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ON SOME CONDITIONS FOR UNIVALENCE

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Abstract

We present some sufficient conditions for univalence in terms of the coefficients of an analytic functions.

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1 Introduction

Let A be the class of analytic functions f in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n \ldots , \qquad z \in U \tag{1}$$

Let S denote the class of functions $f \in A$, f univalent in U. The usual subclasses of S consisting of starlike, convex and uniformly convex functions will be denoted by ST, CV and respectively UCV.

Given the sequence of coefficients (a_n) in (1), how does this sequence influence the geometric properties of f and can we decide if f is univalent in U ? So, it is well-known that if f is given by (1) and

$$\sum_{n=2}^{\infty} n\,|a_n| \le 1 ,$$

then f is univalent in U. The same condition assures that f is a starlike function. (see[1]).

In [2] Goodman gave the sufficient condition

$$\sum_{n=2}^{\infty} 3n(n-1)\,|a_n| \le 1 ,$$

for the function f of the form (1) to be uniformly convex. An improvement of this condition was obtained in [5]. If

$$\sum_{n=2}^{\infty} n(2n-1)\,|a_n| \le 1 ,$$

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then the function f of the form (1) is in UCV.

The above results are related to the univalence of an analytic function f in U. We are interesting if similar conditions can assure the analyticity and the univalence of a family of functions defined by an integral operator. Our considerations are based on the following results.

2 Preliminaries

Theorem 1. **([6]).** *Let $f \in A$, $\alpha \in \mathbb{C}$, $|\alpha - 1| < 1$. If for all $z \in U$*

$$|f'(z) - 1| < 1, \tag{2}$$

then the function

$$F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u) du\right)^{1/\alpha} \tag{3}$$

is analytic and univalent in U, where the principal branch is intended.

Theorem 2. **([3]).** *Let $f \in A$, $\alpha \in \mathbb{C}$, $Re\alpha \geq 1$. If the inequality*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \tag{4}$$

is true for all $z \in U$, then the function F_α defined by (3) is analytic and univalent in U.

Theorem 3. **([4]).** *Let $f \in A$, $\beta \in \mathbb{C}$, $Re\beta > 0$. If*

$$\frac{1 - |z|^{2Re\beta}}{Re\beta} \cdot \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{5}$$

for all $z \in U$, then for all complex numbers α, $Re\alpha \geq Re\beta$, the function F_α defined by (3) is analytic and univalent in U.

3 Main results

Theorem 4. *Let $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. If*

$$\sum_{n=2}^{\infty} n\,|a_n| < 1, \tag{6}$$

then f is univalent in U and for all $\alpha \in \mathbb{C}$, $|\alpha - 1| < 1$, the functions

$$F_\alpha(z) = z \cdot \left[1 + \sum_{n=2}^{\infty} \frac{n a_n \alpha}{\alpha + n - 1} z^{n-1}\right]^{1/\alpha} \tag{7}$$

are analytic and univalent in U.

Proof. For all $z \in U$, the condition (2) of Theorem 1 is verified.

$$|f'(z) - 1\,| = \left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n|a_n| \, < 1.$$

Thus $f(z) = F_1(z)$ is univalent and for every $\alpha \in \mathbb{C}$, $|\alpha - 1| < 1$, the functions F_α defined by (7) are analytic and univalent in U, □

Theorem 5. *Let $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. If*

$$\sum_{n=2}^{\infty} n\,|a_n| \, < \, 1\,, \tag{8}$$

then f is starlike in U and for all $\alpha \in \mathbb{C}$, $Re\alpha \geq 1$, the functions F_α defined by (7) are analytic and univalent in U.

Proof. It is easy to verify that the assumption (4) of Theorem 2 is satisfied. If (8) holds, then $\sum_{n=2}^{\infty} |a_n| \, < \, 1$ and it follows

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{a_2 z + \ldots + (n-1)a_n z^{n-1} + \ldots}{1 + a_2 z + \ldots + a_n z^{n-1} + \ldots} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}$$

The last expression is boundede above by 1 if $\sum_{n=2}^{\infty} n\,|a_n| \, < \, 1$. Since (4) implies $Re\frac{zf'(z)}{f(z)} > 0$ we deduce that f is starlike in U and in view of Theorem 2, the functions F_α are analytic and univalent in U. □

Theorem 6. *Let $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. If*

$$\sum_{n=2}^{\infty} n(n-1)\,|a_n| \, < \, \frac{27 - 6\sqrt{3}}{23} \, = \approx 0.722\,, \tag{9}$$

then f is univalent in U and for all $\alpha \in \mathbb{C}$, $Re\alpha \geq 1$, the functions F_α defined by (7) are analytic and univalent in U.

Proof. First, we note that Theorem 3 improves Becker's univalence criterion. Indeed, for $\beta = 1$, the condition (5) becames

$$(\,1 - |z|^2\,) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \qquad z \in U$$

and assures the univalence of the function f and also of the functions F_α defined by (3) , for all $\alpha \in \mathbb{C}$, $Re\alpha \geq 1$. We consider now the function $h : [\,0,\,1\,] \longrightarrow \mathbb{R}$, $h(x) = x(1 - x^2)$ which has a maximum value in the point $x_0 = \sqrt{3}/3$, namely

$$0 \, < \, h(x) \, \leq \, \frac{2\sqrt{3}}{9}\,, \qquad x \in [\,0,\,1\,].$$

It follows that

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2\sqrt{3}}{9}\cdot\max_{z\in U}\left|\frac{f''(z)}{f'(z)}\right| \leq 1,$$

for

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{3\sqrt{3}}{2} \qquad z\in U. \tag{10}$$

Suppose that $\sum_{n=2}^{\infty} n(n-1)\,|a_n| \leq v < 1$. Then $\sum_{n=2}^{\infty} n\,|a_n| < v$ and

$$\frac{1}{1-\sum_{n=2}^{\infty} n\,|a_n|} < \frac{1}{1-v}$$

For all $z\in U$ we have

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)\,|a_n|}{1-\sum_{n=2}^{\infty} n\,|a_n|}$$

Therefore, the inequality (10) is satisfied if

$$\sum_{n=2}^{\infty} n(n-1)\,|a_n| < \frac{3\sqrt{3}}{2+3\sqrt{3}} = \frac{27-6\sqrt{3}}{23}.$$

Thus, in view of Theorem 3, for all $\alpha\in\mathbb{C}$, $Re\alpha \geq 1$, the functions F_α defined by (7) are analytic and univalent in U. □

The following result improves the bounded (9) given in Theorem 6.

Theorem 7. *Let $f\in A$, $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, $z\in U$. If*

$$\sum_{n=2}^{\infty} n(2n+3\sqrt{3}-2)\,|a_n| < 3\sqrt{3} \tag{11}$$

then f is univalent in U and for all $\alpha\in\mathbb{C}$, $Re\alpha \geq 1$, the functions F_α defined by (7) are analytic and univalent in U.

Proof. In view of Theorem 6, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)\,|a_n|}{1-\sum_{n=2}^{\infty} n\,|a_n|}$$

The last expression is bounded above by $3\sqrt{3}/2$ if $\sum_{n=2}^{\infty} n(2n+3\sqrt{3}-2)\,|a_n| < 3\sqrt{3}$. □

Taking into account the result of paper [5], we can give the following

Corollary 1. *If $\sum_{n=2}^{\infty} n(2n+3\sqrt{3}-2)\,|a_n| \leq 1$, then the function f of the form (1) is in UCV and for all $\alpha\in\mathbb{C}$, $Re\alpha \geq 1$, the functions F_α defined by (7) are analytic and univalent in U.*

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