}{\partial\eta^k}, \tag{14}$$

and their corresponding conjugates, where $H_\alpha = N_j(B_{0\alpha}^j + N_k^j B_\alpha^k)$.

A notable result, obtained just like in [10], Theorem 5.4.1, asserts that the induced (c.n.c.) of the Chern-Finsler (c.n.c.) on $\tilde{M}$ is given by $\tilde{N}_\alpha^\beta = g^{\bar{\gamma}\beta} \frac{\partial^2 \tilde{L}}{\partial\xi^\alpha \partial\bar{\theta}^\gamma}$ and coincides with the intrinsic (c.n.c.) of the $\tilde{M}$ hypersurface.

In order to study the induced linear connection on $\tilde{M}$, as in [10], we consider three types of derivative rules. Firstly, we consider the *coupling connection* $D\Gamma(\tilde{N})$ of N-complex linear connection D on (M, F), defined for any tangent vector $X^i = B_\alpha^i X^\alpha$ by $\tilde{D}X^i = DX^i$, and which for the Chern-Finsler (c.l.c.) case is given by

$$L_{j\alpha}^i = g^{\bar{m}i} \delta_\alpha(g_{j\bar{m}}), \quad C_{j\alpha}^i = g^{\bar{m}i} \dot{\partial}_\alpha(g_{j\bar{m}}), \quad L_{\bar{j}\alpha}^{\bar{i}} = C_{\bar{j}\alpha}^{\bar{i}} = 0. \tag{15}$$

The second type of connection is the *induced tangent connection* $\tilde{D}\Gamma(\tilde{N})$ defined by $\tilde{D}X^\alpha = B_i^\alpha DX^i$ for any tangent vertical field $X^i = B_\alpha^i X^\alpha$. The coefficients of the Chern-Finsler induced tangent connection coincide with the intrinsic Chern-Finsler connection of the holomorphic hypersurface $\tilde{M}$ and are given by

$$\tilde{L}_{\beta\gamma}^\alpha = g^{\bar{\delta}\alpha} \delta_\gamma(g_{\beta\bar{\delta}}), \qquad \tilde{C}_{\beta\gamma}^\alpha = g^{\bar{\delta}\alpha} \dot{\partial}_\gamma(g_{\beta\bar{\delta}}), \qquad \tilde{L}_{\bar{\beta}\gamma}^{\bar{\alpha}} = \tilde{C}_{\bar{\beta}\gamma}^{\bar{\alpha}} = 0. \tag{16}$$

The last type of connection is the *induced normal connection* $D^\perp\Gamma(\tilde{N})$ defined by $D^\perp \underline{X} = N_i DX^i$, for any normal vector field $X^i = N^i \underline{X}$ and if it comes from a Chern-Finsler connection it has the coefficients

$$\begin{aligned} L_\alpha &= N_i(\delta_\alpha N^i + N^j g^{\bar{m}i} \delta_\alpha g_{j\bar{m}}), & L_{\bar{\alpha}} &= N_i \delta_{\bar{\alpha}} N^i, \\ C_\alpha &= N_i(\dot{\partial}_\alpha N^i + N^j g^{\bar{m}i} \dot{\partial}_\alpha g_{j\bar{m}}), & C_{\bar{\alpha}} &= N_i \dot{\partial}_{\bar{\alpha}} N^i. \end{aligned} \tag{17}$$

Further, by using the induced tangent connection $\tilde{D}\Gamma(\tilde{N})$ we introduce the Gauss-Weingarten formulae of the holomorphic hypersurface and with their help we deduce the Gauss, Ricci and Codazzi equation. Thus, the following decomposition takes place

$$\begin{aligned} D_X Y &= \tilde{D}_X Y + H(X,Y), \quad \forall X \in \Gamma(T_C T'\tilde{M}),\ Y \in \Gamma(V_C T'\tilde{M}), \\ D_X W &= -A_W X + D^{\perp}_X W, \quad \forall X \in \Gamma(T_C T'\tilde{M}),\ W \in \Gamma(V_C T'\tilde{M}^{\perp}) \end{aligned} \tag{18}$$

known as the *Gauss-Weingarten formulae*, where $H(X,Y) \in \Gamma(V_C T'\tilde{M}^{\perp})$ is the second fundamental form and $A_W X \in \Gamma(V_C T'\tilde{M})$ represents the shape operator (or Weingarten operator) of the holomorphic hypersurface. With respect to the adapted frame of a (c.n.c.), $\{\delta_\alpha, \dot{\partial}_\alpha, \delta_{\bar{\alpha}}, \dot{\partial}_{\bar{\alpha}}\}$, the second fundamental form H and the shape operator A are well defined by the next set of coefficients

$$\begin{gathered} H(\delta_\beta, \dot{\partial}_\alpha) = H_{\alpha\beta} N, \quad H(\delta_{\bar{\beta}}, \dot{\partial}_\alpha) = H_{\alpha\bar{\beta}} N, \\ H(\dot{\partial}_\beta, \dot{\partial}_\alpha) = K_{\alpha\beta} N, \quad H(\dot{\partial}_{\bar{\beta}}, \dot{\partial}_\alpha) = K_{\alpha\bar{\beta}} N, \\ A(\delta_\beta) = A^\alpha_\beta \dot{\partial}_\alpha, \qquad A(\delta_{\bar{\beta}}) = A^\alpha_{\bar{\beta}} \dot{\partial}_\alpha, \qquad A(\dot{\partial}_\beta) = V^\alpha_\beta \dot{\partial}_\alpha, \qquad A(\dot{\partial}_{\bar{\beta}}) = V^\alpha_{\bar{\beta}} \dot{\partial}_\alpha, \end{gathered}$$

which are given by

$$\begin{aligned} H_{\alpha\beta} &= N_i (B^i_{\alpha\beta} + B^j_\alpha L^i_{j\beta}), & H_{\alpha\bar{\beta}} &= N_i B^j_\alpha L^i_{j\bar{\beta}}, \\ K_{\alpha\beta} &= N_i B^j_\alpha C^i_{j\beta}, & K_{\alpha\bar{\beta}} &= N_i B^j_\alpha C^i_{j\bar{\beta}}, \\ A^\alpha_\beta &= -B^\alpha_i (\delta_\beta N^i + N^j L^i_{j\beta}), & A^\alpha_{\bar{\beta}} &= -B^\alpha_i (\delta_{\bar{\beta}} N^i + N^j L^i_{j\bar{\beta}}), \\ V^\alpha_\beta &= -B^\alpha_i (\dot{\partial}_\beta N^i + N^j C^i_{j\beta}), & V^\alpha_{\bar{\beta}} &= -B^\alpha_i (\dot{\partial}_{\bar{\beta}} N^i + N^j C^i_{j\bar{\beta}}). \end{aligned} \tag{19}$$

In particular, for the Chern-Finsler connection case, using (15), they are as follows

$$\begin{aligned} & H_{\alpha\beta} = N_i (B^i_{\alpha\beta} + B^j_\alpha B^k_\beta L^i_{jk} + B^j_\alpha H_\beta N^k C^i_{jk}), \\ & K_{\alpha\beta} = N_i B^j_\alpha B^k_\beta C^i_{jk}, \qquad H_{\alpha\bar{\beta}} = K_{\alpha\bar{\beta}} = 0, \\ & A^\alpha_{\bar{\beta}} = -B^\alpha_i \delta_{\bar{\beta}} N^i, \qquad V^\alpha_{\bar{\beta}} = -B^\alpha_i \dot{\partial}_{\bar{\beta}} N^i, \qquad A^\alpha_\beta = V^\alpha_\beta = 0. \end{aligned}$$

Since for any metric connection the second fundamental form and the shape operator verify $G(H(X, \dot{\partial}_\alpha), \bar{N}) = G(A_{\bar{N}} X, \dot{\partial}_\alpha)$, we have the following relations

$$\begin{aligned} H_{\alpha\beta} &= g_{\alpha\bar{\gamma}} A^{\bar{\gamma}}_\beta, & H_{\alpha\bar{\beta}} &= g_{\alpha\bar{\gamma}} A^{\bar{\gamma}}_{\bar{\beta}}, \\ K_{\alpha\beta} &= g_{\alpha\bar{\gamma}} V^{\bar{\gamma}}_\beta, & K_{\alpha\bar{\beta}} &= g_{\alpha\bar{\gamma}} V^{\bar{\gamma}}_{\bar{\beta}}. \end{aligned} \tag{20}$$

Then, by considering $\tilde{D}$ and $D^{\perp}$ the induced and normal connection on the holomorphic hypersurface $\tilde{M}$ of a (c.l.c.) connection D and by applying the Gauss-Weingarten formulae, we can obtain, as in [5, 6, 12], the link between curvatures $R(X,Y)Z$ of D and $\tilde{R}(X,Y)Z$ of $\tilde{D}$, known as the *Gauss, H-Codazzi, A-Codazzi* *and Ricci equation* of the holomorphic hypersurface:

$$\begin{aligned}
G\left(R(X,Y)Z,U\right) &= \tilde{G}(\tilde{R}(X,Y)Z,U) + \tilde{G}\left(A_{H(X,Z)}Y - A_{H(Y,Z)}X, U\right),\\
G\left(R(X,Y)Z,\bar{N}\right) &= G\left(\left(D_X H\right)(Y,Z) - \left(D_Y H\right)(X,Z), \bar{N}\right)\\
&\quad + G(H(\tilde{T}(X,Y),Z),\bar{N}),\\
G\left(R(X,Y)N,Z\right) &= \tilde{G}\left(\left(D_Y A\right)(N,X) - \left(D_X A\right)(N,Y), Z\right)\\
&\quad - \tilde{G}(A_N(\tilde{T}(X,Y)),Z),\\
G\left(R(X,Y)N,\bar{N}\right) &= G\left(R^{\perp}(X,Y)N,\bar{N}\right)\\
&\quad + G\left(H\left(Y, A_N X\right) - H\left(X, A_N Y\right), \bar{N}\right),
\end{aligned}$$

for any vector fields X, Y, Z, U tangent to $\tilde{M}$, where $\tilde{T}$ is the torsion of the induced tangent connection $\tilde{D}$ and $R^{\perp}$ is the curvature form relative to the normal Finsler connection $D^{\perp}$.

If we consider that the induced Chern-Finsler connection coincides with the intrinsic Chern-Finsler connection, we can calculate its nonzero curvature coefficients as having the same form as in (3), but in Greek indices, and we obtain

$$\begin{aligned}
T^{\alpha}_{\beta\gamma} &= B^{\alpha}_{i}B^{j}_{\beta}B^{k}_{\gamma}T^{i}_{jk} + B^{\alpha}_{i}N^{k}(B^{j}_{\beta}H_{\gamma} - B^{j}_{\gamma}H_{\beta})Q^{i}_{jk},\\
Q^{\alpha}_{\beta\gamma} &= B^{\alpha}_{i}B^{j}_{\beta}B^{k}_{\gamma}Q^{i}_{jk},\\
\Theta^{\alpha}_{\beta\bar{\gamma}} &= A^{\alpha}_{\bar{\gamma}}H_{\beta} + B^{\alpha}_{i}B^{j}_{\beta}B^{\bar{k}}_{\bar{\gamma}}\Theta^{i}_{j\bar{k}} + B^{\alpha}_{i}B^{j}_{\beta}N^{\bar{k}}H_{\bar{\gamma}}\rho^{i}_{j\bar{k}},\\
\rho^{\alpha}_{\beta\bar{\gamma}} &= C^{\alpha}_{\bar{\gamma}}H_{\beta} + B^{\alpha}_{i}B^{j}_{\beta}B^{\bar{k}}_{\bar{\gamma}}\rho^{i}_{j\bar{k}}.
\end{aligned} \tag{21}$$

In the same manner, using (4) we can obtain the nonzero curvature coefficients of the induced Chern-Finsler connection as

$$\begin{aligned}
R^{i}_{j\bar{\beta}\gamma} &= B^{m}_{\gamma}B^{\bar{k}}_{\bar{\beta}}R^{i}_{j\bar{k}m} + B^{m}_{\gamma}N^{\bar{k}}H_{\bar{\beta}}Q^{i}_{j\bar{k}m} + N^{m}H_{\gamma}B^{\bar{k}}_{\bar{\beta}}P^{i}_{j\bar{k}m} + N^{m}N^{\bar{k}}H_{\gamma}H_{\bar{\beta}}S^{i}_{j\bar{k}m},\\
Q^{i}_{j\bar{\beta}\gamma} &= B^{m}_{\gamma}B^{\bar{k}}_{\bar{\beta}}Q^{i}_{j\bar{k}m} + N^{m}H_{\gamma}B^{\bar{k}}_{\bar{\beta}}S^{i}_{j\bar{k}m},\\
P^{i}_{j\bar{\beta}\gamma} &= B^{m}_{\gamma}B^{\bar{k}}_{\bar{\beta}}P^{i}_{j\bar{k}m} + B^{m}_{\gamma}N^{\bar{k}}H_{\bar{\beta}}S^{i}_{j\bar{k}m},\\
S^{i}_{j\bar{\beta}\gamma} &= B^{m}_{\gamma}B^{\bar{k}}_{\bar{\beta}}S^{i}_{j\bar{k}m},
\end{aligned} \tag{22}$$

Thus, we can state

Theorem 1. *Regarding the induced Chern-Finsler connection tangent to the holomorphic hypersurface* $(\tilde{M}, \tilde{F})$ *by the Chern-Finsler connection, we have:*

i) *the Gauss equations*

$$B^{j}_{\alpha}B^{\mu}_{i}R^{i}_{j\bar{\beta}\gamma} = \tilde{R}^{\mu}_{\alpha\bar{\beta}\gamma} + H_{\alpha\gamma}A^{\mu}_{\bar{\beta}}, \qquad B^{j}_{\alpha}B^{\mu}_{i}Q^{i}_{j\bar{\beta}\gamma} = \tilde{Q}^{\mu}_{\alpha\bar{\beta}\gamma} + H_{\alpha\gamma}V^{\mu}_{\bar{\beta}},$$
$$B^{j}_{\alpha}B^{\mu}_{i}P^{i}_{j\bar{\beta}\gamma} = \tilde{P}^{\mu}_{\alpha\bar{\beta}\gamma} + K_{\alpha\gamma}A^{\mu}_{\bar{\beta}}, \qquad B^{j}_{\alpha}B^{\mu}_{i}S^{i}_{j\bar{\beta}\gamma} = \tilde{S}^{\mu}_{\alpha\bar{\beta}\gamma} + K_{\alpha\gamma}V^{\mu}_{\bar{\beta}};$$

ii) *the H-Codazzi equations*

$$B^j_\alpha N_i R^i_{j\bar\beta\gamma} = -[H_{\alpha\gamma\,|\bar\beta} + H_{\alpha\gamma}N_iN^i{}_{|\bar\beta}] - \Theta^\mu_{\gamma\bar\beta}K_{\alpha\mu},$$
$$B^j_\alpha N_i Q^i_{j\bar\beta\gamma} = -[H_{\alpha\gamma}|_{\bar\beta} + H_{\alpha\gamma}N_iN^i|_{\bar\beta}] - \rho^\mu_{\gamma\bar\beta}K_{\alpha\mu},$$
$$B^j_\alpha N_i P^i_{j\bar\beta\gamma} = -[K_{\alpha\gamma\,|\bar\beta} + K_{\alpha\gamma}N_iN^i{}_{|\bar\beta}],$$
$$B^j_\alpha N_i S^i_{j\bar\beta\gamma} = -[K_{\alpha\gamma}|_{\bar\beta} + K_{\alpha\gamma}N_iN^i|_{\bar\beta}];$$

iii) *the A-Codazzi equations*

$$N^jB^\alpha_iR^i_{j\bar\beta\gamma} = -A^\alpha_{\bar\beta\,|\gamma} + \overset{CF}{L_\gamma}A^\alpha_{\bar\beta}, \qquad N^jB^\alpha_iQ^i_{j\bar\beta\gamma} = -V^\alpha_{\bar\beta\,|\gamma} + \overset{CF}{L_\gamma}V^\alpha_{\bar\beta},$$
$$N^jB^\alpha_iR^i_{j\bar\beta\gamma} = -A^\alpha_{\bar\beta}|_\gamma + \overset{CF}{C_\gamma}A^\alpha_{\bar\beta}, \qquad N^jB^\alpha_iS^i_{j\bar\beta\gamma} = -V^\alpha_{\bar\beta}|_\gamma + \overset{CF}{C_\gamma}V^\alpha_{\bar\beta};$$

iv) *the Ricci equations*

$$N^jN_iR^i_{j\bar\beta\gamma} = R^\perp_{\bar\beta\gamma} - A^\alpha_{\bar\beta}H_{\alpha\gamma}, \qquad N^jN_iQ^i_{j\bar\beta\gamma} = Q^\perp_{\bar\beta\gamma} - V^\alpha_{\bar\beta}H_{\alpha\gamma},$$
$$N^jN_iP^i_{j\bar\beta\gamma} = P^\perp_{\bar\beta\gamma} - A^\alpha_{\bar\beta}K_{\alpha\gamma}, \qquad N^jN_iS^i_{j\bar\beta\gamma} = S^\perp_{\bar\beta\gamma} - V^\alpha_{\bar\beta}K_{\alpha\gamma},$$

where

$$R^\perp_{\bar\beta\gamma} = \delta_\gamma L_{\bar\beta} - \delta_{\bar\beta}L_\gamma - \Theta^\alpha_{\gamma\bar\beta}C_\alpha + \overline{\Theta^\alpha_{\beta\bar\gamma}}C_{\bar\alpha},$$
$$Q^\perp_{\bar\beta\gamma} = \delta_\gamma C_{\bar\beta} - \dot\partial_{\bar\beta}L_\gamma - \rho^\alpha_{\gamma\bar\beta}C_\alpha,$$
$$P^\perp_{\bar\beta\gamma} = \dot\partial_\gamma L_{\bar\beta} - \delta_{\bar\beta}C_\gamma + \overline{\rho^\alpha_{\beta\bar\gamma}}C_{\bar\alpha},$$
$$S^\perp_{\bar\beta\gamma} = \dot\partial_\gamma C_{\bar\beta} - \dot\partial_{\bar\beta}C_\gamma$$

are the curvature coefficients associated to the normal Chern-Finsler connection (17).

3 Some applications of the fundamental formulas

In this section we will use the fundamental formulas of the holomorphic hypersurfaces to prove some of the properties of the holomorphic Finsler hypersurfaces. A first application is a geometric characterization of the holomorphic curvature of the complex Finsler hypersurface.

The sectional holomorphic curvature of a complex Finsler space $(M, L = F^2)$ is a subtle notion in this geometry, defined by the horizontal curvature and being connected with that of $c-$complex geodesic (cf. [1]). This notion has been approached and extended for holomorphic subspaces of a complex Finsler space by [12, 15].

Let $\eta = \eta^i\dot\partial_i$ be the vertical radial vector field and $\chi =: l^h(\eta) = \eta^i\delta_i$ its horizontal lift with respect to Chern-Finsler (c.n.c.). Also, let us take the Sasaki lift (2) the Hermitian metric structure on M and let R be the curvature form of the Chern-Finsler linear connection. According to [1], p.108, [10], p.81, the *holomorphic curvature* of the (M,F) space in direction η is

$$K_F(z,\eta) = \frac{2}{L^2} G(R(\chi,\bar{\chi})\chi,\bar{\chi}). \tag{23}$$

In order to obtain a local expression for the holomorphic curvature, as in [2], we consider $R_{i\bar{j}k\bar{h}} = g_{l\bar{j}} R^l_{i\bar{h}k}$ the local form of the Riemann tensor

$$R(X,\bar{Y};\bar{W},Z) := G(R(X,\bar{Y})Z,\bar{W}), \quad \forall X,Y,Z,W \in \Gamma(T'(T'M)), \tag{24}$$

and take $R_{\bar{j}k} = R_{i\bar{j}k\bar{h}}\eta^i\bar{\eta}^h$ the horizontal Ricci tensor of Chern-Finsler connection. Using the homogeneity condition of the complex Finsler metric and the local expression of the curvature tensor $R^l_{j\bar{h}k}$, (23) yields to

$$K_F(z,\eta) = \frac{2}{L^2} R_{\bar{j}k}\bar{\eta}^j\eta^k, \qquad \text{where } R_{\bar{j}k} = -g_{l\bar{j}}\delta_{\bar{h}}(N^l_k)\bar{\eta}^h. \tag{25}$$

By taking $(\tilde{M},\tilde{L} := \tilde{F}^2)$ a holomorphic hypersurface of the complex Finsler space (M,L), we consider the Ricci curvature $\tilde{R}_{\bar{\beta}\gamma}$ and the holomorphic curvature in direction θ of a point $\tilde{u} = (\xi,\theta)$ on the hypersurface, with respect to the intrinsic Chern-Finsler connection, as

$$K_{\tilde{F}}(\xi,\theta) = \frac{2}{\tilde{L}^2}\tilde{R}_{\bar{\beta}\gamma}\bar{\theta}^\beta\theta^\gamma, \quad \text{with } \tilde{R}_{\bar{\beta}\gamma} = -g_{\alpha\bar{\beta}}\delta_{\bar{\mu}}(\tilde{N}^\alpha_\gamma)\bar{\theta}^\mu. \tag{26}$$

Since the induced tangent Chern-Finsler connection coincides with the intrinsic Chern-Finsler connection of the hypersurface, the study of the link between the holomorphic curvature K_F and $\tilde{K}_{\tilde{F}}$ will be simplified in the complex Finsler case in comparison to the real Finsler case [5]. However, we still have to deal with some inconvenient calculus.

If we consider the holomorphic curvature K_F in direction $\eta(\xi,\theta)$ of a point $u = (z(\xi),\eta(\xi,\theta)) \in T'M$, in order to establish a link between K_F and $\tilde{K}_{\tilde{F}}$, by using the first Gauss equation from Theorem 1 and relations (5), (20), we firstly compute

$$\tilde{R}_{\bar{\beta}\gamma} = \tilde{R}_{\alpha\bar{\beta}\gamma\bar{\mu}}\theta^\alpha\bar{\theta}^\mu = g_{\sigma\bar{\beta}}\tilde{R}^\sigma_{\alpha\bar{\mu}\gamma}\theta^\alpha\bar{\theta}^\mu = g_{\sigma\bar{\beta}}(B^j_\alpha B^\sigma_i R^i_{j\bar{\mu}\gamma} - H_{\alpha\gamma}A^\sigma_{\bar{\mu}})\theta^\alpha\bar{\theta}^\mu.$$

Thus, $$\tilde{R}_{\bar{\beta}\gamma} = g_{\sigma\bar{\beta}}B^\sigma_i\eta^j R^i_{j\bar{\mu}\gamma}\bar{\theta}^\mu - H_{\alpha\gamma}H_{\bar{\beta}\bar{\mu}}\theta^\alpha\bar{\theta}^\mu. \tag{27}$$

By applying this relation into (26) and considering that from (5) and (10) we have $\theta^{\bar{\beta}} = B^{\bar{\beta}}_{\bar{h}}\eta^h$ and $N_h\eta^h = 0$, from (8'), (22) and the homogeneity condition $C^i_{jh}\eta^j = 0$ which implies $P^i_{j\bar{k}m}\eta^j = S^i_{j\bar{k}m}\eta^j = 0$, we finally obtain

$$\begin{aligned} K_{\tilde{F}} &= \tfrac{2}{L^2}[g_{i\bar{h}}\eta^j\bar{\eta}^h R^i_{j\bar{\mu}\gamma}\theta^\gamma\bar{\theta}^\mu - H_{\alpha\gamma}H_{\bar{\beta}\bar{\mu}}\theta^\alpha\bar{\theta}^\mu\bar{\theta}^\beta\theta^\gamma] \\ &= \tfrac{2}{L^2}[g_{i\bar{h}}\eta^j\bar{\eta}^h\eta^m\bar{\eta}^k R^i_{j\bar{k}m} + g_{i\bar{h}}\eta^j\bar{\eta}^h\eta^m N^{\bar{k}}H_{\bar{\mu}}\bar{\theta}^\mu Q^i_{j\bar{k}m} - H_{\alpha\gamma}H_{\bar{\beta}\bar{\mu}}\theta^\alpha\bar{\theta}^\mu\bar{\theta}^\beta\theta^\gamma] \end{aligned}$$

If we use now (25) and $K_F = \frac{2}{L^2}g_{i\bar{h}}R^i_{j\bar{k}m}\eta^j\bar{\eta}^k\eta^m\bar{\eta}^h$, we can state

Theorem 2. *The holomorphic curvatures of the holomorphic hypersurface* $(\tilde{M}, \tilde{L})$ *and of the* (M, L) *space are related by*

$$\tilde{K}_{\tilde{F}}(\tilde{u}) = K_F(u) + \frac{2}{L^2} H_{\bar{\mu}} \bar{\theta}^{\mu} Q^i_{j\bar{k}m} g_{i\bar{h}} \bar{\eta}^h \eta^j \eta^m N^{\bar{k}} - \frac{2}{L^2} H_{\alpha\gamma} H_{\bar{\beta}\bar{\mu}} \theta^{\alpha} \bar{\theta}^{\mu} \bar{\theta}^{\beta} \theta^{\gamma}. \quad (28)$$

It is obvious that $Q^i_{j\bar{k}m} g_{i\bar{h}} = Q_{j\bar{h}m\bar{k}}$ are the components of a complex Riemann tensor. The vanishing of the second fundamental form and of $Q_{j\bar{h}m\bar{k}}$ tensors leads to the equality of the holomorphic curvatures $\tilde{K}_{\tilde{F}} = K_F$, in any arbitrary point of the holomorphic hypersurface.

As we have already pointed out, the notion of holomorphic curvature is related to that of c-complex geodesic, which is a special notion in characterizing the Kobayashi metric. However, we will firstly analyze the totally geodesic property of a hypersurface for the complex geodesic case.

Even if a complex submanifold of a Kähler manifold is also a Kähler manifold, we cannot extend this result to complex Finsler case. By considering the conditions which must be fulfilled for a Finsler space to be Kähler, according to [1] terminology, and using the nonzero torsion coefficients (21), we immediately have

Theorem 3. *A holomorphic hypersurface* $(\tilde{M}, \tilde{F})$ *of a strongly Kähler Finsler manifold* (M, F) *is in its turn strongly Kähler if and only if*

$$B^{\alpha}_i N^k (B^j_{\beta} H_{\gamma} - B^j_{\gamma} H_{\beta}) Q^i_{jk} = 0. \quad (29)$$

A holomorphic hypersurface $(\tilde{M}, \tilde{F})$ *of a Kähler Finsler manifold* (M, F) *is Kähler if and only if*

$$B^{\alpha}_i B^j_{\gamma} H_{\beta} \theta^{\beta} N^k Q^i_{jk} = 0. \quad (30)$$

A holomorphic hypersurface $(\tilde{M}, \tilde{F})$ *of a weakly Kähler Finsler manifold* (M, F) *is weakly Kähler if and only if*

$$\eta_i B^j_{\gamma} H_{\beta} \theta^{\beta} N^k Q^i_{jk} = 0, \quad \textit{where } \eta_i = g_{i\bar{m}} \bar{\eta}^m. \quad (31)$$

This theorem has a simplified form when we restrict to the complex geodesics of a totally geodesic complex Finsler hypersurface $(\tilde{M}, \tilde{F})$ of the Finsler space (M, F), which is defined as a particular case of totally geodesic holomorphic subspaces. Thus, a complex holomorphic hypersurface $(\tilde{M}, \tilde{F})$ of the complex Finsler space (M, F) is called *totally geodesic* if any complex geodesic of $(\tilde{M}, \tilde{F})$ is also a complex geodesic of (M, F).

From [1], p.101, we find that a complex geodesic $\{z^i(t)\}$ of (M, F) space satisfies the following equations

$$\frac{d^2 z^i(t)}{dt^2} + N^i_j(z(t), \eta(t)) \frac{dz^{\beta}(t)}{dt} = \Theta^{*i}, \quad i = 1, \dots, n, \quad (32)$$

where $\eta^i(t) = \frac{dz^i(t)}{dt}$ and $\Theta^{*i} = g^{\bar{m}i} g_{j\bar{k}} (L^{\bar{k}}_{\bar{p}\bar{m}} - L^{\bar{k}}_{\bar{m}\bar{p}}) \eta^j \bar{\eta}^p$ with respect to the Chern-Finsler (c.l.c.). Since $\delta\eta^i = d\eta^i + N^i_j dz^j$, we can express (32) as follows

$$\frac{\delta \eta^i}{dt} = \Theta^{*i}, \quad i = 1, \dots, n.$$

Let us consider now $\{\xi^\alpha(t)\}$ a complex geodesic of a holomorphic hypersurface $(\tilde{M},\tilde{F})$ of (M,F). Then, we have

$$\frac{\delta\theta^\alpha}{dt}=\tilde{\Theta}^{*\alpha},\quad \alpha=1,\ldots,n-1,$$

where $\theta^\alpha(t)=\frac{d\xi^\alpha(t)}{dt}$ and $\tilde{\Theta}^{*\alpha}=g^{\bar{\mu}\alpha}g_{\beta\bar{\gamma}}(\tilde{L}^{\bar{\gamma}}_{\bar{\sigma}\bar{\mu}}-\tilde{L}^{\bar{\gamma}}_{\bar{\mu}\bar{\sigma}})\theta^\beta\bar{\theta}^\sigma$. As $\delta\theta^\alpha=d\theta^\alpha+\tilde{N}^\alpha_\beta d\xi^\beta$, where $\tilde{N}$ is the induced nonlinear connection on $T'\tilde{M}$ which fulfills $\delta\theta^\alpha=B^\alpha_i\delta\eta^i$, we have

$$\tilde{\Theta}^{*\alpha}=B^\alpha_i\Theta^{*i}.$$

Thus, it implies that if (M,F) is weakly Kähler Finsler along its complex geodesic, then $(\tilde{M},\tilde{F})$ hypersurface is also weakly Kähler along its complex geodesic. By the second relation from (13), we have

$$\Theta^{*i}=\frac{\delta\eta^i}{dt}=B^i_\alpha\frac{\delta\theta^\alpha}{dt}+N^iH_\alpha\frac{d\xi^\alpha}{dt}=B^i_\alpha\tilde{\Theta}^{*\alpha}+N^iH_\alpha\theta^\alpha.$$

Theorem 4. *Let $(\tilde{M},\tilde{F})$ be a holomorphic hypersurface of a weakly Kähler Finsler manifold (M,F). Then $(\tilde{M},\tilde{F})$ is totally geodesic if and only if*

$$N^iH_\alpha\theta^\alpha=0,\quad i=1,\ldots,n,$$

holds along any complex geodesic $\{\xi^\alpha(t)\}$ on $(\tilde{M},\tilde{F})$.

Since this condition takes place for any $i=1,\ldots,n$, we get $H_\alpha\theta^\alpha=0$ and using $H_\alpha=H_{\beta\alpha}\theta^\beta$, we can rewrite

Corollary 1. *Let $(\tilde{M},\tilde{F})$ be a holomorphic hypersurface of a weakly Kähler Finsler manifold (M,F). Then $(\tilde{M},\tilde{F})$ is totally geodesic if and only if the horizontal components of the second fundamental form satisfy*

$$(H_{\alpha\beta}-H_{\beta\alpha})\theta^\alpha\theta^\beta=0$$

along any complex geodesic $\{\xi^\alpha(t)\}$ of $(\tilde{M},\tilde{F})$ hypersurface.

Remark. If $L=h_{i\bar{j}}(z)\eta^i\bar{\eta}^j$ comes from a Kähler metric on M and $\tilde{M}$ is one of its holomorphic hypersurfaces, then the metric tensor $g_{i\bar{j}}=h_{i\bar{j}}(z)$ is independent of the direction $\eta\in\widetilde{T'M}$ and $Q^i_{jk}=C^i_{jk}\equiv 0$. Then, from (19) we get $K_{\alpha\beta}=0$, and the three relations from Theorem 3 hold identically. In this case, our results coincide with the classic results of submanifolds, and implicitly holomorphic hypersurfaces defined on complex Kähler manifolds. Theorem 2, 3 and Corollary 1 show the importance of horizontal components of the second fundamental form H in the investigation of holomorphic hypersurfaces theory of complex Finsler spaces.

Further, we approach the study of c-complex geodesics and circumstances under which the holomorphic hypersurface of a complex Finsler space is totally geodesic. We recall from [1, 10] a characterization of this notion.

A c-complex geodesic, $c \in \mathbb{R}$ is a geodesic of the Finsler complex space (M, F) which is the image via holomorphic maps of a geodesic on unitary disc Δ with Poincaré metric. According to [1], Theorem 3.1.10 (ii), through any point (z, η) there exists a c-complex geodesic φ if and only if the complex Finsler space (M, F) is weakly Kähler and it fulfils along the curve φ the following condition

$$vT(\varphi'^h, \overline{\varphi'^h}) = cG(\varphi'^h, \overline{\varphi'^h}) l^v(\varphi'),$$

where φ' represents the tangent application of φ and $\varphi'^h = l^h(\varphi')$ the stands for the horizontal lift of $\varphi'(0)$ vector. Locally, on φ curve this condition implies

$$\delta_{\bar{j}}(N^l_k)\bar{\eta}^j \eta^k = cL\eta^l. \tag{33}$$

Definition 2. *A holomorphic hypersurface* $(\tilde{M}, \tilde{L})$ *is called* $c-$totally geodesic immersed *in* (M, L) *if any of its* $c-$*complex geodesic curves is a* $c-$*complex geodesic of the* (M, L) *complex Finsler space.*

This notion is more special to that of totally geodesic from real case. Further we will characterize the $c-$complex totally geodesic immersed hypersurfaces, in brief $(c.t.g)$.

By using (3), the (33) condition becomes $\Theta^l_{k\bar{j}}\bar{\eta}^j\eta^k = cL\eta^l$. From (21) and (5), it follows that

$$\Theta^\alpha_{\beta\bar{\gamma}}\theta^\beta\theta^{\bar{\gamma}} = A^\alpha_{\bar{\gamma}} H_\beta \theta^\beta \theta^{\bar{\gamma}} + B^\alpha_i \Theta^i_{j\bar{k}} \eta^j \bar{\eta}^k + B^\alpha_i H_{\bar{\gamma}} \bar{\theta}^\gamma \rho^i_{j\bar{k}} \eta^j N^{\bar{k}}.$$

Considering this relation, if we have the equalities $A^\alpha_{\bar{\gamma}} H_\beta \theta^\beta \theta^{\bar{\gamma}} + B^\alpha_i H_{\bar{\gamma}} \bar{\theta}^\gamma \rho^i_{j\bar{k}} \eta^j N^{\bar{k}} = 0$ or $A^\alpha_{\bar{\gamma}} \bar{\theta}^\gamma = B^\alpha_i \rho^i_{j\bar{k}} \eta^j N^{\bar{k}} = 0$, using (5), we easily obtain that (33) takes place on (M, L). Then it is automatically fulfilled in the corresponding points of $(\tilde{M}, \tilde{L})$.

Conversely, if $\Theta^\alpha_{\beta\bar{\gamma}} = c\tilde{L}\theta^\alpha$, by applying the above relation it results that

$$A^\alpha_{\bar{\gamma}} H_\beta \theta^\beta \theta^{\bar{\gamma}} + B^\alpha_i \Theta^i_{j\bar{k}} \eta^j \bar{\eta}^k + B^\alpha_i H_{\bar{\gamma}} \bar{\theta}^\gamma \rho^i_{j\bar{k}} \eta^j N^{\bar{k}} = c\tilde{L}\theta^\alpha,$$

which contracted with B^l_α and using the last relation from (10), it becomes

$$\Theta^l_{j\bar{k}} \eta^j \bar{\eta}^k = cL\eta^l + N^l N_i \Theta^i_{j\bar{k}} \eta^j \bar{\eta}^k - B^l_\alpha [A^\alpha_{\bar{\gamma}} H_\beta \theta^\beta + B^\alpha_i H_{\bar{\gamma}} \bar{\theta}^\gamma \rho^i_{j\bar{k}} \eta^j N^{\bar{k}}] \bar{\theta}^\gamma.$$

Thus, the $(c.t.g)$ condition (33) implies the vanishing of the last part from the previous expression

$$N^l N_i \Theta^i_{j\bar{k}} \eta^j \bar{\eta}^k = B^l_\alpha [A^\alpha_{\bar{\gamma}} H_\beta \theta^\beta + B^\alpha_i H_{\bar{\gamma}} \rho^i_{j\bar{k}} \eta^j N^{\bar{k}}] \bar{\theta}^\gamma.$$

By contracting this relation with N_l or B^σ_l, we obtain the following equivalent conditions

$$N_i \Theta^i_{j\bar{k}} \eta^j \bar{\eta}^k = 0 \quad \text{and} \quad A^\alpha_{\bar{\gamma}} \bar{\theta}^\gamma H_\beta \theta^\beta + B^\alpha_i H_{\bar{\gamma}} \bar{\theta}^\gamma \rho^i_{j\bar{k}} \eta^j N^{\bar{k}} = 0, \ \forall \alpha = \overline{1, n-1}. \tag{34}$$

Therefore, by taking into account the conditions under which the weakly Kähler property transmits from Theorem 3, we can state

Theorem 5. *The holomorphic hypersurface $(\tilde{M}, \tilde{L})$ is (c.t.g) immersed in the complex Finsler manifold (M, L) if and only if the (34) and (31) conditions are fulfilled.*

Although the complex Finsler geometry is quite important for some applications, there are not many examples of complex Finsler metrics. Thus, only few examples which verify this Theorem are known. In order to verify the Theorem conditions, let us start by recalling the Kobayashi pseudo-metric F_K which is defined by $F_K(z, \eta) = \inf \frac{1}{r}$, where the infimum is taken over all $r > 0$ such that $H_M(r) = \{f : \Delta_r \to M \mid f$ is holomorphic and $f(0) = z, f'(z) = \eta\}$ is non-empty. This is a pseudo-distance because it may not be positive definite. When it is positive definite it is said to be Kobayashi hyperbolic [14] or with Kobayashi metric [1].

Proposition 1. *Any holomorphic hypersurface of a Kobayashi hyperbolic space is also Kobayashi hyperbolic.*

The idea of demonstration for holomorphic subspaces, consequently for hypersurface as well, is presented in [12]. Thus, if $\tilde{M}$ is a holomorphic hypersurface of M then $H_{\tilde{M}}(r) \subset H_M(r)$ and so $F_K(z, \eta) \leq \tilde{F}_K(z, \eta)$. In particular, if F_K is positive definite then $\tilde{F}_K$ is also positive definite.

A characterization of Kobayashi metric is given by Theorem 3.1.15 from [1]. Thus, if $F : T'M \to \mathbb{R}^+$ is smooth and complete, strongly pseudoconvex (i.e. $\det(g_{i\bar{j}}) > 0$) and condition (33) takes place for $c = -2$, then F_K is the Kobayashi metric on M. The above observations do not involve the induced metric and the situations when the induced Kobayashi metric coincides with the Kobayashi intrinsic metric of the hypersurface rarely occur. For example, this happens iff we have the strongly pseudoconvexity $g_{\alpha\bar{\beta}}$.

Next, from [10] we recall that a complex Finsler space is locally Minkowski if at any point there exist local charts such that the metric depends only on direction, i.e. $g_{i\bar{j}}(\eta)$, and hence $N^i_j = L^i_{jk} = 0$ in the chosen local charts. Then, it results that $T^i_{jk} = 0$ and $\Theta^i_{j\bar{k}} = \rho^i_{j\bar{k}} = 0$. The (*c.t.g*) hypersurface condition reduces to the vanishing of $A^{\alpha}_{\bar{\gamma}}\bar{\theta}^{\gamma} H_{\beta}\theta^{\beta} = 0$ and of (31). For example, $Q^i_{jk}\eta_i B^j_{\gamma} N^k = A^{\alpha}_{\bar{\gamma}}\bar{\theta}^{\gamma} = 0$ or $H_{\beta}\theta^{\beta} = 0$ are sufficient conditions for this requirement. One simple example of locally Minkowski metric is the complex version of Antonelli-Shimada metric $L_{AS} = \{\sum(\eta^i\bar{\eta}^i)^m\}^{1/m}$, $m \geq 2$.

Another important notion considered in [1] is the strongly Kähler Finsler space, i.e. $T^i_{jk} = 0$, and thus the horizontal part of $d\Phi$ vanishes, where $\Phi = \sqrt{-1} g_{j\bar{k}} dz^j \otimes d\bar{z}^k$ is the horizontal Kähler form. The vanishing of the vertical part of $d\Phi$ implies $\Theta^i_{j\bar{k}} = 0$. Recall that in [10] a complex Finsler space with the metric depending only on position, $g_{i\bar{j}}(z)$, is named purely Hermitian. We know here the classical examples of Bergman, Fubini-Study or euclidean metrics which are strongly Kähler. In a purely Hermitian Finsler space we have $Q^i_{jk} = C^i_{jk} = g^{\bar{m}i}\dot{\partial}_k g_{j\bar{m}} = 0$ and the notion of strongly Kähler coincides with that of weakly Kähler. From (34) and (31) we easily obtain

Proposition 2. *If (M, L) is a purely Hermitian complex Finsler space with constant Kähler form, then any holomorphic hypersurface $(\tilde{M}, \tilde{L})$ with the last part of (34) relation satisfied is (c.t.g).*

Next, after these remarks regarding the $(c.t.g)$ conditions for Kobayashi metric, implicitly related to $c-$complex geodesic notion, for locally Minkowski and purely Hermitian metrics, we propose to go back to the (28) link between the holomorphic curvatures. A first problem is to obtain the conditions under which the $(\tilde{M}, \tilde{L})$ hypersurface is of constant holomorphic curvature. Regarding this, we will offer an answer for a special class of complex Finsler spaces, namely the generalized Einstein spaces, denoted by $(g.E)$. According to [3], a complex Finsler space (M, L) is *generalized Einstein* if the horizontal Ricci tensor $R_{\bar{j}k}$ is proportional to the angular metric $h_{k\bar{j}} = Lg_{k\bar{j}} + \eta_k \bar{\eta}_j$, where $\eta_k = g_{k\bar{m}} \bar{\eta}^m$. The angular metric is invertible and this class of $(g.E)$ spaces is a bit more general than that of purely Hermitian metrics; they coincide in the Kählerian case.

Theorem 6 ([3])**.** *Let (M, F) be a (g.E) complex Finsler space. Then:*

i) *$R_{\bar{j}k} = \frac{1}{4} K_F h_{k\bar{j}}$ and the holomorphic curvature K_F depends only on z;*

ii) *If (M, F) is connected, weakly Kähler and of dimension $n \geq 2$, then it is of constant holomorphic curvature (Shur type theorem).*

Conversely, a purely Hermitian complex Finsler metric which is Kähler of constant holomorphic curvature is $(g.E)$.

Now, we will apply these results to the holomorphic hypersurface case. Firstly, let us consider $\theta_\alpha = g_{\alpha\bar{\beta}} \bar{\theta}^\beta$ and using (8) and (5), we easily deduce $\theta_\alpha = B^i_\alpha \eta_i$. Thus, the angular metric induced on the hypersurface $h_{\alpha\bar{\beta}} = \tilde{L} g_{\alpha\bar{\beta}} + \theta_\alpha \bar{\theta}_\beta$, becomes $h_{\alpha\bar{\beta}} = B^i_\alpha B^{\bar{j}}_{\bar{\beta}} h_{i\bar{j}}$.

Let us assume that (M, F) is a $(g.E)$ complex Finsler space, i.e. $R_{\bar{j}k} = \frac{1}{4} K_F h_{k\bar{j}}$. By replacing this into (27) formula and using (22), (5), the observation $P^i_{j\bar{k}m} \eta^j = S^i_{j\bar{k}m} \eta^j = 0$, $g_{\alpha\bar{\beta}} B^\alpha_i = g_{i\bar{p}} B^{\bar{p}}_{\bar{\beta}}$ and $R_{\bar{p}m} = g_{i\bar{p}} R^i_{j\bar{k}m} \eta^j \bar{\eta}^m$, we get

$$\begin{aligned} \tilde{R}_{\bar{\beta}\gamma} &= B^m_\gamma B^{\bar{p}}_{\bar{\beta}} R_{\bar{p}m} + g_{\alpha\bar{\beta}} B^\alpha_i \eta^j \bar{\theta}^\mu H_{\bar{\mu}} B^m_\gamma N^{\bar{k}} Q^i_{j\bar{k}m} - H_{\alpha\gamma} H_{\bar{\beta}\bar{\mu}} \theta^\alpha \bar{\theta}^\mu \\ &= \tfrac{1}{4} K_F h_{\alpha\bar{\beta}} + g_{\alpha\bar{\beta}} B^\alpha_i \eta^j \bar{\theta}^\mu H_{\bar{\mu}} B^m_\gamma N^{\bar{k}} Q^i_{j\bar{k}m} - H_{\alpha\gamma} H_{\bar{\beta}\bar{\mu}} \theta^\alpha \bar{\theta}^\mu. \end{aligned}$$

Using now K_F from (28) and $H_{\alpha\gamma} \theta^\alpha = H_\gamma$ we obtain

$$\tilde{R}_{\bar{\beta}\gamma} = \frac{1}{4} \tilde{K}_{\tilde{F}} h_{\gamma\bar{\beta}} + S_{\bar{\beta}\gamma}, \tag{35}$$

where

$$\begin{aligned} S_{\bar{\beta}\gamma} = \; & g_{i\bar{h}} \eta^j H_{\bar{\mu}} \bar{\theta}^\mu N^{\bar{k}} Q^i_{j\bar{k}m} [B^{\bar{h}}_{\bar{\beta}} B^m_\gamma - \tfrac{1}{2L^2} \bar{\eta}^h \eta^m h_{\gamma\bar{\beta}}] \\ & + \tfrac{1}{2L^2} H_\alpha H_{\bar{\mu}} \theta^\alpha \bar{\theta}^\mu h_{\gamma\bar{\beta}} - H_\gamma H_{\bar{\beta}\bar{\mu}} \bar{\theta}^\mu. \end{aligned}$$

We notice that if $g_{i\bar{j}}$ depends only of the position variables z, then $g_{\alpha\bar{\beta}}$ depends only on ξ. Thus,

Theorem 7. *Let (M, L) be a (g.E) complex Finsler space and $(\tilde{M}, \tilde{L})$ one of its holomorphic surfaces. Then:*

i) *$(\tilde{M}, \tilde{L})$ is (g.E) if and only if $S_{\bar{\beta}\gamma} = 0$.*

ii) *If $S_{\bar{\beta}\gamma} = 0$ and $\tilde{M}$ is connected of dimension $n - 1 \geq 2$, then $(\tilde{M}, \tilde{L})$ is of constant holomorphic curvature.*

iii) *If we assume that (M, L) is purely Hermitian and connected and $(\tilde{M}, \tilde{L})$ is Kähler, i.e. $T^{\alpha}_{\beta\gamma}\theta^{\beta} = 0$, with constant holomorphic curvature, then $(\tilde{M}, \tilde{L})$ is (g.E).*

References

[1] Abate, M., Patrizio, G., *Finsler Metrics - A Global Approach*, Lecture Notes in Math., **1591**, Springer-Verlag, 1994.

[2] Aldea, N., *On Chern complex linear connection*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **45(93)** (2002), no. 3-4, 119–131 (2003).

[3] Aldea, N., *Complex Finsler spaces of constant holomorphic curvature*, Differential geometry and its applications, 179–190, Matfyzpress, Prague, 2005.

[4] Aldea, N., Munteanu, G., *On complex Douglas spaces*, J. Geom. Phys., **66** (2013), 80-93

[5] Bejancu, A., *On the theory of Finsler subspaces*, Finslerian Geom., Kluwer Acad. Publ., **109**(2000), 111-129.

[6] Bejancu, A., Faran, H.R., *The geometry of pseudo-Finsler submanifolds*, Kluwer Acad. Publ., 2000.

[7] Chen, B.Y, Shen, Y., *Kähler Finsler metrics are actually strongly Kähler*, Chin. Ann. Math. Ser. B, **30**, 2(2009), 173-178.

[8] Matsumoto, M., *The induced and intrinsic Finsler connections of a hypersurface and Finslerien projective geometry,* J. Math. Kyoto Univ. **25** (1985), no. 1, 107–144.

[9] Matsumoto, M., *Theory of Y-extremal and minimal hypersurfaces in a Finsler space. On Wegener's and Barthel's theories*, J. Math. Kyoto Univ. **26** (1986), no. 4, 647–665.

[10] Munteanu, Gh., *Complex Spaces in Finsler, Lagrange and Hamilton Geometries*, Kluwer Acad. Publ., 2004.

[11] Munteanu, Gh. *Minimal hypersurfaces of a complex Finsler space,* Bull. Transilv. Univ. Braşov Ser. B (N.S.) **14(49)** (2007), suppl., 175–185.

[12] Munteanu, Gh., *The equations of a holomorphic subspace in a complex Finsler space,* Period. Math. Hungar. **55** (2007), no. 1, 97–112.

[13] Munteanu, Gh., *Totally geodesic holomorphic subspaces,* Nonlinear Anal. Real World Appl. **8** (2007), no. 4, 1132–1143.

[14] Wong, P.-M., *A survey of complex Finsler geometry.* Finsler geometry, Sapporo 2005, In memory of Makoto Matsumoto, 375–433, Adv. Stud. Pure Math., 48, Math. Soc. Japan, Tokyo, 2007.

[15] Zhong, C., *On the fundamental formulas of complex Finsler submanifolds,* J. Geom. Phys. **58** (2008), no. 4, 423–449.

[16] Zhong, C., *Holomorphic curvature of complex Finsler submanifolds,* Sci. China Math. **53** (2010), no. 2, 261–274.