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ON A TYPE OF P-SASAKIAN MANIFOLDS

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Abstract

In this paper, we investigate pseudo-projectively symmetric P-Sasakian manifolds and P-Sasakian manifolds satisfying the condition $R(X,\xi)\cdot P = P(X,\xi)\cdot R$. Next we study P-Sasakian manifolds satisfying the curvature condition $P \cdot S = 0$ and pseudoprojectively flat P-Sasakian manifolds. Further, we discuss about ϕ-projectively semisymmetric and locally ϕ-projectively symmetric P-Sasakian manifolds. Finally, we construct an example of a 5-dimensional P-Sasakian manifold to verify some results.

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1 Introduction

An n-dimensional differentiable manifold M is said to admit an almost paracontact structure (ϕ, ξ, η), introduced by Satō [15], where ϕ is a $(1,1)$- tensor field, ξ is a vector field and η is a 1-form if

$$\phi^2 X = X - \eta(X)\xi,\ \phi\xi = 0,\ \eta(\xi) = 1,\ \eta(\phi X) = 0, \tag{1}$$

for all vector field X on M. Moreover, if M admits a Riemannian metric g such that

$$g(X,\xi) = \eta(X),\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2}$$

then (ϕ, ξ, η, g) is called almost paracontact metric structure and M an almost paracontact metric manifold [15]. If (ϕ, ξ, η, g) satisfy the following equations:

$$\begin{aligned} &d\eta = 0,\ \nabla_X \xi = \phi X, \\ &(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \end{aligned} \tag{3}$$

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then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold [1]. Especially, a P-Sasakian manifold M is called a special para-Sasakian manifold or briefly a SP-Sasakian manifold [16] if M admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{4}$$

We define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \tag{5}$$

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \tag{6}$$

respectively, where $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M, A is the symmetric $(0,2)$-tensor, R is the Riemannian curvature tensor of type $(1,3)$ and ∇ is the Levi-Civita connection.

Let M be an n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. Here P is defined by [18]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{7}$$

for all $X, Y, Z \in \chi(M)$, where R is the curvature tensor and S is the Ricci tensor of type $(0,2)$. In fact, M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

For a $(0,k)$-tensor field T, $k \geq 1$, on (M^n, g) we define the tensors $R \cdot T$ and $Q(g,T)$ by

$$\begin{aligned}(R(X,Y)\cdot T)(X_1, X_2, ..., X_k) &= -T(R(X,Y)X_1, X_2, ..., X_k) \\ &\quad -T(X_1, R(X,Y)X_2, ..., X_k) \\ &\quad -... - T(X_1, X_2, ..., R(X,Y)X_k)\end{aligned} \tag{8}$$

and

$$\begin{aligned}Q(g,T)(X_1, X_2, ..., X_k; X, Y) &= -T((X\wedge Y)X_1, X_2, ..., X_k) \\ &\quad -T(X_1, (X\wedge Y)X_2, ..., X_k) \\ &\quad -... - T(X_1, X_2, ..., (X\wedge Y)X_k),\end{aligned} \tag{9}$$

respectively [20].

A Riemannian or a semi-Riemannian manifold is said to pseudosymmetric [20] if $R \cdot R$ and $Q(g,R)$ are linearly dependent. That is, $R \cdot R = L_R Q(g,R)$, where L_R is some function on M.

If the tensors $R \cdot P$ and $Q(g, P)$ are linearly dependent then M^n is called Pseudo-projectively symmetric. This is equivalent to

$$R \cdot P = L_P Q(g, P), \tag{10}$$

holding on the set $U_P = \{x \in M : P \neq 0 \text{ at } x\}$, where L_P is some function on U_P.

Furthermore we define the tensors $P \cdot R$ and $P \cdot S$ on (M^n, g) by

$$\begin{aligned}(P(X,Y) \cdot R)(U,V)W &= P(X,Y)R(U,V)W - R(P(X,Y)U,V)W \\ &\quad -R(U,P(X,Y)V)W - R(U,V)P(X,Y)W \end{aligned} \tag{11}$$

and

$$(P(X,Y) \cdot S)(U,V) = -S(P(X,Y)U,V) - S(U,P(X,Y)V), \tag{12}$$

respectively.

The notion of Weyl projective semi-symmetric manifold is defined by $R(X,Y) \cdot P = 0$, where $R(X,Y)$ is considered as a derivation of tensor algebra at each point of the manifold.

An almost paracontact Riemannian manifold M is said to be an η-Einstein manifold if the Ricci tensor S satisfies condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M is an Einstein manifold.

De and Tarafdar [7] studied P-Sasakian manifolds satisfying the condition $R(X,Y) \cdot R = 0$. In [6], De and Pathak studied P-Sasakian manifolds satisfying the conditions $R(X,Y) \cdot P = 0$ and $R(X,Y) \cdot S = 0$. Özgür [14] studied Weyl-pseudosymmetric P-Sasakian manifolds and also P-Sasakian manifolds satisfying the condition $C \cdot S = 0$. Also P-Sasakian manifolds have been studied by several authors such as Adati and Miyazawa [2], Deshmukh and Ahmed [8], De et al [4, 5], Sharfuddin, Deshmukh, Husain [17], Matsumoto et al [11, 12], Mihai [13], Mandal and De [10] and many others.

Motivated by the above studies, we characterize P-Sasakian manifolds satisfying certain curvature conditions on the projective curvature tensor. The paper is organized as follows: After preliminaries in section 3, we first study the characterizations of the pseudo-projectively symmetric P-Sasakian manifold and it is proved that the manifold is an Einstein manifold. Section 4 is devoted to the study of P-Sasakian manifolds satisfying the condition $R(X,\xi) \cdot P = P(X,\xi) \cdot R$ and in this case we have shown that the square of the Ricci tensor is the linear sum of the Ricci tensor and the metric tensor. Section 5 deals with P-Sasakian manifolds satisfying the curvature condition $P \cdot S = 0$ and it is proved that such a manifold satisfies $P \cdot S = 0$ if and only if it is an Einstein manifold. In section 6, we discuss about pseudoprojectively flat P-Sasakian manifolds and such manifolds are necessarily Einstein. ϕ-projectively semisymmetric and locally ϕ-projectively symmetric P-Sasakian manifolds are studied in section 7 and section 8 respectively and in both the cases we prove that the manifolds are Einstein. Finally, we construct an example of a 5-dimensional P-Sasakian manifold to verify some results.

2 Preliminaries

In a P-Sasakian manifold the following relations hold ([1],[14]):

$$S(X,\xi) = -(n-1)\eta(X),\ Q\xi = -(n-1)\xi, \tag{13}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{14}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{15}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{16}$$

$$\eta(R(X,Y)\xi) = 0, \tag{17}$$

for any vector fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor. Using equations (15) and (13) we obtain from (7)

$$P(X,\xi)Z = g(X,Z)\xi + \frac{1}{n-1}S(X,Z)\xi. \tag{18}$$

Definition 1. *A P-Sasakian manifold $(M^n, g), n > 3$, is said to be ϕ-projectively semisymmetric if*

$$P(X,Y)\cdot\phi = 0$$

on M, for any vector fields $X, Y \in \chi(M)$.

According to Takahashi [19] we have the following definition.

Definition 2. *A P-Sasakian manifold (M^n, g) is said to be ϕ-projectively symmetric, if it satisfies*

$$\phi^2((\nabla_W P)(X,Y)Z) = 0,$$

for any vector fields X, Y, Z and $W \in \chi(M)$. Moreover, if the vector fields W, X, Y, Z are orthogonal to ξ, then the manifold is called locally ϕ-projectively symmetric.

3 Pseudo-projectively symmetric P-Sasakian manifolds

In this section we study pseudo-projectively symmetric manifolds, that is, the manifolds satisfying the condition $R(X,Y)\cdot P = L_P Q(g,P)$. At first we prove the following theorem.

Theorem 1. *Let M be an n-dimensional, $n \geq 3$, P-Sasakian manifold. If M is pseudo-projectively symmetric, then M is an Einstein manifold, or $L_P = -1$ holds on M.*

Proof. Assume that $M(n \geq 3)$ is a pseudo-projectively symmetric P-Sasakian manifold and $X, Y, U, V, W \in \chi(M)$. We have

$$(R(X,Y)\cdot P)(U,V)W = L_P Q(g,P)(U,V)W. \tag{19}$$

From (8) and (9) we have

$$R(X,Y)P(U,V)W - P(R(X,Y)U,V)W - P(U,R(X,Y)V)W$$
$$-P(U,V)R(X,Y)W = L_P[(X \wedge Y)P(U,V)W - P((X \wedge Y)U,V)W$$
$$-P(U,(X \wedge Y)V)W - P(U,V)(X \wedge Y)W]. \tag{20}$$

Substituting $Y = \xi$ in (20) yields

$$R(X,\xi)P(U,V)W - P(R(X,\xi)U,V)W - P(U,R(X,\xi)V)W$$
$$-P(U,V)R(X,\xi)W = L_P[(X \wedge \xi)P(U,V)W - P((X \wedge \xi)U,V)W$$
$$-P(U,(X \wedge \xi)V)W - P(U,V)(X \wedge \xi)W]. \tag{21}$$

With the help of (15) and (6) we get from (21)

$$[1 + L_P][\eta(P(U,V)W)X - g(X,P(U,V)W)\xi - \eta(U)P(X,V)W$$
$$+g(X,U)P(\xi,V)W - \eta(V)P(U,X)W + g(X,V)P(U,\xi)W$$
$$-\eta(W)P(U,V)X + g(X,W)P(U,V)\xi] = 0. \tag{22}$$

Taking inner product of (22) with ξ we obtain

$$[1 + L_P][\eta(P(U,V)W)\eta(X) - g(X,P(U,V)W) - \eta(U)\eta(P(X,V)W)$$
$$+g(X,U)\eta(P(\xi,V)W) - \eta(V)\eta(P(U,X)W) + g(X,V)\eta(P(U,\xi)W)$$
$$-\eta(W)\eta(P(U,V)X) + g(X,W)\eta(P(U,V)\xi)] = 0. \tag{23}$$

Now putting $U = W = e_i$ in (23), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i = 1,2,...,n$ we get

$$[1 + L_P][S(V,X) + (n-1)g(V,X)] = 0. \tag{24}$$

Then either $L_P = -1$ or, the manifold is an Einstein manifold. This completes the proof of the theorem.
By the above discussion we have the following:

Corollary 1. *Every n-dimensional ($n \geq 3$) pseudo-projectively symmetric P-Sasakian manifold is of the form $R \cdot P = -Q(g,P)$, provided the manifold is non-Einstein.*

In particular, if $L_p = 0$, then pseudo-projectively symmetric manifold reduces to a projectively symmetric manifold, that is, $R \cdot P = 0$. Hence in this case we get the following:

Corollary 2. *A projectively symmetric P-Sasakian manifold is an Einstein manifold.*

The above corollary have been proved by De and Pathak in their paper [6].

4 P-Sasakian manifolds satisfying the condition $R(X,\xi)\cdot P = P(X,\xi)\cdot R$

In this section we characterize the P-Sasakian manifolds satisfying the condition $R(X,\xi)\cdot P = P(X,\xi)\cdot R$. Let M be an n-dimensional ($n \geq 3$) P-Sasakian manifold satisfying the condition

$$R(X,\xi)\cdot P = P(X,\xi)\cdot R. \tag{25}$$

Making use of (8) and (11) we get from (25)

$$\begin{aligned}
&R(X,\xi)P(U,V)W - P(R(X,\xi)U,V)W \\
&-P(U,R(X,\xi)V)W - P(U,V)R(X,\xi)W \\
&= P(X,\xi)R(U,V)W - R(P(X,\xi)U,V)W \\
&-R(U,P(X,\xi)V)W - R(U,V)P(X,\xi)W.
\end{aligned} \tag{26}$$

Using (18) in (26) we have

$$\begin{aligned}
&[g(X,P(U,V)W)\xi - \eta(P(U,V)W)X - g(X,U)P(\xi,V)W + \eta(U)P(X,V)W \\
&-g(X,V)P(U,\xi)W + \eta(V)P(U,X)W - g(X,W)P(U,V)\xi + \eta(W)P(U,V)X] \\
&= [g(X,R(U,V)W)\xi + \frac{1}{n-1}S(X,R(U,V)W)\xi - g(X,U)\eta(W)V \\
&+g(X,U)g(V,W)\xi - \frac{1}{n-1}S(X,U)\eta(W)V + \frac{1}{n-1}S(X,U)g(V,W)\xi \\
&-g(X,V)g(U,W)\xi + g(X,V)\eta(W)U - \frac{1}{n-1}S(X,V)g(U,W)\xi \\
&+\frac{1}{n-1}S(X,V)\eta(W)U - g(X,W)\eta(U)V + g(X,W)\eta(V)U \\
&-\frac{1}{n-1}S(X,W)\eta(U)V + \frac{1}{n-1}S(X,W)\eta(V)U].
\end{aligned} \tag{27}$$

Taking the inner product of (27) with ξ and using (18) we get

$$\begin{aligned}
&[g(X,P(U,V)W) - \eta(P(U,V)W)\eta(X) + g(X,U)g(V,W) \\
&+\frac{1}{n-1}g(X,U)S(V,W) + \eta(U)\eta(P(X,V)W) - g(X,V)g(U,W) \\
&-\frac{1}{n-1}g(X,V)S(U,W) + \eta(V)\eta(P(U,X)W) + \eta(W)\eta(P(U,V)X)] \\
&= [g(X,R(U,V)W) + \frac{1}{n-1}S(X,R(U,V)W) - g(X,U)\eta(W)\eta(V) \\
&+g(X,U)g(V,W) - \frac{1}{n-1}S(X,U)\eta(W)\eta(V) + \frac{1}{n-1}S(X,U)g(V,W) \\
&-g(X,V)g(U,W) + g(X,V)\eta(W)\eta(U) \\
&-\frac{1}{n-1}S(X,V)g(U,W) + \frac{1}{n-1}S(X,V)\eta(W)\eta(U)].
\end{aligned} \tag{28}$$

Let $\{e_i\}(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point. Now taking summation over $i = 1, 2, ..., n$ of the relation (28) for $U = W = e_i$ gives

$$\begin{aligned}[-\frac{n}{n-1}S(V,X) - ng(V,X)] &= [-\frac{1}{n-1}S^2(V,X) + \frac{3-2n}{n-1}S(V,X) \\ &\quad +(2-n)g(V,X)]. \end{aligned} \tag{29}$$

This implies

$$S^2(V,X) = (3-n)S(V,X) + 2(n-1)g(V,X).$$

Here the $(0,2)$-tensor S^2 is defined by $S^2(X,Y) = S(QX,Y)$. This leads to the following:

Theorem 2. *If M be an n-dimensional $(n \geq 3)$ P-Sasakian manifold satisfying the condition $R(X,\xi)\cdot P = P(X,\xi)\cdot R$, then the square S^2 of the Ricci tensor S is the linear combination of the Ricci tensor and the metric tensor g, that is,*

$$S^2(V,X) = (3-n)S(V,X) + 2(n-1)g(V,X).$$

Lemma 1. *[9] Let A be a symmetric $(0,2)$-tensor at a point x of a semi-Riemannian manifold (M,g) of dimension $n > 1$, and let $T = g \barwedge A$ be the Kulkarni-Nomizu product of g and A. Then the relation*

$$T \cdot T = \alpha Q(g,T),\ \alpha \in \mathbb{R}$$

is true at x if and only if the following condition

$$A^2 = \alpha A + \lambda g,\ \lambda \in \mathbb{R}$$

holds at x.

From Theorem 2 and Lemma 1 we have the following:

Corollary 3. *Let (M^n,g) be a P-Sasakian manifold satisfying the condition $R(X,\xi)\cdot P = P(X,\xi)\cdot R$, then $T\cdot T = \alpha Q(g,T)$, where $T = g \barwedge S$ and $\alpha = (3-n)$.*

5 P-Sasakian manifolds satisfying the condition $P\cdot S = 0$

In this section we consider a P-Sasakian manifold satisfying the curvature condition

$$(P(X,Y)\cdot S)(U,V) = 0.$$

By (12) and the above equation we have

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0. \tag{30}$$

Substituting $X = U = \xi$ in the above equation we obtain

$$S(P(\xi, Y)\xi, V) + S(\xi, P(\xi, Y)U) = 0. \tag{31}$$

Making use of (13) and (18) in (31) yields

$$(n-1)\eta(P(\xi, Y)V) = 0. \tag{32}$$

For $n > 1$ we have from (32)

$$\eta(P(\xi, Y)V) = 0. \tag{33}$$

Again using (18) we obtain from the above equation

$$S(Y, V) = -(n-1)g(Y, V), \tag{34}$$

for any vector fields $Y, V \in \chi(M)$. Therefore, the manifold M is an Einstein one. Conversely, if the manifold is an Einstein manifold of the form (34) then it is obvious that $S(P(X, Y)U, V) + S(U, P(X, Y)V) = 0$, for any $X, Y, U, V \in \chi(M)$, that is, $P \cdot S = 0$. Hence, we can state the following:

Theorem 3. *A P-Sasakian manifold $M^n (n > 1)$ satisfies the curvature condition $P \cdot S = 0$ if and only if the manifold is an Einstein one.*

6 Pseudoprojectively flat P-Sasakian manifolds

A P-Sasakian manifold is said to be pseudoprojectively flat [3] if the following condition holds

$$g(P(\phi X, Y)Z, \phi W) = 0. \tag{35}$$

From (7) and (35) we have

$$\begin{aligned} \widetilde{R}(\phi X, Y, Z, \phi W) &= \frac{1}{n-1}[S(Y, Z)g(\phi X, \phi W) \\ &\quad - S(\phi X, Z)g(Y, \phi W)], \end{aligned} \tag{36}$$

where $\widetilde{R}(\phi X, Y, Z, \phi W) = g(R(\phi X, Y)Z, \phi W))$ for all $X, Y, Z, W \in \chi(M)$. Replacing X and W by ϕX and ϕW respectively, we obtain from (36)

$$\begin{aligned} \widetilde{R}(\phi^2 X, Y, Z, \phi^2 W) &= \frac{1}{n-1}[S(Y, Z)g(\phi^2 X, \phi^2 W) \\ &\quad - S(\phi^2 X, Z)g(Y, \phi^2 W)]. \end{aligned} \tag{37}$$

Making use of (1) we get from (37)

$$\begin{aligned} &\widetilde{R}(X, Y, Z, W) - \eta(W)\eta(R(X, Y)Z) \\ &-\eta(X)g(R(\xi, Y)Z, W) + \eta(X)\eta(W)\eta(R(\xi, Y)Z) \\ &= \frac{1}{n-1}[S(Y, Z)\{g(X, W) - \eta(X)\eta(W)\} \\ &-\{S(X, Z) + (n-1)\eta(X)\eta(Z)\}\{g(Y, W) - \eta(Y)\eta(W)\}]. \end{aligned} \tag{38}$$

Setting $X = W = e_i$, where $\{e_i\}(1 \leq i \leq n)$ is an orthonormal basis of the tangent space at any point, in (38) and hence using (15) implies

$$S(Y,Z) + g(Y,Z) - \eta(Y)\eta(Z) = \frac{1}{n-1}[(n-1)S(Y,Z) - S(Y,Z) - (n-1)\eta(Y)\eta(Z)], \tag{39}$$

which implies

$$S(Y,Z) = -(n-1)g(Y,Z),$$

for any $Y, Z \in \chi(M)$. By the above discussions we have the following:

Theorem 4. *A pseudoprojectively flat P-Sasakian manifold is an Einstein manifold.*

7 ϕ-projectively semisymmetric P-Sasakian manifolds

Let M be an n-dimensional $(n > 3)$ ϕ-projectively semisymmetric P-Sasakian manifold. Therefore $P(X,Y) \cdot \phi = 0$ implies

$$(P(X,Y) \cdot \phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0, \tag{40}$$

for any vector fields X, Y and $Z \in \chi(M)$.
From (7) we have

$$P(X,Y)\phi Z = R(X,Y)\phi Z - \frac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y], \tag{41}$$

and

$$\phi P(X,Y)Z = \phi R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y]. \tag{42}$$

Substituting (41) and (42) in (40) yields

$$R(X,Y)\phi Z - \phi R(X,Y)Z - \frac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y - S(Y,Z)\phi X + S(X,Z)\phi Y] = 0. \tag{43}$$

Letting $X = \xi$ in (43) we have

$$R(\xi,Y)\phi Z - \phi R(\xi,Y)Z - \frac{1}{n-1}[S(Y,\phi Z)\xi + S(\xi,Z)\phi Y] = 0. \tag{44}$$

Making use of (15) and (13) in (44) we obtain

$$S(Y,\phi Z)\xi = -(n-1)g(Y,\phi Z)\xi. \tag{45}$$

Taking inner product of (45) with ξ we get

$$S(Y,\phi Z) = -(n-1)g(Y,\phi Z). \tag{46}$$

Replacing Z by ϕZ in (46) and using (1) we have

$$S(Y,Z) = -(n-1)g(Y,Z),$$

which implies that the manifold M is an Einstein one. Therefore from the above discussions we can state the following:

Theorem 5. *An n-dimensional $(n > 3)$ ϕ-projectively semisymmetric P-Sasakian manifold is an Einstein one.*

8 Locally ϕ-projectively symmetric P-Sasakian manifolds

Let M be an n-dimensional locally ϕ-projectively symmetric P-Sasakian manifold. Therefore

$$\phi^2((\nabla_W P)(X,Y)Z) = 0, \tag{47}$$

for any vector fields X, Y, Z and W orthogonal to ξ. Substituting $X = \xi$ in (47) implies

$$\phi^2((\nabla_W P)(\xi,Y)Z) = 0, \tag{48}$$

for any vector fields Y, Z and W orthogonal to ξ. From (18) we have

$$P(\xi,Y)Z = -g(Y,Z)\xi - \frac{1}{n-1}S(Y,Z)\xi. \tag{49}$$

Taking the covariant differentiation along any arbitrary vector field $W \in \chi(M)$ of (49) we obtain

$$\begin{aligned}(\nabla_W P)(\xi,Y)Z &= -g(Y,Z)\nabla_W\xi - \frac{1}{n-1}(\nabla_W S)(Y,Z)\xi \\ &\quad -\frac{1}{n-1}S(Y,Z)\nabla_W\xi,\end{aligned} \tag{50}$$

which implies

$$\begin{aligned}(\nabla_W P)(\xi,Y)Z &= -g(Y,Z)\phi W - \frac{1}{n-1}(\nabla_W S)(Y,Z)\xi \\ &\quad -\frac{1}{n-1}S(Y,Z)\phi W.\end{aligned} \tag{51}$$

Applying ϕ^2 both sides of (51) yields

$$\phi^2((\nabla_W P)(\xi,Y)Z) = -g(Y,Z)\phi W - \frac{1}{n-1}S(Y,Z)\phi W. \tag{52}$$

In view of (48) and (52) we have

$$S(Y,Z)\phi W = -(n-1)g(Y,Z)\phi W. \tag{53}$$

Putting $W = \phi W$ in the above equation and noticing that W is orthogonal to ξ implies

$$S(Y,Z)W = -(n-1)g(Y,Z)W, \tag{54}$$

from which it follows that

$$S(Y,Z) = -(n-1)g(Y,Z).$$

Hence we can state the following:

Theorem 6. *If an n-dimensional P-Sasakian manifold (M^n, g) is locally ϕ-projectively symmetric then the manifold is an Einstein one.*

9 Example of a 5-dimensional P-Sasakian manifold

We consider the 5-dimensional manifold $M = \{(x,y,z,u,v) \in \mathbb{R}^5, (x,y,z,u,v) \neq (0,0,0,0,0)\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$.
We choose the vector fields

$$e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z},\ e_4 = \frac{\partial}{\partial u},\ e_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$

which are linearly independent at each point of M.
Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j;\ i,j = 1,2,3,4,5. \end{cases}$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_5),$$

for any $Z \in \chi(M)$.
Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_1,\ \phi(e_2) = e_2,\ \phi(e_3) = e_3,\ \phi(e_4) = e_4,\ \phi(e_5) = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_5) = 1,\ \phi^2 Z = Z - \eta(Z)e_5$$

and

$$g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U),$$

for any vector fields $Z, U \in \chi(M)$. Thus for $e_5 = \xi$, the structure (ϕ, ξ, η, g) defines an almost paracontact metric structure on M.
Then we have

$$\begin{aligned} &[e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = e_1, \\ &[e_2, e_3] = [e_2, e_4] = 0, [e_2, e_5] = e_2, \\ &[e_3, e_4] = 0, [e_3, e_5] = e_3, [e_4, e_5] = e_4. \end{aligned}$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{55}$$

Taking $e_5 = \xi$ and using (55), we get the following:

$$\begin{aligned}
&\nabla_{e_1} e_1 = -e_5,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = e_1,\\
&\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -e_5,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = e_2,\\
&\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -e_5,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = e_3,\\
&\nabla_{e_4} e_1 = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = -e_5,\ \nabla_{e_4} e_5 = e_4,\\
&\nabla_{e_5} e_1 = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = 0.
\end{aligned}$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensor as follows:

$$\begin{aligned}
&R(e_1,e_2)e_1 = e_2,\ R(e_1,e_2)e_2 = -e_1,\ R(e_1,e_3)e_1 = e_3,\ R(e_1,e_3)e_3 = -e_1,\\
&R(e_1,e_4)e_1 = e_4,\ R(e_1,e_4)e_4 = -e_1,\ R(e_1,e_5)e_1 = e_5,\ R(e_1,e_5)e_5 = -e_1,\\
&R(e_2,e_3)e_2 = e_3,\ R(e_2,e_3)e_3 = -e_2,\ R(e_2,e_4)e_2 = e_4,\ R(e_2,e_4)e_4 = -e_2,\\
&R(e_2,e_5)e_2 = e_5,\ R(e_2,e_5)e_5 = -e_2,\ R(e_3,e_4)e_3 = e_4,\ R(e_3,e_4)e_4 = -e_3,\\
&R(e_3,e_5)e_3 = e_5,\ R(e_3,e_5)e_5 = -e_3,\ R(e_4,e_5)e_4 = e_5,\ R(e_4,e_5)e_5 = -e_4.
\end{aligned}$$

Using the above expressions we have

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = S(e_5,e_5) = -4.$$

Clearly

$$R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\},$$

for any $X, Y, Z \in \chi(M)$, where $k = -1$. Thus the manifold is of constant curvature, which implies that the manifold is an Einstein manifold of the form $S(X,Y) = -4g(X,Y)$. Hence Theorem 3 is verified.

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