Bulletin of the *Transilvania* University of Braşov • Vol 8(57), No. 2 - 2015
Series III: Mathematics, Informatics, Physics, 1-12

DISTORTION BOUNDS FOR A NEW SUBCLASS OF ANALYTIC FUNCTIONS AND THEIR PARTIAL SUMS

Ibrahim AKTAŞ[1] and Halit ORHAN[2]

Abstract

In the present paper, we introduce a new subclass of functions which are analytic in the open unit disk. Also, we obtain coefficient inequalities for functions belonging to this class. Futhermore, we give some results associated with distortions bounds. In addition to that, we investigate lower bounds for partial sums of functions belonging to this class.

2000 *Mathematics Subject Classification:* 30C45.
Key words: $p-valent$ function, starlike function, distortion bounds, partial sums.

1 Introduction

Let A_p denote the class of functions $f(z)$ in the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \tag{1}$$

which are analytic and $p - valent$ in the open unit disk $\Delta = \{z : z \in C, |z| < 1\}$ and $p \in N = \{1, 2, 3, ...\}$. Further, by A and $S^*(\alpha)$, we denote the following classes:

$$A = \left\{ f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1} : f \text{ is analytic in } \Delta \right\}$$

and

$$S^*(\alpha) = \left\{ f \in A : \Re e \left(\frac{zf'(z)}{f(z)} \right) > \alpha, 0 \leq \alpha < 1, z \in \Delta \right\}$$

[1]Corresponding author, Department of Mathematical Engineering, Faculty of Engineering and Natural Science, Gümüşhane University , Gümüşhane-Turkey, e-mail: aktasibrahim38@gmail.com
[2]Department of Mathematics, Faculty of Science, Ataturk University, Erzurum-Turkey, e-mail: orhanhalit607@gmail.com

respectively. We know that, $S^*(\alpha)$ is a familiar subclass of A consisting of functions which are *starlike of order* α in Δ. For the function classes $S^*(\alpha)$ was given coefficient inequality by Silverman [6] in 1975. In 1991, Altıntas [1] gave coefficient inequality for a subclass of certain starlike functions with negative coefficients. Owa, Ochiai and Srivastava [4] introduced the subclass $M(\alpha)$ of the class A and they proved some theorems relations with coeficient inequality for this class in 2006. Kamali [7] defined a new subclass $M(\alpha,\lambda,\Omega)$ of the class A and he investigated some properties for this subclass in 2013.

For the functions $f(z)$ belonging to the class A_p, we define the operator D as follows:

$$\begin{aligned}
D^0(f(z)) &= f(z)\\
D^1(f(z)) &= D(f(z)) = \frac{z}{p}f^{'}(z) = z^p + \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)a_{k+p}z^{k+p}\\
D^2(f(z)) &= D(D^1(f(z))) = z^p + \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^2 a_{k+p}z^{k+p}\\
&\vdots\\
D^{\Omega}(f(z)) &= D(D^{\Omega-1}(f(z))) = z^p + \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega} a_{k+p}z^{k+p}.
\end{aligned}$$

We note that, for $p=1$ we obtain *Salagean differential* operator which was defined by Salagean [5].

In this work, we introduce a new subclass $M_p(\alpha,\lambda,\Omega)$ of the class A_p consisting of functions $f(z)$ such that

$$\left|\frac{(1-\lambda)D^{\Omega}(f(z))+\lambda D^{\Omega+1}(f(z))}{(1-\lambda)\frac{z}{p}(D^{\Omega}(f(z)))'+\lambda\frac{z}{p}(D^{\Omega+1}(f(z)))'}-\frac{p+\lambda}{2\alpha}\right|<\frac{p+\lambda}{2\alpha} \tag{2}$$

$z\in\Delta, 0<\alpha<1, 0\leq\lambda<1, \Omega\in N_0=\{0,1,2,3,...\}, p\in N=\{1,2,3,...\}.$

2 The Coefficient Inequality for the class $M_p(\alpha,\lambda,\Omega)$

Theorem 1. *Let* $0<\alpha<1, 0\leq\lambda<1, \Omega\in N_0=\{0,1,2,3,...\}$ *and* $p\in N=\{1,2,3,...\}$. *If* $f(z)\in A_p$ *satisfies the following coefficient inequality:*

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left\{\left|2\alpha-(p+\lambda)\left(\frac{k+p}{p}\right)\right|+(p+\lambda)\left(\frac{k+p}{p}\right)\right\}\times$$
$$\times\left(1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right)|a_{k+p}|$$

$$\leq (p+\lambda) - |2\alpha - (p+\lambda)| = \{ \begin{array}{cc} 2\alpha & ;\ \ 0 < \alpha \leq \frac{p+\lambda}{2} \\ 2\,(p+\lambda-\alpha) & ; \frac{p+\lambda}{2} \leq \alpha < p+\lambda \end{array} \tag{3}$$

then $f(z) \in M_p\,(\alpha, \lambda, \Omega)$.

Proof. In view of condition (2), we should show that

$$\left| \left(\frac{2\alpha}{p+\lambda} \right) \frac{(1-\lambda) D^{\Omega}(f(z)) + \lambda D^{\Omega+1}(f(z))}{(1-\lambda)\frac{z}{p}(D^{\Omega}(f(z)))' + \lambda \frac{z}{p}(D^{\Omega+1}(f(z)))'} - 1 \right| < 1. \tag{4}$$

We can see that

$$\left| \left(\frac{2\alpha}{p+\lambda} \right) \frac{(1-\lambda) D^{\Omega}(f(z)) + \lambda D^{\Omega+1}(f(z))}{(1-\lambda)\frac{z}{p}(D^{\Omega}(f(z)))' + \lambda \frac{z}{p}(D^{\Omega+1}(f(z)))'} - 1 \right|$$

$$= \left| \frac{A+B}{(p+\lambda)\,z^p + \sum\limits_{k=1}^{\infty} (p+\lambda) \left(\frac{k+p}{p}\right)^{\Omega+1} \left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right] a_{k+p} z^{k+p}} \right| =: T,$$

where

$$A := [2\alpha - (p+\lambda)]\, z^p$$
$$B := \sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} \left[2\alpha - (p+\lambda)\left(\frac{k+p}{p}\right)\right] \left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right] a_{k+p} z^{k+p}.$$

Then

$$T \leq$$
$$\frac{|2\alpha-(p+\lambda)| + \sum\limits_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} \left|2\alpha-(p+\lambda)\left(\frac{k+p}{p}\right)\right| \left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right] |a_{k+p}|\,|z|^k}{(p+\lambda) - \sum\limits_{k=1}^{\infty} (p+\lambda)\left(\frac{k+p}{p}\right)^{\Omega+1} \left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right] |a_{k+p}|\,|z|^k}$$

$$< \frac{|2\alpha-(p+\lambda)| + \sum\limits_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} \left|2\alpha-(p+\lambda)\left(\frac{k+p}{p}\right)\right| \left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right] |a_{k+p}|}{(p+\lambda) - \sum\limits_{k=1}^{\infty} (p+\lambda)\left(\frac{k+p}{p}\right] |a_{k+p}|} \tag{5}$$

With coefficient inequality (3), we can obtain

$$\sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} \left|2\alpha-(p+\lambda)\left(\frac{k+p}{p}\right)\right| \left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right] |a_{k+p}|$$

$$\leq (p+\lambda) - |2\alpha - (p+\lambda)| - \sum_{k=1}^{\infty} (p+\lambda) \left(\frac{p+k}{p}\right)^{\Omega+1} \left[1-\lambda+\lambda\left(\frac{p+k}{p}\right)\right] |a_{p+k}| . \tag{6}$$

By using the inequality (6) in (5), we get

$$\left| \left(\frac{2\alpha}{p+\lambda}\right) \frac{(1-\lambda)D^{\Omega}(f(z)) + \lambda D^{\Omega+1}(f(z))}{(1-\lambda)\frac{z}{p}(D^{\Omega}(f(z)))' + \lambda\frac{z}{p}(D^{\Omega+1}(f(z)))'} - 1 \right|$$
$$< \frac{C-D}{(p+\lambda) - \sum\limits_{k=1}^{\infty} (p+\lambda) \left(\frac{k+p}{p}\right)^{\Omega+1} \left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right] |a_{k+p}|} = 1,$$

where

$$C := |2\alpha - (p+\lambda)| + (p+\lambda) - |2\alpha - (p+\lambda)|$$
$$D := \sum_{k=1}^{\infty} (p+\lambda) \left(\frac{k+p}{p}\right)^{\Omega+1} \left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right] |a_{k+p}| ,$$

or

$$\left| \frac{(1-\lambda)D^{\Omega}(f(z)) + \lambda D^{\Omega+1}(f(z))}{(1-\lambda)\frac{z}{p}(D^{\Omega}(f(z)))' + \lambda\frac{z}{p}(D^{\Omega+1}(f(z)))'} - \frac{p+\lambda}{2\alpha} \right| < \frac{p+\lambda}{2\alpha}$$

that is $f(z) \in M_p(\alpha,\lambda,\Omega)$. □

Theorem 2. *If $f(z) \in M_p(\alpha,\lambda,\Omega)$, then*

$$\Re e \left\{ \frac{(1-\lambda)\frac{z}{p}(D^{\Omega}(f(z)))' + \lambda\frac{z}{p}(D^{\Omega+1}(f(z)))'}{(1-\lambda)D^{\Omega}(f(z)) + \lambda D^{\Omega+1}(f(z))} \right\} > \frac{\alpha}{p+\lambda}.$$

Proof. Let $f(z) \in M_p(\alpha,\lambda,\Omega)$, $H(z) = \frac{\frac{z}{p}F'(z)}{F(z)}$ and $F(z) = (1-\lambda)D^{\Omega}(f(z)) + \lambda D^{\Omega+1}(f(z))$. In this case, we can write that

$$\left| \frac{1}{H(z)} - \frac{p+\lambda}{2\alpha} \right| < \frac{p+\lambda}{2\alpha}.$$

After some calculations, we get

$$\left| \frac{1}{H(z)} - \frac{p+\lambda}{2\alpha} \right| < \frac{p+\lambda}{2\alpha} \Longrightarrow \left| \frac{1}{H(z)} - \frac{p+\lambda}{2\alpha} \right|^2 < \left(\frac{p+\lambda}{2\alpha}\right)^2$$

$$\left| \frac{2\alpha - (p+\lambda) H(z)}{2\alpha H(z)} \right|^2 < \left(\frac{p+\lambda}{2\alpha}\right)^2 \Longrightarrow (2\alpha - (p+\lambda) H(z))^2 < (p+\lambda)^2 |H(z)|^2$$

$$\Longrightarrow (2\alpha - (p+\lambda) H(z)) \overline{(2\alpha - (p+\lambda) H(z))} < (p+\lambda)^2 H(z)\overline{(H(z))}$$

$$\Longrightarrow 4\alpha^2 - 2\alpha\,(p+\lambda)\,\overline{H(z)} - 2\alpha\,(p+\lambda)\,H(z) + (p+\lambda)^2\,H(z)\overline{(H(z))}$$
$$< (p+\lambda)^2\,H(z)\overline{(H(z))}$$

$$\Longrightarrow 4\alpha^2 < 2\alpha\,(p+\lambda)\left\{H(z) + \overline{H(z)}\right\}. \tag{7}$$

We know that for each $z \in C$,

$$z + \overline{z} = 2\Re ez. \tag{8}$$

If we consider equality (8) in inequality (7), then we obtain

$$2\alpha < (p+\lambda)\,2\Re eH(z) \Longrightarrow \Re eH(z) > \frac{\alpha}{p+\lambda}.$$

This is desired. □

Remark 1. *Let $p = 1$. If $f(z) \in M_1\left(\alpha, \lambda, \Omega\right)$, then $F(z) \in S^*(\frac{\alpha}{1+\lambda})$.*

3 Some Results Associated with Distortion Bounds

In recent years, S. Owa, K. Ochiai and H. M. Srivastava [4] have introduced the integro-differential operator for an analytic function $f(z)$ which is denoted in the form of $I_s f(z)$ and defined as shown below:

$$I_{-1}f(z) = f^{'}(z),\ \ I_0 f(z) = f(z)$$

and

$$I_s f(z) = \int_0^z I_{s-1} f(t)dt$$

for $s \in N = \{1, 2, 3, ...\}$.

Let us denote by $M_p^*(\alpha, \lambda, \Omega)$ the subclass of the class $M_p\left(\alpha, \lambda, \Omega\right)$ which satisfies the coefficient inequality (3) for some α and which consists of the $f(z) \in M_p\left(\alpha, \lambda, \Omega\right)$.

By definition in (1), we can write

$$I_s f(z) = \frac{p!}{(s+p)!} z^{s+p} + \sum_{k=1}^{\infty} \frac{(p+k)!}{(s+p+k)!} a_{p+k} z^{s+p+k} \tag{9}$$

Now, by integro-differential operator, we can obtain the following results about the distortion bounds for the functions belonging to the subclass $M_p^*(\alpha, \lambda, \Omega)$.

Theorem 3. *If $f(z) \in M_p^*(\alpha, \lambda, \Omega)$, then we have the following inequality:*

$$\frac{p!}{(s+p)!}\left|z\right|^{p+s} -$$
$$-\frac{(p+1)!\left\{(p+\lambda) - \left|2\alpha - (p+\lambda)\right|\right\}}{(s+p+1)!\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha - (p+\lambda)\left(\frac{p+1}{p}\right)\right| + (p+\lambda)\left(\frac{p+1}{p}\right)\right\}}\left|z\right|^{p+s+1}$$
$$\leq \left|I_s f(z)\right|$$

$$\leq \frac{p!}{(s+p)!}|z|^{p+s}$$
$$+\frac{(p+1)!\left\{(p+\lambda)-|2\alpha-(p+\lambda)|\right\}}{(s+p+1)!\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}|z|^{p+s+1}$$

for $z\in\Delta, s\in N\cup\{-1,0\}, 0<\alpha<1, 0\leq\lambda<1$ and $\Omega\in N_0=N\cup\{0\}$.

Proof. After taking the absolute value in both sides of equality (9) and applying the triangle inequality, we can denote that

$$\begin{aligned}|I_sf(z)| &= \left|\frac{p!}{(s+p)!}z^{s+p}+\sum_{k=1}^{\infty}\frac{(p+k)!}{(s+p+k)!}a_{k+p}z^{s+p+k}\right| \qquad (10)\\ &\leq \frac{p!}{(s+p)!}|z|^{s+p}+|z|^{s+p+1}\sum_{k=1}^{\infty}\frac{(p+k)!}{(s+p+k)!}|a_{k+p}|\,z^{k-1}\\ &< \frac{p!}{(s+p)!}|z|^{s+p}+|z|^{s+p+1}\sum_{k=1}^{\infty}\frac{(p+k)!}{(s+p+k)!}|a_{k+p}|\,.\end{aligned}$$

Besides, we can write

$$\begin{aligned}&\frac{1}{(p+1)!}(s+p+1)!\left(\frac{p+1}{p}\right)^{\Omega}\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}\times\\ &\quad\times\left(1+\frac{\lambda}{p}\right)\sum_{k=1}^{\infty}\frac{(p+k)!}{(s+p+k)!}|a_{k+p}|\\ &\leq\left(\frac{p+1}{p}\right)^{\Omega}\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}\left(1+\frac{\lambda}{p}\right)\sum_{k=1}^{\infty}|a_{k+p}|\\ &\leq\sum_{k=1}^{\infty}\left(\frac{p+1}{p}\right)^{\Omega}\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}\times\\ &\quad\times\left(1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right)|a_{k+p}|\\ &\leq(p+\lambda)-|2\alpha-(p+\lambda)|\end{aligned}$$

or

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{(p+k)!}{(s+p+k)!}|a_{k+p}|\\ &\quad\leq\frac{(p+1)!\left\{(p+\lambda)-|2\alpha-(p+\lambda)|\right\}}{(s+p+1)!\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}.\end{aligned}$$

Using the last step of inequality in (10), we obtain

$$
\begin{aligned}
&|I_s f(z)| \leq \frac{p!}{(s+p)!}\,|z|^{s+p} \\
&+\frac{(p+1)!\left\{(p+\lambda)-|2\alpha-(p+\lambda)|\right\}}{(s+p+1)!\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}\,|z|^{s+p+1}
\end{aligned}
\tag{11}
$$

As similar implementations above, we can write

$$
\begin{aligned}
&|I_s f(z)| \geq \frac{p!}{(s+p)!}\,|z|^{s+p} \\
&-\frac{(p+1)!\left\{(p+\lambda)-|2\alpha-(p+\lambda)|\right\}|z|^{s+p+1}}{(s+p+1)!\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}
\end{aligned}
\tag{12}
$$

By joining (11) and (12), we obtain

$$
\begin{aligned}
&\frac{p!}{(s+p)!}\,|z|^{s+p} - \\
&-\frac{(p+1)!\left\{(p+\lambda)-|2\alpha-(p+\lambda)|\right\}|z|^{s+p+1}}{(s+p+1)!\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}} \\
&\leq |I_s f(z)| \\
&\leq \frac{p!}{(s+p)!}\,|z|^{s+p} \\
&+\frac{(p+1)!\left\{(p+\lambda)-|2\alpha-(p+\lambda)|\right\}|z|^{s+p+1}}{(s+p+1)!\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}
\end{aligned}
$$

$\square$

Remark 2. *If we choose $\lambda = \Omega = 0$ and $p = 1$ in Theorem 3, then we have the same results given by Owa, Ochiai and Srivastava [4].*

Setting $s = -1, 0, 1$ in Theorem 3, we get the following Corollary.1.

Corollary 1. *If $f(z) \in M_p^*(\alpha, \lambda, \Omega)$, then the following inequalities are obtained.*

$$
p\,|z|^{p-1}-\frac{(p+1)\left\{(p+\lambda)-|2\alpha-(p+\lambda)|\right\}}{\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}\,|z|^{p} \leq \left|f'(z)\right|
$$

$$
\leq p\,|z|^{p-1}+\frac{(p+1)\left\{(p+\lambda)-|2\alpha-(p+\lambda)|\right\}}{\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}\,|z|^{p}
$$

for $s=-1$,

$$|z|^p-\frac{\{(p+\lambda)-|2\alpha-(p+\lambda)|\}}{\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}|z|^{p+1}\leq|f(z)|$$

$$\leq|z|^p+\frac{\{(p+\lambda)-|2\alpha-(p+\lambda)|\}}{\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}|z|^{p+1}$$

for $s=0$ and

$$\frac{|z|^{p+1}}{p+1}-\frac{\{(p+\lambda)-|2\alpha-(p+\lambda)|\}|z|^{p+2}}{(p+2)\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}}$$

$$\leq|I_1f(z)|$$

$$\leq\frac{|z|^{p+1}}{p+1}+\frac{\{(p+\lambda)-|2\alpha-(p+\lambda)|\}|z|^{p+2}}{(p+2)\left(\frac{p+1}{p}\right)^{\Omega}\left(1+\frac{\lambda}{p}\right)\left\{\left|2\alpha-(p+\lambda)\left(\frac{p+1}{p}\right)\right|+(p+\lambda)\left(\frac{p+1}{p}\right)\right\}},$$

for $s=1$.

Remark 3. *Putting $p=1$ in Corollary 1, we get the same result given by Kamali [7].*

4 Partial Sums for the Class $M_p(\alpha,\lambda,\Omega)$

In this section following the earlier works by Silverman [8], Deniz and Orhan [9], N. C. Cho. at al. [10] and others (see also [11], [12]) on partial sums of analytic functions, we study the ratio of real parts of functions involving (1) and their sequence of partial sums defined by

$$\begin{aligned} f_n(z) &= z^p &&; n=k+p-1 \\ f_n(z) &= z^p+\sum_{k=1}^{n}a_{k+p}z^{k+p} &&; n=k+p,k+p+1,... \end{aligned} \tag{13}$$

and determine sharp lower bounds for

$$\Re e\left\{\frac{f(z)}{f_n(z)}\right\},\Re e\left\{\frac{f_n(z)}{f(z)}\right\}.$$

Theorem 4. *Let $f(z)\in M_p(\alpha,\lambda,\Omega)$ and $f_n(z)$ be given by (1) and (13), respectively. Suppose also that*

$$\sum_{k=1}^{\infty}\phi_k(p,\lambda,\alpha,\Omega)|a_{k+p}|\leq\delta$$

where

$$\begin{aligned}\phi_k &= \phi_k(p,\lambda,\alpha,\Omega)\\ &= \left(\frac{k+p}{p}\right)^{\Omega}\left\{\left|2\alpha-(p+\lambda)\left(\frac{k+p}{p}\right)\right|+(p+\lambda)\left(\frac{k+p}{p}\right)\right\}\times\\ &\quad\times\left[1-\lambda+\lambda\left(\frac{k+p}{p}\right)\right]\end{aligned}$$

and $\delta=\delta(p,\lambda,\alpha)=(p+\lambda)-|2\alpha-(p+\lambda)|$.

Then, we have

$$\Re e\left\{\frac{f(z)}{f_n(z)}\right\}\geq\frac{\phi_{n+1}-\delta}{\phi_{n+1}} \tag{14}$$

and

$$\Re e\left\{\frac{f_n(z)}{f(z)}\right\}\geq\frac{\phi_{n+1}}{\phi_{n+1}+\delta}. \tag{15}$$

This results are sharp for every n *with the extremal functions given by*

$$f(z)=z^p+\frac{\delta}{\phi_{n+1}}z^{n+p+1}. \tag{16}$$

Proof. In order to prove (14), it suffices to show that

$$\frac{\varphi_{n+1}}{\delta}\left\{\frac{f(z)}{f_n(z)}-\frac{\varphi_{n+1}-\delta}{\varphi_{n+1}}\right\}\prec\frac{1+z}{1-z}\ (z\in\Delta). \tag{17}$$

We can write

$$\begin{aligned}\frac{\varphi_{n+1}}{\delta}\left\{\frac{f(z)}{f_n(z)}-\frac{\varphi_{n+1}-\delta}{\varphi_{n+1}}\right\} &= \frac{\varphi_{n+1}}{\delta}\left\{\frac{z^p+\sum\limits_{k=1}^{\infty}a_{k+p}z^{k+p}}{z^p+\sum\limits_{k=1}^{n}a_{k+p}z^{k+p}}-\frac{\varphi_{n+1}-\delta}{\varphi_{n+1}}\right\}\\ &= \frac{1+\frac{\varphi_{n+1}}{\delta}\sum\limits_{k=n+1}^{\infty}a_{k+p}z^k+\sum\limits_{k=1}^{n}a_{k+p}z^k}{1+\sum\limits_{k=1}^{n}a_{k+p}z^k}=\frac{1+w(z)}{1-w(z)}.\end{aligned}$$

Then

$$w(z)=\frac{\frac{\varphi_{n+1}}{\delta}\sum\limits_{k=n+1}^{\infty}a_{k+p}z^k}{2+2\sum\limits_{k=1}^{n}a_{k+p}z^k+\frac{\varphi_{n+1}}{\delta}\sum\limits_{k=n+1}^{\infty}a_{k+p}z^k}.$$

Obviously $w(0)=0$ and

$$|w(z)|\leq\frac{\frac{\varphi_{n+1}}{\delta}\sum\limits_{k=n+1}^{\infty}|a_{k+p}|}{2-2\sum\limits_{k=1}^{n}|a_{k+p}|-\frac{\varphi_{n+1}}{\delta}\sum\limits_{k=n+1}^{\infty}|a_{k+p}|}.$$

Now $|w(z)| \leq 1$ if and only if

$$\frac{\varphi_{n+1}}{\delta}\sum_{k=n+1}^{\infty}|a_{k+p}|+\sum_{k=1}^{n}|a_{k+p}|\leq 1 \tag{18}$$

It is suffices to show that the left hand side of (18) is bounded above by $\sum\limits_{k=1}^{\infty}\frac{\varphi_k|a_{k+p}|}{\delta}$ which is equivalent to

$$\sum_{k=1}^{n}(\varphi_k-\delta)|a_{k+p}|+\sum_{k=n+1}^{\infty}(\varphi_k-\varphi_{n+1})|a_{k+p}|\geq 0.$$

To see that the function f given by (16) gives the sharp result, we obseve for $z=|z|\,e^{\frac{\pi i}{n+1}}$ that

$$\frac{f(z)}{f_n(z)}=1+\frac{\delta}{\varphi_{n+1}}z^{n+1}=1+\frac{\delta}{\varphi_{n+1}}\left(|z|\,e^{\frac{\pi i}{n+1}}\right)^{n+1}\to\frac{\phi_{n+1}-\delta}{\phi_{n+1}}.$$

We thus complete the proof of inequality (14). $\square$

The proof of inequality (15) can be made similar to that of (14), here we choose to omit the analogous details.

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