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# ON THE CONVERGENCE ORDER FOR SOME METHODS OF SIMULTANEOUSLY ROOT FINDING FOR POLYNOMIALS USING A COMPUTER ALGEBRA SYSTEM

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**Abstract**

In this paper several methods for finding the roots of a polynomial simultaneously are revised, together with a general method for proving their convergence. The theorem used states a lower bound of the convergence order, too. The conditions of this theorem may be easily verified using a Computer Algebra System, like Mathematica. The verification is carried out on a particular polynomial.

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## 1 Introduction

Several methods are known to compute simultaneously the roots of a polynomial [3], [5], [9], [10]. The direct proofs of the convergence are usualy very technical. Conditions of the initial approximations to ensure the safe convergence of these methods are known, too.

Our purpose is to point out a simple procedure to verify the convergence order of a method for finding the roots of a polynomial simultaneously using a Computer Algebra System (CAS). We remind a result enabling to establish the convergence of such methods and a lower bound of their convergence order [3].

Let $\lim_{k\to\infty} x_k = x_*$. If $\lim_{k\to\infty} \frac{\|x_{k+1}-x_*\|}{\|x_k-x_*\|^r} = \rho$, with $0 < \rho < \infty$, then $r$ is the convergence order of the sequence $(x_k)_{k\in\mathbb{N}}$.

Applied to several methods for finding the roots of a polynomial simultaneously, the procedure to verify their convergence order is carried out with *Mathematica.*

## 2 The convergence framework

Let $\Omega \in \mathbb{C}^n$ be an open convex subset, $T : \Omega \to \mathbb{C}^n$, $T(z) = (T_1(z), \ldots, T_n(z))^T$ an $m$ times differentiable operator such that $T^{(m)}(z)$ is continuous and the sequence $(z^{(k)})_{k\in\mathbb{N}}$

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defined by

$$z^{(k+1)} = T(z^{(k)}), \quad z^{(k)} = (z_1^{(k)}, z_2^{(k)}, \ldots, z_n^{(k)})^T \qquad \Leftrightarrow \tag{1}$$
$$\Leftrightarrow \qquad z_i^{(k+1)} = T_i(z^{(k)}), \forall \quad i \in \{1, 2, \ldots, n\}, \ k \in \mathbb{N}.$$

In $\mathbb{C}^n$ we shall use the *max* norm $\|z\| = \max\{|z_1|, |z_2|, \ldots, |z_n|\}$.

The main ingredient of the convergence theorem is this well known result

**Theorem 1.** *[1] Let $X, Y$ be normed spaces, $D$ an open convex subset of $X$ and $T : D \to Y$ an $m$ times Frèchet differentiable operator. Then, for any $x, y \in D$*

$$\|T(y) - T(x) - \sum_{j=1}^{m-1} \frac{1}{j!} T^{(j)}(x) \underbrace{(y-x)\ldots(y-x)}_{j \ times} \| \le \frac{\|y-x\|^m}{m!} \sup_{\zeta \in [x,y]} \|T^{(m)}(\zeta)\|. \tag{2}$$

Using this result, we have

**Theorem 2.** *Let $\alpha \in \Omega$. If*

1. $T(\alpha) = \alpha$,
2. $T'(\alpha) = T''(\alpha) = \ldots = T^{(m-1)}(\alpha) = 0$

*then there exists $r > 0$ such that for any $z^{(0)} \in \mathbb{C}^n$, $\|z^{(0)} - \alpha\| < r$, the sequence $z^{(k+1)} = T(z^{(k)})$, $k \in \mathbb{N}$, (1) converges to $\alpha$.*

*Proof.* Let $r_0 > 0$ be such that $V_0 = \{z \in \mathbb{C}^n : \|z - \alpha\| \le r_0\} \subset \Omega$ and $C_0 = \max_{z \in V_0} \|T^{(m)}(z)\|$.

There exists $0 < r \le r_0$ such that

$$\frac{C_0 r^m}{m!} < r \quad \Leftrightarrow \quad \left(\frac{C_0}{m!}\right)^{\frac{1}{m-1}} r < 1.$$

We denote $V = \{z \in \mathbb{C}^n : \|z - \alpha\| \le r\}$. If $z \in V$, then (2) and the present hypothesys implies

$$\|T(z) - \alpha\| = \|T(z) - T(\alpha) - \sum_{j=1}^{m-1} \frac{1}{j!} T^{(j)}(\alpha) \underbrace{(z-\alpha)\ldots(z-\alpha)}_{j \text{ times}} \| \le$$
$$\le \frac{1}{m!} \|z - \alpha\|^m \sup_{\zeta \in [\alpha, z]} \|T^{(m)}(\zeta)\| \le \frac{C_0 r^m}{m!} < r,$$

thus $T(z) \in V$.

For $z = z^{(k)}$ from the above relations we obtain

$$\|z^{(k+1)} - \alpha\| = \|T(z^{(k)}) - \alpha\| \le \frac{C_0}{m!} \|z^{(k)} - \alpha\|^m. \tag{3}$$

Using recursively the inequality (3), we find

$$\|z^{(k)} - \alpha\| \le \frac{C_0}{m!} \|z^{(k-1)} - \alpha\|^m \le \frac{C_0}{m!} (\frac{C_0}{m!} \|z^{(k-2)} - \alpha\|^m)^m =$$

$$= (\frac{C_0}{m!})^{1+m}\|z^{(k-2)} - \alpha\|^{m^2} \leq \ldots \leq (\frac{C_0}{m!})^{1+m+\ldots+m^{k-1}}\|z^{(0)} - \alpha\|^{m^k} <$$

$$< (\frac{C_0}{m!})^{\frac{m^k}{m-1}}\|z^{(0)} - \alpha\|^{m^k} \leq \left((\frac{C_0}{m!})^{\frac{1}{m-1}} r\right)^{m^k} \to 0,$$

for $k \to \infty$. □

From the inequality (3) it results that the convergence order of the sequence $(z^{(k)})_{k\in\mathbb{N}}$ is at least $m$.

## 3 Simultaneous root finding methods for polynomials

Let $P \in \mathbb{C}[X],\ P(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n$, be the polynomial having all the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ simple. We denote by $\alpha$ the vector whose components are the roots of $P$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$.

There exists a neighbourhood of $\alpha$ in $\mathbb{C}^n$, such that the components of any element of that neighborhood are distinct complex numbers.

Denoting $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T$, $\mathbf{z}$ will be a vector from $\mathbb{C}^n$ while $z$ will be a complex number.
If $z_1, \ldots, z_n$ are complex numbers then

$$u(z) = \prod_{j=1}^{n}(z - z_j), \qquad u_i(z) = \frac{u(z)}{z - z_i} = \prod_{\substack{j=1\\ j\neq i}}^{n}(z - z_j) \qquad Q_i(\mathbf{z}) = \prod_{\substack{j=1\\ j\neq i}}^{n}(z_i - z_j).$$

In order to find the roots of the polynomial $P$ simultaneously, i.e. to solve the equation

$$P(z) = 0, \tag{4}$$

there are considered sequences of type (1).

We revisit some methods of this kind.

**Durand-Kerner's method** [3][10]**.** The equality $P(z) = (z - \alpha_1)\ldots(z - \alpha_n)$ is rewritten as

$$z - \alpha_i = \frac{P(z)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z - \alpha_j)} \quad \text{or} \quad \alpha_i = z - \frac{P(z)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z - \alpha_j)}. \tag{5}$$

If $z^{(k)} = (z_1^{(k)}, z_2^{(k)}, \ldots, z_n^{(k)})^T$ approximates $\alpha$, then substituting in the right side of (5) the components of $\alpha$ with the corresponding components of $z^{(k)}$, the formula (5) suggests the recurrences

$$z_i^{(k+1)} = z_i^{(k)} - \frac{P(z_i^{(k)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i^{(k)} - z_j^{(k)})} = z_i^{(k)} - \frac{P(z_i^{(k)})}{Q_i(z^{(k)})}, \qquad i \in \{1, 2, \ldots, n\},\ k \in \mathbb{N}.$$

The function $T_i(\mathbf{z})$ is

$$T_i(\mathbf{z}) = z_i - \frac{P(z_i)}{Q_i(\mathbf{z})}.$$

It is obvious that $T_i(\alpha) = \alpha_i$. We compute the partial derivates of the function $T_i(\mathbf{z})$.

$$\frac{\partial T_i(\mathbf{z})}{\partial z_i} = 1 - \frac{P'(z_i)}{Q_i(\mathbf{z})} + \frac{P(z_i)}{Q_i^2(\mathbf{z})} \frac{\partial Q_i(\mathbf{z})}{\partial z_i}.$$

Because $P'(\alpha_i) = \prod_{\substack{j=1\\j\neq i}}^n (\alpha_i - \alpha_j) = Q_i(\alpha)$, it results that $\frac{\partial T_i(\alpha)}{\partial z_i} = 0$.

For $i \neq j$,

$$\frac{\partial T_i(\mathbf{z})}{\partial z_j} = \frac{P(z_i)}{Q_i^2(\mathbf{z})} \frac{\partial Q_i(\mathbf{z})}{\partial z_j},$$

hence $\frac{\partial T_i(\alpha)}{\partial z_j} = 0$.

Consequently $T'(\alpha) = 0$, and the convergence order of the sequence $(z^{(k)})_{k\in\mathbb{N}}$ is at least 2.

In [2], the method is deduced using Newton's method to solve the nonlinear equations arrising from the Viète's formulas.

If we substitute, on the right hand side of (5), $\alpha_j$ with $z_j^{(k)} - \frac{P(z_j^{(k)})}{Q_j(z^{(k)})}$ then it rezults the improved Durand-Kerner method, [6], [7]

$$z_i^{(k+1)} = z_i^{(k)} - \frac{P(z_i^{(k)})}{\prod_{\substack{j=1\\j\neq i}}^n \left( z_i^{(k)} - z_j^{(k)} + \frac{P(z_j^{(k)})}{Q_j(z^{(k)})} \right)}, \qquad i \in \{1, 2, \ldots, n\},\ k \in \mathbb{N}.$$

**Ehrlich's method** [3]**.** Let $z_1, \ldots, z_n$ be distinct complex numbers. In order to compute $\alpha_i$, Newton's method is applied to the equation $\frac{P(z)}{u_i(z)} = 0$. We have

$$\left(\frac{P(z)}{u_i(z)}\right)' = \frac{P'(z)}{u_i(z)} - \frac{P(z)}{u_i(z)} \frac{u_i'(z)}{u_i(z)} = \frac{P'(z)}{u_i(z)} - \frac{P(z)}{u_i(z)} \sum_{\substack{j=1\\j\neq i}}^n \frac{1}{z - z_j},$$

For $z = z_i$, we assume that $P'(z_i) = u_i(z_i)$ – valid if $z_i = \alpha_i, \forall i$ – and then

$$\left(\frac{P(z)}{u_i(z)}\right)' |_{z=z_i} \approx 1 - \frac{P(z_i)}{u_i(z_i)} \sum_{\substack{j=1\\j\neq i}}^n \frac{1}{z_i - z_j} = 1 - \frac{P(z_i)}{Q_i(\mathbf{z})} \sum_{\substack{j=1\\j\neq i}}^n \frac{1}{z_i - z_j}.$$

Newton's method becomes

$$z_i^{(k+1)} = z_i^{(k)} - \frac{\frac{P(z_i^{(k)})}{Q_i(z^{(k)})}}{1 - \frac{P(z_i^{(k)})}{Q_i(z^{(k)})} \sum_{\substack{j=1\\j\neq i}}^n \frac{1}{z_i^{(k)} - z_j^{(k)}}} = z_i^{(k)} - \frac{P(z_i^{(k)})}{Q_i(z^{(k)}) - P(z_i^{(k)}) \sum_{\substack{j=1\\j\neq i}}^n \frac{1}{z_i^{(k)} - z_j^{(k)}}},$$

$i \in \{1,\ldots,n\},\ k \in \mathbb{N}$ and where $z^{(k)} = (z_1^{(k)}, z_2^{(k)}, \ldots, z_n^{(k)})$.

**Nourein's method** [5],[6]**.** Again, let $z_1,\ldots,z_n$ be distinct complex numbers. The $n-1$ degree polynomial $P(z) - u(z)$ is equal to the Lagrange interpolating polynomial $L(P_{n-1}; z_1,\ldots,z_n; P-u)(z) = L(P_{n-1}; z_1,\ldots,z_n; P)(z)$

$$P(z) - u(z) = L(P_{n-1}; z_1,\ldots,z_n; P)(z) = \sum_{j=1}^{n} P(z_j)\frac{u(z)}{(z-z_j)u'(z_j)}.$$

For $z = \alpha_i$ it results

$$-1 = \frac{P(z_i)}{(\alpha_i - z_i)u'(z_i)} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{P(z_j)}{(\alpha_i - z_j)u'(z_j)}$$

from where we find

$$\alpha_i - z_i = -\frac{\frac{P(z_i)}{u_i(z_i)}}{1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{P(z_j)}{(\alpha_i - z_j)u'(z_j)}}.$$

Denoting $\triangle z_i = \alpha_i - z_i$, the above relation may be rewritten as

$$\triangle z_i = -\frac{\frac{P(z_i)}{u_i(z_i)}}{1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{P(z_j)}{(z_i + \triangle z_i - z_j)u'(z_j)}}. \tag{6}$$

$\triangle z_i$ appears both on the left and right side of (6).

If on the right hand side of (6) we neglect $\triangle z_i$ then using the same reasoning as it was done at the Durand-Kerner's method we obtain the recurrences [5]

$$z_i^{(k+1)} = z_i^{(k)} - \frac{\frac{P(z_i^{(k)})}{Q_i(z^{(k)})}}{1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{P(z_j^{(k)})}{(z_i^{(k)} - z_j^{(k)})Q_j(z^{(k)})}}, \qquad i \in \{1,\ldots,n\},\ k \in \mathbb{N}.$$

Another approach is to substitute on the right hand side of (6) $\triangle z_i$ with $-\frac{P(z_i)}{u'(z_i)}$, [6], [7]. It results the recurrences (the Nourein's improved method)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{\frac{P(z_i^{(k)})}{Q_i(z^{(k)})}}{1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{P(z_j^{(k)})}{(z_i^{(k)} - \frac{P(z_i^{(k)})}{Q_i(z^{(k)})} - z_j^{(k)})Q_j(z^{(k)})}}, \qquad i \in \{1,\ldots,n\},\ k \in \mathbb{N}.$$

**Wang-Zheng's method** [9]**.** The recurrences for this fourth convergence order method are

$$z_i^{(k+1)} = z_i^{(k)} - \frac{1}{\frac{P'(z_i^{(k)})}{P(z_i^{(k)})} - \frac{P''(z_i^{(k)})}{2P'(z_i^{(k)})} - \frac{P(z_i^{(k)})}{2P'(z_i^{(k)})}\left[\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z_i^{(k)} - z_j^{(k)}}\right)^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{(z_i^{(k)} - z_j^{(k)})^2}\right]},$$

$i \in \{1,\ldots,n\},\ k \in \mathbb{N}$.

## 4 Verification of the convergence order using CAS

We consider the polynomial $P(z) = (z-a)(z-b)(z-c)$ and the first component $T_1(\mathbf{z})$ of a method to compute the roots of the polynomial $P(z)$, $z^{(k+1)} = T(z^{(k)})$, simultaneously.

In order to verify the conditions of the theorem 2, thanks to the symmetries and the fact that the differentials of an operator $T(z)$ are expressed through the partial derivates of its components $T_i(z)$, it is sufficient to compute the derivates

$$\begin{array}{ccccccc}
\frac{\partial T_1(\mathbf{z})}{\partial z_1} & \frac{\partial T_1(\mathbf{z})}{\partial z_2} & & & & & \\
\frac{\partial^2 T_1(\mathbf{z})}{\partial z_1^2} & \frac{\partial^2 T_1(\mathbf{z})}{\partial z_1 \partial z_2} & \frac{\partial^2 T_1(\mathbf{z})}{\partial z_2^2} & \frac{\partial^2 T_1(\mathbf{z})}{\partial z_2 \partial z_3} & & & \\
\frac{\partial^3 T_1(\mathbf{z})}{\partial z_1^3} & \frac{\partial^3 T_1(\mathbf{z})}{\partial z_1^2 \partial z_2} & \frac{\partial^3 T_1(\mathbf{z})}{\partial z_1 \partial z_2^2} & \frac{\partial^3 T_1(\mathbf{z})}{\partial z_2^3} & \frac{\partial^3 T_1(\mathbf{z})}{\partial z_2^2 \partial z_3} & & \\
\frac{\partial^4 T_1(\mathbf{z})}{\partial z_1^4} & \frac{\partial^4 T_1(\mathbf{z})}{\partial z_1^3 \partial z_2} & \frac{\partial^4 T_1(\mathbf{z})}{\partial z_1^2 \partial z_2^2} & \frac{\partial^4 T_1(\mathbf{z})}{\partial z_1 \partial z_2^3} & \frac{\partial^4 T_1(\mathbf{z})}{\partial z_2^4} & \frac{\partial^4 T_1(\mathbf{z})}{\partial z_2^3 \partial z_3} & \frac{\partial^4 T_1(\mathbf{z})}{\partial z_2^2 \partial z_3^2} \\
\vdots & & & & & &
\end{array}$$

The rows of the above table must be computed until the first no zero element is found. The index of that row gives the lower bound of the convergence order.

The expresion of $T_1(z_1, z_2, z_3)$ and the *Mathematica* codes are given for each of the above considered methods:

- **Durand-Kerner's method.**

$$T_1(z_1, z_2, z_3) = z_1 - \frac{P(z_1)}{(z_1 - z_2)(z_1 - z_3)}$$

The *Mathematica* codes are

```
In[1]:=
   T1[z1,z2,z3]:=
    z1-(z1-a)*(z1-b)*(z1-c)/((z1-z2)*(z1-z3))
In[2]:=
   D[T1[z1,z2,z3],z1]/.{z1->a,z2->b,z3->c}
Out[2]:= 0
In[3]:=
   D[T1[z1,z2,z3],z2]/.{z1->a,z2->b,z3->c}
Out[3]:= 0
In[4]:=
   Simplify[D[T1[z1,z2,z3],z1,z2]/.
       {z1->a,z2->b,z3->c}]
```

Out[4]:= $\frac{1}{-a+b}$

- **Improved Durand-Kerner's method.**

$$T_1(z_1, z_2, z_3) = z_1 -$$

$$-\frac{P(z_1)}{(z_1 - z_2 + \frac{P(z_2)}{(z_2-z_1)(z_2-z_3)})(z_1 - z_3 + \frac{P(z_3)}{(z_3-z_1)(z_3-z_2)})}$$

The *Mathematica* codes are

```
In[1]:=
   P[z_]:=(z-a)*(z-b)*(z-c)
   T1[z1,z2,z3]:=
    z1-
    P[z1]/((z1 - z2 + P[z2]/((z2 - z1)*(z2 - z3)))*
        (z1 - z3 + P[z3]/((z3 - z1)*(z3 - z2))))
In[2]:=
   D[T1[z1,z2,z3],z1]/.{z1->a,z2->b,z3->c}
Out[2]:= 0
In[3]:=
   D[T1[z1,z2,z3],z2]/.{z1->a,z2->b,z3->c}
Out[3]:= 0
In[4]:=
   Simplify[D[T1[z1,z2,z3],{z1,2}]/.
        {z1->a,z2->b,z3->c}]
Out[4]:= 0
In[5]:=
   Simplify[D[T1[z1,z2,z3],z1,z2]/.
        {z1->a,z2->b,z3->c}]
Out[5]:= 0
In[6]:=
   Simplify[D[T1[z1,z2,z3],{z2,2}]/.
        {z1->a,z2->b,z3->c}]
Out[6]:= 0
In[7]:=
   Simplify[D[T1[z1,z2,z3],z2,z3]/.
        {z1->a,z2->b,z3->c}]
Out[7]:= 0
In[8]:=
   Simplify[D[T1[z1,z2,z3],{z1,3}]/.
        {z1->a,z2->b,z3->c}]
Out[8]:= 0
In[9]:=
   Simplify[D[T1[z1,z2,z3],{z1,2},z2]/.
        {z1->a,z2->b,z3->c}]
```

Out[9]:= $-\frac{2}{(a-b)^2}$

- **Erlich's method.**

$$T_1(z_1,z_2,z_3) = z_1-$$
$$-\frac{P(z_1)}{(z_1-z_2)(z_1-z_3) - P(z_1)\left(\frac{1}{z_1-z_2}+\frac{1}{z1-z3}\right)}$$

The *Mathematica* codes are

```
In[1]:=
   T1[z1,z2,z3]:=
    z1-(z1-a)*(z1-b)*(z1-c)/((z1-z2)*(z1-z3)-
    (z1-a)*(z1-b)*(z1-c)*
    (1/(z1-z2)+1/(z1-z3)))
In[2]:=
   D[T1[z1,z2,z3],z1]/.{z1->a,z2->b,z3->c}
Out[2]:= 0
In[3]:=
   D[T1[z1,z2,z3],z2]/.{z1->a,z2->b,z3->c}
Out[3]:= 0
In[4]:=
   Simplify[D[T1[z1,z2,z3],{z1,2}]/.{z1->a,z2->b,z3->c}]
```

Out[4]:= $\frac{2(-2a+b+c)}{(a-b)(a-c)}$

- **Nourein's method**

$$T_1(z_1,z_2,z_3)=z_1-\frac{P(z_1)}{(z_1-z_2)(z_1-z_3)\left[1+\frac{P(z_2)}{(z_2-z_1)(z_2-z_3)(z_1-z_2)}+\frac{P(z_3)}{(z_3-z_1)(z_3-z_2)(z_1-z_3)}\right]}=$$

$$=z_1-\frac{P(z_1)}{(z_1-z_2)(z_1-z_3)+\frac{(z_1-z_3)P(z_2)}{(z_2-z_1)(z_2-z_3)}+\frac{(z_1-z_2)P(z_3)}{(z_3-z_1)(z_3-z_2)}}$$

The first non zero derivates is given only. The *Mathematica* codes are

```
In[1]:=
   T1[z1,z2,z3]:=
    z1-(z1-a)*(z1-b)*(z1-c)/((z1-z2)*(z1-z3)+
    (z2-a)*(z2-b)*(z2-c)*(z1-z3)/((z2-z1)*(z2-z3))+
    (z3-a)*(z3-b)*(z3-c)*(z1-z2)/((z3-z1)*(z3-z2)))
In[2]:=
   D[T1[z1,z2,z3],z1]/.{z1->a,z2->b,z3->c}
Out[2]:= 0
. . . . .
In[7]:=
   Simplify[D[T1[z1,z2,z3],{z1,2},z2]/.
       {z1->a,z2->b,z3->c}]
```

Out[7]:= $-\frac{2}{(a-b)^2}$

- **Improved Nourein's method**

$$T_1(z_1,z_2,z_3)=z_1-\frac{P(z_1)}{(z_1-z_2)(z_1-z_3)}\cdot$$

$$\cdot\frac{1}{1+\frac{P(z_2)}{(z_2-z_1)(z_2-z_3)(z_1-\frac{P(z1)}{(z1-z2)(z1-z3)}-z_2)}+\frac{P(z_3)}{(z_3-z_1)(z_3-z_2)(z_1-\frac{P(z1)}{(z1-z2)(z1-z3)}-z_3)}}=$$

$$=z_1-\frac{P(z_1)}{(z_1-z_2)(z_1-z_3)+\frac{(z_1-z_3)P(z_2)}{(z_1-\frac{P(z1)}{(z1-z2)(z1-z3)}-z_2)(z_3-z_2)}+\frac{(z_1-z_2)P(z_3)}{(z_1-\frac{P(z1)}{(z1-z2)(z1-z3)}-z_3)(z_2-z_3)}}$$

The *Mathematica* codes are

```
In[1]:=
   P[z_]:=(z-a)*(z-b)*(z-c)
   DQ1:=-P[z1]/((z1-z2)*(z1-z3))
   T1[z1_,z2_,z3_]:=
    z1-P[z1]/((z1-z2)*(z1-z3)+
    P[z2]*(z1-z3)/((z3-z2)*(z1+DQ1-z2))+
    P[z3]*(z1-z2)/((z2-z3)*(z1+DQ1-z3)))
In[2]:=
   D[T1[z1,z2,z3],z1]/.{z1->a,z2->b,z3->c}
Out[2]:= 0
. . . . .
In[13]:=
   Simplify[D[T1[z1,z2,z3],{z1,2},{z2,2}]/.
      {z1->a,z2->b,z3->c}]
```

Out[13]:= $\frac{4}{(a-b)^3}$

- **Wang-Zheng's method.**

$$T_1(z_1,z_2,z_3) = z_1 -$$

$$-\frac{2P(z_1)P'(z_1)}{2P'^2(z_1) - P(z_1)P''(z_1) - 2P^2(z_1)\left(\frac{1}{(z_1-z_2)^2} + \frac{1}{(z_1-z_2)(z_1-z_3)} + \frac{1}{(z_1-z_3)^2}\right)}$$

The *Mathematica* codes are

```
In[1]:=
   P[x_]:=x^3-(a+b+c)*x*x+(a*b+b*c+c*a)*x-a*b*c
   D1P[x_]:=3*x*x-2*(a+b+c)*x+a*b+b*c+c*a
   D2P[x_]:=6*x-2*(a+b+c)
In[2]:=
   T1[z1,z2,z3]:=
    z1-2*P[z1]*D1P[z1]/
      (2*D1P[z1]*D1P[z1]-P[z1]*D2P[z1]-
         2*P[z1]*P[z1]*
    (1/(z1-z2)^2+1/((z1-z2)*(z1-z3))+1/(z1-z3)^2))
In[3]:=
   D[T1[z1,z2,z3],z1]/.{z1->a,z2->b,z3->c}
Out[3]:= 0
. . . . .
In[14]:=
   Simplify[D[T1[z1,z2,z3],{z1,3},z2]/.
     {z1->a,z2->b,z3->c}]
```

Out[14]:= $\frac{6(-3a+b+2c)}{(a-b)^3(a-c)}$

For the comparision purpose, the lower bound of the convergence order, the computing complexity of a component are given in Table 1.

| Method | Order | Complexity |
|---|---|---|
| Durand-Kerner | 2 | $O(n)$ |
| Improved Durand-Kerner | 3 | $O(n^2)$ |
| Ehrlich | 2 | $O(n)$ |
| Nourein | 3 | $O(n^2)$ |
| Improved Nourein | 4 | $O(n^2)$ |
| Wang-Zheng | 4 | $O(n)$ |

Table 1.

Finally, we mention another way to compute the roots of a polynomial simultaneously[4]: The so called companion matrix is associated to the polynomial – a matrix whose characteristic polynomial is the original polynomial – and the eigenvalues of that matrix are computed using the QR algorithm.

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