Bulletin of the *Transilvania* University of Braşov • Vol 4(53), No. 1 - 2011
Series III: Mathematics, Informatics, Physics, 99-106

ESTIMATES WITH OPTIMAL CONSTANTS USING PEETRE's K-FUNCTIONALS ON SIMPLEX

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Abstract

We present general estimates of the degree of approximation by positive linear operators with the K-functionals $K_{1,p}^{\infty}$, $1 \leq p \leq \infty$.

2010 *Mathematics Subject Classification:* 41A36, 41A25.
Key words: Positive linear operators, Peetre's K-functional, degree of approximation.

1 Indroduction

Let $S = \left\{\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d | x_1, \cdots, x_d \geq 0,\, x_1 + \cdots + x_d \leq 1\right\}$ be the simplex in $\mathbb{R}^d$, $d \in \mathbb{N}$. We consider the K-functional $K_{1,p}^{s}(f,t) = K^s\left(f, t; \mathbf{C}(S), \mathbf{C}_p^1(S)\right)$, $t > 0$, $1 \leq s, p \leq \infty$ defined for the Banach space of continuous functions on S

$$(\mathbf{C}(S), \|\cdot\|), \; \|f\| = \max_{\mathbf{x} \in S} |f(\mathbf{x})|$$

and the semi-Banach subspace of functions of class C^1 on S

$$\left(\mathbf{C}_p^1(S), |\cdot|_{C_p^1}\right), \; |f|_{C_p^1} = \|\nabla f\|_p = \left\| \left(\left\| \frac{\partial f}{\partial x_1} \right\|, \cdots, \left\| \frac{\partial f}{\partial x_d} \right\| \right) \right\|_p$$

by

$$K_{1,p}^{s}(f,t) = \inf_{g \in \mathbf{C^1}(S)} \left(\|f - g\|^s + t^s \|\nabla g\|_p^s \right)^{\frac{1}{s}}, \; 1 \leq s < \infty$$

and

$$K_{1,p}^{\infty}(f,t) = \inf_{g \in \mathbf{C^1}(S)} \max\left(\|f - g\|, t \|\nabla g\|_p \right).$$

In fact

$$K_{1,p}^{s}(f,t) = \inf_{g \in \mathbf{C^1}(S)} \left\| \left(\|f - g\|, t \|\nabla g\|_p \right) \right\|_s, \; 1 \leq s, \, p \leq \infty,$$

where $\|\cdot\|_s$, $1 \leq s < \infty$, is the Minkowski norm in $\mathbb{R}^2$ and $\|\cdot\|_\infty$ is the Chebychev norm in $\mathbb{R}^2$, respectively.

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In Section 2 general estimates with the K-functional $K_{1,p}^{s}$ are established.

We use the notation e_0 for the function $e_0 : \mathbb{R}^d \to \mathbb{R}$, $e_0(\mathbf{x}) = 1$, e_1 for the function $e_1 : \mathbb{R}^d \to \mathbb{R}^d$, $e_1(\mathbf{x}) = \mathbf{x}$ and $\pi_k : \mathbb{R}^d \to \mathbb{R}$ for the projection on the component k, $k \in \{1, \ldots, d\}$.

2 General estimates with $K_{1,p}^{s}$, $1 \le p,\ s \le \infty$

Theorem 1. *Let $L : \mathbf{C}(S) \longrightarrow \mathbf{C}(S)$ a positive linear operator and $f \in \mathbf{C}(S)$. Then for every $\mathbf{x} \in S$ and $t > 0$ we have*

$$\begin{aligned} |L(f,\mathbf{x}) - f(\mathbf{x})| &\le |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + \\ &+ \left(2L(e_0,\mathbf{x}) + \frac{L\left(\|e_1 - \mathbf{x}e_0\|_q, \mathbf{x}\right)}{t} \right) \cdot K_{1,p}^{\infty}(f,t), \end{aligned} \tag{1}$$

where q is such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Conversely, if there exists A, B, $C \ge 0$ such that

$$\begin{aligned} |L(f,\mathbf{x}) - f(\mathbf{x})| &\le A \cdot |f(\mathbf{x})| \, |L(e_0,\mathbf{x}) - 1| + \\ &+ \left(B \cdot L(e_0,\mathbf{x}) + C\frac{L\left(\|e_1 - \mathbf{x}e_0\|_q, \mathbf{x}\right)}{t} \right) \cdot K_{1,p}^{\infty}(f,t) \end{aligned} \tag{2}$$

holds for all positive linear operator, any $f \in \mathbf{C}(S)$, any $\mathbf{x} \in S$ and any $t > 0$, then $A \ge 1$, $C \ge 1$ and $B \ge 2$.

Proof. Let $g \in \mathbf{C^1}(S)$. We have

$$\begin{aligned} |f(\mathbf{y}) - f(\mathbf{x})| &\le 2\|f-g\| + |g(\mathbf{y}) - g(\mathbf{x})| \\ &= 2\|f-g\| + |dg(\mathbf{c})(\mathbf{y}-\mathbf{x})| \\ &\le 2\|f-g\| + \sum_{i=1}^{d} \left\| \frac{\partial g}{\partial x_i} \right\| \cdot |y_i - x_i| \\ &\le 2\|f-g\| + \frac{\|\mathbf{y}-\mathbf{x}\|_q}{t} \cdot t\|\nabla g\|_p \\ &\le \left(2 + \frac{\|\mathbf{y}-\mathbf{x}\|_q}{t}\right) \cdot \max\left\{ \|f-g\|, t\|\nabla g\|_p \right\}, \end{aligned}$$

where $\mathbf{c} \in [\mathbf{x}, \mathbf{y}]$. Since g is arbitrary it follows that

$$|f(\mathbf{y}) - f(\mathbf{x})| \le \left(2 + \frac{\|\mathbf{y}-\mathbf{x}\|_q}{t}\right) \cdot K_{1,p}^{\infty}(f,t).$$

Then

$$\begin{aligned}
|L(f,\mathbf{x})-f(\mathbf{x})| &\le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1|+|L(f-f(\mathbf{x})e_0,\mathbf{x})| \\
&\le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1|+L(|f-f(\mathbf{x})e_0|,\mathbf{x}) \\
&\le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1|+L\left(2e_0+\frac{\|e_1-\mathbf{x}e_0\|_q}{t},\mathbf{x}\right)\cdot K_{1,p}^{\infty}(f,t) \\
&= |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| \\
&\quad+\left(2L(e_0,\mathbf{x})+\frac{L\left(\|e_1-\mathbf{x}e_0\|_q,\mathbf{x}\right)}{t}\right)\cdot K_{1,p}^{\infty}(f,t),
\end{aligned}$$

which is (1).

For the converse part, if we choose $L(h,\mathbf{x})=0$ and $f=e_0$ and replace in (2) we obtain $A\ge 1$.

If we choose $L(h,\mathbf{x})=h(1,0,\dots,0)$, $f(\mathbf{x})=x_1$, $\mathbf{x}=\mathbf{0}$ and replace in (2) we obtain $1\le Bt+C$, $(\forall)t>0$ (we use $K_{1,p}^{\infty}(f,t)\le t$). Passing to the limit $t\to 0$ we obtain $C\ge 1$.

If we choose $L(h,\mathbf{x})=h(1,0,\dots,0)$, $f(\mathbf{x})=2x_1-1$, $\mathbf{x}=0$ and replace in (2) we obtain $2\le B+\frac{C}{t}$, $(\forall)t>0$ (we use $K_{1,p}^{\infty}(f,t)\le\|f\|=1$). Passing to the limit $t\to\infty$ we obtain $B\ge 2$. □

For the estimate with $K_{1,p}^{s}$, $1\le s<\infty$ we use the estimate (1) and the following relation between the K-functionals [1], [5]:

Lemma 1. *Let $1\le s<\infty$. Then for $f\in\mathbf{C}(S)$ and $t>0$ we have*

$$K_{1,p}^{s}(f,t)=\inf_{u>0}\left(1+\frac{t^s}{u^s}\right)^{\frac{1}{s}}K_{1,p}^{\infty}(f,u). \tag{3}$$

Proof. "$\le$:" Let $g\in\mathbf{C^1}(S)$ and $u>0$. We have

$$\begin{aligned}
K_{1,p}^{s}(f,t) &\le \left(\|f-g\|^s+t^s\|\nabla g\|_p^s\right)^{\frac{1}{s}} \\
&\le \left(1+\frac{t^s}{u^s}\right)^{\frac{1}{s}}\max\left\{\|f-g\|,\,u\|\nabla g\|_p\right\}.
\end{aligned}$$

Since g is arbitrary this implies $K_{1,p}^{s}(f,t)\le\left(1+\frac{t^s}{u^s}\right)^{\frac{1}{s}}K_{1,p}^{\infty}(f,u)$. Also since u is arbitrary the above inequality implies $K_{1,p}^{s}(f,t)\le\inf_{u>0}\left(1+\frac{t^s}{u^s}\right)^{\frac{1}{s}}K_{1,p}^{\infty}(f,u)$.

"$\ge$": Let $\varepsilon>0$. We can choose $g\in\mathbf{C^1}(S)$ such that $\|f-g\|\ne 0$, $\|\nabla g\|_p\ne 0$ and

$$K_{1,p}^{s}(f,t)+C\varepsilon\ge\left(\|f-g\|^s+t^s\|\nabla g\|_p^s\right)^{\frac{1}{s}}$$

where $C = C(t)$. Indeed, by definition of the K-functional there exists $g_1 \in \mathbf{C^1}(S)$ such that

$$K_{1,p}^s(f,t) + \varepsilon \geq \left(\|f - g_1\|^s + t^s \|\nabla g_1\|_p^s \right)^{\frac{1}{s}} .$$

So, if $\|f - g_1\| \neq 0$ and $\|\nabla g_1\|_p \neq 0$ we choose $g = g_1$.

If $\|f - g_1\| = 0$ and $\|\nabla g_1\|_p \neq 0$ we choose $g = g_1 + \varepsilon e_0$. Then $\|f - g\| = \|f - g_1 - \varepsilon e_0\| = \varepsilon \neq 0$, $\|\nabla g\|_p = \|\nabla g_1\|_p \neq 0$ and

$$\left(\|f - g\|^s + t^s \|\nabla g\|_p^s \right)^{\frac{1}{s}} = \left(\varepsilon^s + t^s \|\nabla g_1\|_p^s \right)^{\frac{1}{s}} \leq K_{1,p}^s(f,t) + 2\varepsilon.$$

If $\|f - g_1\| \neq 0$ and $\|\nabla g_1\|_p = 0$ we choose $g = g_1 + C_1 \varepsilon \pi_1$, where C_1 is a constant for which $\|f - g\| \neq 0$. We can suppose that $0 < C_1 < 1$. Then $\|f - g\| = \|f - g_1 - C_1 \varepsilon \pi_1\| \leq \|f - g_1\| + C_1 \varepsilon \|\pi_1\|$, $\|\nabla g\|_p = C_1 \varepsilon \neq 0$ and

$$\begin{aligned} \left(\|f - g\|^s + t^s \|\nabla g\|_p^s \right)^{\frac{1}{s}} &\leq \|f - g\| + t \|\nabla g\|_p \\ &\leq \|f - g_1\| + C_1 \varepsilon (1 + t) \leq K_{1,p}^s(f,t) + \varepsilon (2 + t) . \end{aligned}$$

If $\|f - g_1\| = 0$ and $\|\nabla g_1\|_p = 0$ we choose $g = g_1 + \varepsilon \pi_1$. Then $\|f - g\| = \|f - g_1 - \varepsilon \pi_1\| = \varepsilon \|\pi_1\| \neq 0$, $\|\nabla g\|_p = \varepsilon \neq 0$ and

$$\begin{aligned} \left(\|f - g\|^s + t^s \|\nabla g\|_p^s \right)^{\frac{1}{s}} &\leq \|f - g\| + t \|\nabla g\|_p \\ &= \varepsilon (1 + t) \leq K_{1,p}^s(f,t) + \varepsilon (1 + t) . \end{aligned}$$

On the assumption above we have

$$\begin{aligned} \inf_{u>0} \left(1 + \frac{t^s}{u^s} \right)^{\frac{1}{s}} K_{1,p}^{\infty}(f,u) &\leq \left(1 + t^s \frac{\|\nabla g\|_p^s}{\|f - g\|^s} \right)^{\frac{1}{s}} K_{1,p}^{\infty} \left(f, \frac{\|f - g\|}{\|\nabla g\|_p} \right) \\ &\leq \left(1 + t^s \frac{\|\nabla g\|_p^s}{\|f - g\|^s} \right)^{\frac{1}{s}} \cdot \max \left\{ \|f - g\| , \frac{\|f - g\|}{\|\nabla g\|_p} \|\nabla g\|_p \right\} \\ &= \left(\|f - g\|^s + t^s \|\nabla g\|_p^s \right)^{\frac{1}{s}} \leq K_{1,p}^s(f,t) + C\varepsilon . \end{aligned}$$

Since ε is arbitrary this implies the inequality. □

Corollary 1. Under the conditions of theorem we have

$$\begin{aligned} |L(f,\mathbf{x}) - f(\mathbf{x})| &\leq |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + \\ &+ \max \left\{ 2L(e_0,\mathbf{x}) , \frac{L\left(\|e_1 - \mathbf{x} e_0\|_q , \mathbf{x} \right)}{t} \right\} K_{1,p}^1(f,t) \end{aligned} \tag{4}$$

and

$$|L(f,\mathbf{x})-f(\mathbf{x})| \le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| + \\ +\left(2^{s'}L(e_0,\mathbf{x})^{s'}+\frac{L\left(\|e_1-\mathbf{x}e_0\|_q,\mathbf{x}\right)^{s'}}{t^{s'}}\right)^{\frac{1}{s'}} K^s_{1,p}(f,t) \tag{5}$$

for $1<s<\infty$, $s'=\dfrac{s}{s-1}$.

Conversely,

- if $(\exists)A,\ B,\ C\ge 0$ such that

$$|L(f,\mathbf{x})-f(\mathbf{x})| \le A\cdot|f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| + \\ +\max\left\{B\cdot L(e_0,\mathbf{x}),\ C\cdot\frac{L\left(\|e_1-\mathbf{x}e_0\|_q,\mathbf{x}\right)}{t}\right\} K^1_{1,p}(f,t)$$

  holds for all positive linear operator, any $f\in\mathbf{C}(S)$, any $\mathbf{x}\in S$ and any $t>0$, then $A\ge 1$, $C\ge 1$ and $B\ge 2$.

- if $(\exists)A,\ B,\ C\ge 0$ such that

$$|L(f,\mathbf{x})-f(\mathbf{x})| \le A\cdot|f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| + \\ +\left(B\cdot L(e_0,\mathbf{x})^{s'}+C\cdot\frac{L\left(\|e_1-\mathbf{x}e_0\|_q,\mathbf{x}\right)^{s'}}{t^{s'}}\right)^{\frac{1}{s'}} K^s_{1,p}(f,t)$$

  holds for all positive linear operator, any $f\in\mathbf{C}(S)$, any $\mathbf{x}\in S$ and any $t>0$, then $A\ge 1$, $C\ge 1$ and $B\ge 2^{s'}$.

Proof. Let $u>0$. For $s=1$ we have:

$$\begin{aligned}|L(f,\mathbf{x})-f(\mathbf{x})| &\le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| + \\ &\quad+\left(2L(e_0,\mathbf{x})+\frac{L\left(\|e_1-\mathbf{x}e_0\|_q,\mathbf{x}\right)}{u}\right)\cdot K^\infty_{1,p}(f,u)\\ &\le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| + \\ &\quad+\max\left\{2L(e_0,\mathbf{x}),\ \frac{L\left(\|e_1-\mathbf{x}e_0\|_q,\mathbf{x}\right)}{t}\right\}\left(1+\frac{t}{u}\right)K^\infty_{1,p}(f,u),\end{aligned}$$

from whence (4).

For $1 < s < \infty$, we denote by $s' = \dfrac{s}{s-1}$ and by Hölder's inequality we have:

$$\begin{aligned}|L(f,\mathbf{x}) - f(\mathbf{x})| &\leq |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + \\ &+ \left(2L(e_0,\mathbf{x}) + \frac{L\left(\|e_1 - \mathbf{x}e_0\|_q, \mathbf{x}\right)}{u}\right) \cdot K_{1,p}^{\infty}(f,u) \\ &\leq |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + \\ &+ \left(2^{s'} L(e_0,\mathbf{x})^{s'} + \frac{L\left(\|e_1 - \mathbf{x}e_0\|_q, \mathbf{x}\right)^{s'}}{t^{s'}}\right)^{\frac{1}{s'}} \left(1 + \frac{t^s}{u^s}\right)^{\frac{1}{s}} K_{1,p}^{\infty}(f,u),\end{aligned}$$

from whence (5).

For the converse part we make the same choices like in Theorem 1. □

Remark 1. *The estimates (1), (4) and (5) are valid for any compact convex domain of* $\mathbb{R}^d$, $d \in \mathbb{N}$.

Example 1. *The Bernstein operators are defined by*

$$B_n(f,\mathbf{x}) = \sum_{k_1+\cdots+k_d=0}^{n} f\left(\frac{k_1}{n},\ldots,\frac{k_d}{n}\right) b_{n,\mathbf{k}}(\mathbf{x}) \tag{6}$$

where

$$b_{n,\mathbf{k}}(\mathbf{x}) = \frac{n!}{k_1!\cdots k_d! \cdot \left(n - \sum\limits_{i=1}^{d} k_i\right)!} x_1^{k_1}\cdots x_d^{k_d} \left(1 - \sum_{i=1}^{d} x_i\right)^{n - \sum\limits_{i=1}^{d} k_i}$$

with $n \in \mathbb{N}$ *and* $k_1,\ldots,k_d \in \mathbb{N} \cup \{0\}$. *We have*

$$B_n\left(\|e_1 - \mathbf{x}e_0\|_2, \mathbf{x}\right) \leq \sqrt{B_n\left(\|e_1 - \mathbf{x}e_0\|_2^2, \mathbf{x}\right)} = \sqrt{\frac{\sum\limits_{i=1}^{d} x_i(1-x_i)}{n}} \leq \sqrt{\frac{d}{4n}}$$

and from (1) it results:

$$|B_n(f,\mathbf{x}) - f(\mathbf{x})| \leq \left(2 + \sqrt{\frac{d}{4}}\right) K_{1,2}^{\infty}\left(f, \frac{1}{\sqrt{n}}\right).$$

Other estimates with $K_{1,p}^{s}$, $1 < s < \infty$ are obtained using the Hölder-type inequality for positive linear operators.

Theorem 2. *Let $L : \mathbf{C}(S) \longrightarrow \mathbf{C}(S)$ a positive linear operator and $f \in \mathbf{C}(S)$. Then for every $\mathbf{x} \in S$ and $t > 0$ we have*

$$\begin{aligned}|L(f,\mathbf{x}) - f(\mathbf{x})| &\le |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + \\ &+ \left(2^{s'} L(e_0,\mathbf{x}) + \frac{L\left(\|e_1 - \mathbf{x}e_0\|_q^{s'}, \mathbf{x}\right)}{t^{s'}}\right)^{\frac{1}{s'}} \cdot L(e_0,\mathbf{x})^{\frac{1}{s}} \cdot K_{1,p}^s(f,t)\end{aligned} \tag{7}$$

holds for $1 < s < \infty$, $s' = \dfrac{s}{s-1}$ and $q : \dfrac{1}{p} + \dfrac{1}{q} = 1$.

Conversely, if $(\exists) A, B, C \ge 0$ such that

$$\begin{aligned}|L(f,\mathbf{x}) - f(\mathbf{x})| &\le A \cdot |f(\mathbf{x})| \, |L(e_0,\mathbf{x}) - 1| + \\ &+ \left(B \cdot L(e_0,\mathbf{x}) + C\frac{L\left(\|e_1 - \mathbf{x}e_0\|_q^{s'}, \mathbf{x}\right)}{t^{s'}}\right)^{\frac{1}{s'}} \cdot L(e_0,\mathbf{x})^{\frac{1}{s}} \cdot K_{1,p}^s(f,t)\end{aligned} \tag{8}$$

holds for all positive linear operator, any $f \in \mathbf{C}(S)$, any $\mathbf{x} \in S$ and any $t > 0$ then $A \ge 1$, $C \ge 1$ and $B \ge 2^{s'}$.

Proof. Let $g \in \mathbf{C^1}(S)$. We have

$$\begin{aligned}|f(\mathbf{y}) - f(\mathbf{x})| &\le 2\|f - g\| + \frac{\|\mathbf{y} - \mathbf{x}\|_q}{t} \cdot t\|\nabla g\|_p \\ &\le \left\|\left(2, \frac{\|\mathbf{y} - \mathbf{x}\|_q}{t}\right)\right\|_{s'} \cdot \left\|\left(\|f - g\|, t\|\nabla g\|_p\right)\right\|_s\end{aligned}$$

where $1 < s, s' < \infty : \dfrac{1}{s} + \dfrac{1}{s'} = 1$. Since g is arbitrary follows that

$$|f(\mathbf{y}) - f(\mathbf{x})| \le \left\|\left(2, \frac{\|\mathbf{y} - \mathbf{x}\|_q}{t}\right)\right\|_{s'} \cdot K_{1,p}^s(f,t).$$

Then

$$\begin{aligned}|L(f,\mathbf{x}) - f(\mathbf{x})| &\le |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + L(|f - f(\mathbf{x})e_0|, \mathbf{x}) \\ &\le |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + \\ &\quad + L\left(\left\|\left(2e_0, \frac{\|e_1 - \mathbf{x}e_0\|_q}{t}\right)\right\|_{s'}, \mathbf{x}\right) \cdot K_{1,p}^s(f,t) \\ &\le |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + \\ &\quad + L\left(\left\|\left(2e_0, \frac{\|e_1 - \mathbf{x}e_0\|_q}{t}\right)\right\|_{s'}^{s'}, \mathbf{x}\right)^{\frac{1}{s'}} \cdot L(e_0,\mathbf{x})^{\frac{1}{s}} \cdot K_{1,p}^s(f,t) \\ &= |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + \\ &\quad + \left(2^{s'} L(e_0,\mathbf{x}) + \frac{L\left(\|e_1 - \mathbf{x}e_0\|_q^{s'}, \mathbf{x}\right)}{t^{s'}}\right)^{\frac{1}{s'}} \cdot L(e_0,\mathbf{x})^{\frac{1}{s}} \cdot K_{1,p}^s(f,t),\end{aligned}$$

which is (7).

For the converse part we make the same choices like in Theorem 1. □

Remark 2. *From*

$$K_{1,p}^{\infty}(f,t) \le K_{1,p}^{s}(f,t) \le 2^{\frac{1}{s}} K_{1,p}^{\infty}(f,t)$$

it follows that

$$\lim_{s\to\infty} K_{1,p}^{s}(f,t) = K_{1,p}^{\infty}(f,t).$$

Passing to the limit $s \to \infty$ *(in this case* $s' \to 1$*) in (7) we obtain (1).*

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