

UNIQUENESS AND WEIGHTED SHARING OF ENTIRE FUNCTIONS

Pulak SAHOO ¹

Abstract

With the aid of weighted sharing method we study the uniqueness of entire functions concerning nonlinear differential polynomials sharing one value. Though the main concern of the paper is to improve a result in [4] but as a consequence of the main result we also improve and supplement some former results.

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1 Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane \mathbb{C} . For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same set of zeros with the same multiplicities and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities.

It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h , we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$ ($r \rightarrow \infty, r \notin E$). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o\{T(r)\}$ ($r \rightarrow \infty, r \notin E$).

In 1959, Hayman [7] proved the following theorem.

Theorem A. *Let f be a transcendental entire function and $n(\geq 1)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

To establish the corresponding uniqueness theorem, Fang and Hua [5] proved the following theorem.

Theorem B. *Let f and g be two nonconstant entire functions, $n \geq 6$ be an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

¹Department of Mathematics, *Netaji Subhas Open University*, 1, Woodburn Park, Kolkata-700020, West Bengal, India, e-mail:sahoopulak1@gmail.com

Fang [6] investigated the uniqueness of entire functions corresponding to general differential polynomials and obtained the following results.

Theorem C. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem D. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

In 2008, Zhang-Lin [19] extended the results of Fang [6] corresponding to more general differential polynomials. They proved the following theorems.

Theorem E. *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n \geq 2k + m^* + 4$, and λ, μ be constants such that $|\lambda| + |\mu| \neq 0$. If $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 CM, then one of the following holds:
(i) If $\lambda\mu \neq 0$, then $f^d(z) \equiv g^d(z)$, $d = \text{GCD}(n, m)$; especially when $d = 1$, $f \equiv g$.
(ii) If $\lambda\mu = 0$, then either $f \equiv tg$, where t is a constant satisfying $t^{n+m^*} = 1$ or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying*

$$(-1)^k \lambda^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$$

or

$$(-1)^k \mu^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$$

and m^* is defined by $m^* = \chi_\mu m$, where

$$\chi_\mu = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu \neq 0. \end{cases}$$

Theorem F. *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n > 2k + m + 4$. If $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 CM, then either $f \equiv g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(w_1, w_2) = w_1^n (w_1 - 1)^m - w_2^n (w_2 - 1)^m$.*

Now the following question arises: is it really possible to relax in any way the nature of sharing the value 1 in the above results?

The notion of weighted sharing of values is used in [17] to deal this problem. We now explain the notion in the following definition which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1. [10, 11] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .*

The definition implies that if f, g share a value a with weight k , then z_0 is an a -point of f with multiplicity $m(\leq k)$ if and only if it is an a -point of g with multiplicity $m(\leq k)$

and z_0 is an a -point of f with multiplicity $m(> k)$ if and only if it is an a -point of g with multiplicity $n(> k)$, where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for any integer $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

Using the idea mentioned above, Zhang and Lu [17] proved the following theorem.

Theorem G. Let f and g be two nonconstant entire functions, and let $n(\geq 1), k(\geq 1), l(\geq 0)$ be three integers. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n > 2k + 4$ or $l = 1$ and $n > 3k + 6$ or $l = 0$ and $n > 5k + 7$, then the conclusion of Theorem C holds.

In 2008, Banerjee [3] proved the following theorem which improves Theorem G.

Theorem H. Let f and g be two nonconstant entire functions and $n(\geq 1), k(\geq 1), l(\geq 0)$ be three integers. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share (b, l) for a nonzero constant b . If $l \geq 2$ and $n > 2k + 4$ or if $l = 1$ and $n > \frac{5k+9}{2}$ or if $l = 0$ and $n > 5k + 7$, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$ or $f \equiv tg$ for some n -th root of unity t .

Recently X.Y. Zhang, J.F. Chen and W.C. Lin [18] investigated the uniqueness of entire functions concerning some general differential polynomials. They proved the following result.

Theorem I. Let f and g be two nonconstant entire functions and let n, m and k be three positive integers with $n \geq 2k + 3m + 5$. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$ or $P(z) = c_0$, where $a_0(\neq 0), a_1, \dots, a_{m-1}, a_m(\neq 0), c_0(\neq 0)$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share 1 CM, then

(i) when $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$, either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n + m, \dots, n + m - i, \dots, n), a_{m-i} \neq 0$ for some $i = 0, 1, 2, \dots, m$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(x, y) = x^n (a_m x^m + a_{m-1} x^{m-1} + \dots + a_0) - y^n (a_m y^m + a_{m-1} y^{m-1} + \dots + a_0);$$

(ii) when $P(z) = c_0$, either $f(z) = c_1 / c_0^{\frac{1}{n}} e^{cz}, g(z) = c_2 / c_0^{\frac{1}{n}} e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.

Now one may ask the following questions which are the motivation of the paper.

Question 1. What can be said about the relation between two nonconstant entire functions f and g , if we use the notion of weighted sharing of values in place of the CM sharing value in Theorem E, Theorem F and Theorem I ?

Question 2. Whether one can deduce a generalised result which includes the results of Zhang-Lu [17] and Banerjee [3] ?

Regarding the above mentioned questions recently Chen-Zhang-Lin-Chen [4] and Liu [12] proved the following theorems respectively.

Theorem J. [4] Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 5k + 13$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share $(1, 0)$, then $f \equiv g$.

Theorem K. [12] Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers such that $n > 5k + 4m + 9$. If $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share $(1, 0)$, then the conclusion of Theorem F holds.

In the paper, we will prove the following theorem which will not only improve Theorem E, Theorem F and Theorem I by relaxing the nature of sharing the value 1 but also improve Theorem G, Theorem J and Theorem K by reducing the lower bound of n . Our result will improve and supplement Theorem H also. We now state the main result of the paper.

Theorem 1. Let f and g be two nonconstant entire functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$ or $P(z) = c_0$, where $a_0(\neq 0)$, $a_1, \dots, a_{m-1}, a_m(\neq 0)$, $c_0(\neq 0)$ are complex constants. Let $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, l)$ where $l \geq 0$ is an integer. Then the conclusions (i) and (ii) of Theorem I hold in each of the following cases:

- (a) $l \geq 2$ and $n > 2k + m + 4$;
- (b) $l = 1$ and $n > \frac{5k+3m+9}{2}$;
- (c) $l = 0$ and $n > 5k + 4m + 7$.

Corollary 1. Under the condition of Theorem 1, we set $P(z) = \mu z^m + \lambda$, where λ and μ are two constants such that $|\lambda| + |\mu| \neq 0$ and $m(\geq 1)$ is an integer. We assume that one of the following conditions hold:

- (a) $l \geq 2$ and $n > 2k + m^* + 4$;
- (b) $l = 1$ and $n > \frac{5k+3m^*+9}{2}$;
- (c) $l = 0$ and $n > 5k + 4m^* + 7$,

where m^* is defined as in Theorem E. Then the conclusions (i) and (ii) of Theorem E hold.

Corollary 2. Under the condition of Theorem 1, we set $P(z) = (z-1)^m$. Then

- (i) when $m = 0$ and $l \geq 2$, $n > 2k + 4$ or $l = 1$, $n > \frac{5k+9}{2}$ or $l = 0$, $n > 5k + 7$, then the conclusion of Theorem C holds;
- (ii) when $m \geq 1$ and $l \geq 2$, $n > 2k + m + 4$ or $l = 1$, $n > \frac{5k+3m+9}{2}$ or $l = 0$, $n > 5k + 4m + 7$, then either $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = (n + m, \dots, n + m - i, \dots, n + 1, n)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) = 0$, where

$$R(x, y) = x^n(x-1)^m - y^n(y-1)^m.$$

Remark 1. Theorem 1 is an improvement of Theorem I.

Remark 2. Since Theorem H can be obtained as a special case of Corollary 2, Corollary 2 improves and supplements Theorem H.

Remark 3. Corollary 1 is an improvement of Theorem E.

Remark 4. *Corollary 2 improves and supplements Theorem F, Theorem J and Theorem K.*

Remark 5. *Corollary 2 improves Theorem G for $m = 0$ and $l = 1$.*

Though the standard definitions and notations of the value distribution theory are available in [8, 14], we explain some definitions and notations which are used in the paper.

Definition 2. [9] *For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f | = 1)$ the counting function of simple a points of f . For a positive integer m we denote by $N(r, a; f | \leq m)$ the counting function of those a -points of f (counted with multiplicities) whose multiplicities are not greater than m . By $\bar{N}(r, a; f | \leq m)$ we denote the corresponding reduced counting function.*

In an analogous manner we define $N(r, a; f | \geq m)$ and $\bar{N}(r, a; f | \geq m)$.

Definition 3. [11] *Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then*

$$N_k(r, a; f) = \bar{N}(r, a; f) + \bar{N}(r, a; f | \geq 2) + \dots + \bar{N}(r, a; f | \geq k).$$

Clearly $N_1(r, a; f) = \bar{N}(r, a; f)$.

Definition 4. [3] *Let $a, b \in \mathbb{C} \cup \{\infty\}$ and p be a positive integer. Then we denote by $\bar{N}(r, a; f | \geq p | g = b)$ ($\bar{N}(r, a; f | \geq p | g \neq b)$) the reduced counting function of those a -points of f with multiplicities $\geq p$, which are the b -points (not the b -points) of g .*

Definition 5. [1, 2] *Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p , a 1-point of g with multiplicity q . For a positive integer k , $\bar{N}_{f>k}(r, 1; g)$ denotes the reduced counting function of those 1-points of f and g such that $p > q = k$. In an analogous way we can define $\bar{N}_{g>k}(r, 1; f)$.*

Definition 6. [1, 2] *Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p , a 1-point of g with multiplicity q . We denote by $\bar{N}_L(r, 1; f)$ the counting function of those 1-points of f and g , where $p > q$, by $N_E^{(1)}(r, 1; f)$ the counting function of those 1-points of f and g , where $p = q = 1$ and by $N_E^{(2)}(r, 1; f)$ the counting function of those 1-points of f and g , where $p = q \geq 2$, each point in these counting functions is counted only once. Similarly we can define $\bar{N}_L(r, 1; g)$, $N_E^{(1)}(r, 1; g)$ and $N_E^{(2)}(r, 1; g)$.*

2 Lemmas and Propositions

In this section we present some lemmas and propositions which will be needed in the sequel.

Proposition 1. [18] *Let f be a transcendental entire function, and n, m, k be three positive integers such that $n \geq k + 2$, and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_2 z^2 + a_1 z + a_0$, where $a_0, a_1, a_2, \dots, a_{m-1}, a_m$ are complex constants. Then $[f^n P(f)]^{(k)} = 1$ has infinitely many solutions.*

Proposition 2. [18] Let f and g be two nonconstant entire functions and let n, k be two positive integers with $n > k$, and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_2 z^2 + a_1 z + a_0$ be a nonzero polynomial, where $a_0, a_1, a_2, \dots, a_{m-1}, a_m$ are complex constants. If $[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv 1$, then $P(z)$ is reduced to a nonzero monomial, that is, $P(z) = a_i z^i \neq 0$ for some $i = 0, 1, 2, \dots, m$; further, $f(z) = c_1 / a_i^{\frac{1}{n+i}} e^{cz}$, $g(z) = c_2 / a_i^{\frac{1}{n+i}} e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k (c_1 c_2)^{n+i} [(n+i)c]^{2k} = 1.$$

Lemma 1. [13] Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \dots + a_n f^n$, where $a_0, a_1, a_2, \dots, a_n$ are constants and $a_n \neq 0$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2. [8, 14] Let f be a nonconstant entire function, and let k be a positive integer. Then for any non-zero finite complex number c

$$\begin{aligned} T(r, f) &\leq N(r, 0; f) + N\left(r, c; f^{(k)}\right) - N\left(r, 0; f^{(k+1)}\right) + S(r, f) \\ &\leq N_{k+1}(r, 0; f) + \overline{N}\left(r, c; f^{(k)}\right) - N_0\left(r, 0; f^{(k+1)}\right) + S(r, f), \end{aligned}$$

where $N_0(r, 0; f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 3. [16] Let f be a nonconstant meromorphic function and p, k be two positive integers. Then

$$N_p\left(r, 0; f^{(k)}\right) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 4. [4] Let f and g be two nonconstant entire functions and let n, k be two positive integers with $n > k$. If $[f^n]^{(k)} [g^n]^{(k)} \equiv 1$, then $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.

Lemma 5. [2] Let f and g be two nonconstant meromorphic functions that share $(1, 0)$. Then

$$\begin{aligned} \overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + N_E^{(2)}(r, 1; f) - \overline{N}_{f>1}(r, 1; g) - \overline{N}_{g>1}(r, 1; f) \\ \leq N(r, 1; g) - \overline{N}(r, 1; g). \end{aligned}$$

Lemma 6. [15] Let f and g be two nonconstant meromorphic functions sharing $(1, 0)$. Then

$$\overline{N}_L(r, 1; f) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 7. [2] Let f and g share $(1, 0)$. Then

- (i) $\overline{N}_{f>1}(r, 1; g) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\oplus}(r, 0; f') + S(r, f)$;
 - (ii) $\overline{N}_{g>1}(r, 1; f) \leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) - N_{\oplus}(r, 0; g') + S(r, g)$,
- where $N_{\oplus}(r, 0; f')$ ($N_{\oplus}(r, 0; g')$) denotes the counting function of those zeros of f' (g') which are not the zeros of $f(f-1)$ ($g(g-1)$).

Lemma 8. [1] *Let f and g be two nonconstant meromorphic functions that share $(1, 1)$. Then*

$$2\bar{N}_L(r, 1; f) + 2\bar{N}_L(r, 1; g) + N_E^{(2)}(r, 1; f) - \bar{N}_{f>2}(r, 1; g) \leq N(r, 1; g) - \bar{N}(r, 1; g).$$

Lemma 9. [2] *Let f and g share $(1, 1)$. Then*

$$\bar{N}_{f>2}(r, 1; g) \leq \frac{1}{2}\bar{N}(r, 0; f) + \frac{1}{2}\bar{N}(r, \infty; f) - \frac{1}{2}N_{\oplus}(r, 0; f') + S(r, f),$$

where $N_{\oplus}(r, 0; f')$ is defined as in Lemma 7.

Lemma 10. [8, 14] *Let f be a transcendental meromorphic function, and let $a_1(z), a_2(z)$ be two distinct meromorphic functions such that $T(r, a_i(z)) = S(r, f)$, $i=1, 2$. Then*

$$T(r, f) \leq \bar{N}(r, \infty; f) + \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + S(r, f).$$

3 Proof of the Theorem

Proof of Theorem 1. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_2 z^2 + a_1 z + a_0$, where $a_0 (\neq 0)$, $a_1, a_2, \dots, a_{m-1}, a_m (\neq 0)$ are complex constants. We consider $F(z) = f^n P(f)$ and $G(z) = g^n P(g)$. Then $F^{(k)}$ and $G^{(k)}$ share $(1, l)$. Let

$$H = \left(\frac{F^{(k+2)}}{F^{(k+1)}} - \frac{2F^{(k+1)}}{F^{(k)} - 1} \right) - \left(\frac{G^{(k+2)}}{G^{(k+1)}} - \frac{2G^{(k+1)}}{G^{(k)} - 1} \right). \quad (3.1)$$

We assume that $H \not\equiv 0$. Let $l \geq 1$. Suppose that z_0 is a simple 1-point of $F^{(k)}$. Then z_0 is a simple 1-point of $G^{(k)}$. So from (3.1) we see that z_0 is a zero of H . Thus

$$\begin{aligned} N\left(r, 1; F^{(k)} \mid = 1\right) &\leq N(r, 0; H) \leq T(r, H) + O(1) \\ &\leq N(r, \infty; H) + S(r, F) + S(r, G). \end{aligned} \quad (3.2)$$

From (3.1) we know that poles of H possibly result from those zeros of $F^{(k+1)}$ and $G^{(k+1)}$ which are not common 1-points of $F^{(k)}$ and $G^{(k)}$ and from those common 1-points of $F^{(k)}$ and $G^{(k)}$ such that each such point has different multiplicity related to $F^{(k)}$ and $G^{(k)}$. Thus

$$\begin{aligned} N(r, \infty; H) &\leq \bar{N}(r, 0; F^{(k)} \mid \geq 2) + \bar{N}(r, 0; G^{(k)} \mid \geq 2) + \bar{N}_L\left(r, 1; F^{(k)}\right) \\ &\quad + \bar{N}_L\left(r, 1; G^{(k)}\right) + \bar{N}_{\otimes}\left(r, 0; F^{(k+1)}\right) + \bar{N}_{\otimes}\left(r, 0; G^{(k+1)}\right), \end{aligned} \quad (3.3)$$

where $\bar{N}_{\otimes}(r, 0; F^{(k+1)})$ denotes the reduced counting function of those zeros of $F^{(k+1)}$ which are not the zeros of $F^{(k)}$ ($F^{(k)} - 1$). $\bar{N}_{\otimes}(r, 0; G^{(k+1)})$ is defined similarly. Now we consider the following three cases.

Case 1. Let $l \geq 2$. By (3.2) and (3.3) we obtain

$$\begin{aligned}
\bar{N}(r, 1; F^{(k)}) &\leq N(r, 1; F^{(k)} | = 1) + \bar{N}(r, 1; F^{(k)} | \geq 2) \\
&\leq \bar{N}(r, 0; F^{(k)} | \geq 2) + \bar{N}(r, 0; G^{(k)} | \geq 2) + \bar{N}_L(r, 1; F^{(k)}) \\
&\quad + \bar{N}_L(r, 1; G^{(k)}) + \bar{N}(r, 1; F^{(k)} | \geq 2) + \bar{N}_\otimes(r, 0; F^{(k+1)}) \\
&\quad + \bar{N}_\otimes(r, 0; G^{(k+1)}) + S(r, F) + S(r, G).
\end{aligned} \tag{3.4}$$

From (3.4) and Lemma 2 we obtain

$$\begin{aligned}
T(r, F) + T(r, G) &\leq N_{k+1}(r, 0; F) + N_{k+1}(r, 0; G) + \bar{N}(r, 0; F^{(k)} | \geq 2) \\
&\quad + \bar{N}(r, 0; G^{(k)} | \geq 2) + \bar{N}_L(r, 1; F^{(k)}) + \bar{N}_L(r, 1; G^{(k)}) \\
&\quad + \bar{N}(r, 1; F^{(k)} | \geq 2) + \bar{N}(r, 1; G^{(k)}) + \bar{N}_\otimes(r, 0; F^{(k+1)}) \\
&\quad + \bar{N}_\otimes(r, 0; G^{(k+1)}) - N_0(r, 0; F^{(k+1)}) - N_0(r, 0; G^{(k+1)}) \\
&\quad + S(r, F) + S(r, G).
\end{aligned} \tag{3.5}$$

It is clear that

$$\begin{aligned}
&N_{k+1}(r, 0; F) + \bar{N}(r, 0; F^{(k)} | \geq 2) + \bar{N}_\otimes(r, 0; F^{(k+1)}) \\
&\leq N_{k+1}(r, 0; F) + \bar{N}(r, 0; F^{(k)} | \geq 2 | F = 0) \\
&\quad + \bar{N}(r, 0; F^{(k)} | \geq 2 | F \neq 0) + \bar{N}_\otimes(r, 0; F^{(k+1)}) \\
&\leq N_{k+1}(r, 0; F) + \bar{N}(r, 0; F | \geq k + 2) + N_0(r, 0; F^{(k+1)}) \\
&\leq N_{k+2}(r, 0; F) + N_0(r, 0; F^{(k+1)}).
\end{aligned} \tag{3.6}$$

A similar result holds for G also. Since G is entire, we have

$$\begin{aligned}
&\bar{N}(r, 1; F^{(k)} | \geq 2) + \bar{N}_L(r, 1; F^{(k)}) + \bar{N}_L(r, 1; G^{(k)}) + \bar{N}(r, 1; G^{(k)}) \\
&\leq \bar{N}(r, 1; G^{(k)} | = 2) + 2\bar{N}_L(r, 1; F^{(k)}) + 2\bar{N}_L(r, 1; G^{(k)}) \\
&\quad + \bar{N}_E^{(3)}(r, 1; G^{(k)}) + \bar{N}(r, 1; G^{(k)}) \\
&\leq N(r, 1; G^{(k)}) \\
&\leq T(r, G^{(k)}) + O(1) \\
&\leq T(r, G) + S(r, G).
\end{aligned} \tag{3.7}$$

So from (3.5), (3.6) and (3.7) we obtain

$$T(r, F) \leq N_{k+2}(r, 0; F) + N_{k+2}(r, 0; G) + S(r, F) + S(r, G).$$

From this and using Lemma 1 we obtain

$$(n+m)T(r, f) \leq (2k+2m+4)T(r) + S(r). \quad (3.8)$$

Similarly

$$(n+m)T(r, g) \leq (2k+2m+4)T(r) + S(r). \quad (3.9)$$

Combining (3.8) and (3.9) we get

$$(n-2k-m-4)T(r) \leq S(r),$$

which contradicts the fact that $n > 2k+m+4$.

Case 2. Let $l = 1$. In view of Lemmas 3, 8, 9, (3.2) and (3.3) we obtain

$$\begin{aligned} \overline{N}(r, 1; F^{(k)}) + \overline{N}(r, 1; G^{(k)}) &\leq N(r, 1; F^{(k)} | = 1) + \overline{N}_L(r, 1; F^{(k)}) \\ &\quad + \overline{N}_L(r, 1; G^{(k)}) + \overline{N}_E^{(2)}(r, 1; F^{(k)}) + \overline{N}(r, 1; G^{(k)}) \\ &\leq N(r, 1; F^{(k)} | = 1) + N(r, 1; G^{(k)}) - \overline{N}_L(r, 1; F^{(k)}) \\ &\quad - \overline{N}_L(r, 1; G^{(k)}) + \overline{N}_{F^{(k)} > 2}(r, 1; G^{(k)}) \\ &\leq \overline{N}(r, 0; F^{(k)} | \geq 2) + \overline{N}(r, 0; G^{(k)} | \geq 2) \\ &\quad + \frac{1}{2}\overline{N}(r, 0; F^{(k)}) + T(r, G^{(k)}) + \overline{N}_\otimes(r, 0; F^{(k+1)}) \\ &\quad + \overline{N}_\otimes(r, 0; G^{(k+1)}) + S(r, F) + S(r, G) \\ &\leq \overline{N}(r, 0; F^{(k)} | \geq 2) + \overline{N}(r, 0; G^{(k)} | \geq 2) \\ &\quad + \frac{1}{2}N_{k+1}(r, 0; F) + T(r, G) + \overline{N}_\otimes(r, 0; F^{(k+1)}) \\ &\quad + \overline{N}_\otimes(r, 0; G^{(k+1)}) + S(r, F) + S(r, G). \end{aligned} \quad (3.10)$$

Using (3.6) and (3.10) we obtain from Lemma 2 that

$$T(r, F) \leq N_{k+2}(r, 0; F) + N_{k+2}(r, 0; G) + \frac{1}{2}N_{k+1}(r, 0; F) + S(r, F) + S(r, G).$$

From this and using Lemma 1 we obtain

$$(n+m)T(r, f) \leq \left(\frac{5k+5m+9}{2} \right) T(r) + S(r). \quad (3.11)$$

Similarly

$$(n+m)T(r, g) \leq \left(\frac{5k+5m+9}{2} \right) T(r) + S(r). \quad (3.12)$$

From (3.11) and (3.12) we get

$$\left(n - \frac{5k + 3m + 9}{2}\right) T(r) \leq S(r),$$

which contradicts our assumption that $n > \frac{5k+3m+9}{2}$.

Case 3. Let $l = 0$. In this case (3.2) changes to

$$\begin{aligned} N_E^1(r, 1; F^{(k)}) \leq N(r, 0; H) &\leq T(r, H) + O(1) \\ &\leq N(r, \infty; H) + S(r, F) + S(r, G). \end{aligned} \quad (3.13)$$

Using Lemmas 3, 5, 6, 7, (3.3), (3.6) and (3.13) we obtain

$$\begin{aligned} \overline{N}(r, 1; F^{(k)}) + \overline{N}(r, 1; G^{(k)}) &\leq N_E^1(r, 1; F^{(k)}) + \overline{N}_L(r, 1; F^{(k)}) + \overline{N}_L(r, 1; G^{(k)}) \\ &\quad + \overline{N}_E^2(r, 1; F^{(k)}) + \overline{N}(r, 1; G^{(k)}) \\ &\leq \overline{N}(r, 0; F^{(k)} \mid \geq 2) + \overline{N}(r, 0; G^{(k)} \mid \geq 2) \\ &\quad + \overline{N}_L(r, 1; F^{(k)}) + T(r, G^{(k)}) + \overline{N}_{F^{(k)} > 1}(r, 1; G^{(k)}) \\ &\quad + \overline{N}_{G^{(k)} > 1}(r, 1; F^{(k)}) + \overline{N}_{\otimes}(r, 0; F^{(k+1)}) \\ &\quad + \overline{N}_{\otimes}(r, 0; G^{(k+1)}) + S(r, F) + S(r, G) \\ &\leq \overline{N}(r, 0; F^{(k)} \mid \geq 2) + \overline{N}(r, 0; G^{(k)} \mid \geq 2) \\ &\quad + 2N_{k+1}(r, 0; F) + N_{k+1}(r, 0; G) + T(r, G) \\ &\quad + \overline{N}_{\otimes}(r, 0; F^{(k+1)}) + \overline{N}_{\otimes}(r, 0; G^{(k+1)}) \\ &\quad + S(r, F) + S(r, G) \\ &\leq N_{k+1}(r, 0; F) + N_{k+2}(r, 0; F) + N_{k+2}(r, 0; G) \\ &\quad + T(r, G) + N_0(r, 0; F^{(k+1)}) + N_0(r, 0; G^{(k+1)}) \\ &\quad + S(r, F) + S(r, G). \end{aligned} \quad (3.14)$$

Using Lemma 2 we get

$$\begin{aligned} T(r, F) &\leq 2N_{k+1}(r, 0; F) + N_{k+2}(r, 0; F) + N_{k+1}(r, 0; G) + N_{k+2}(r, 0; G) \\ &\quad + S(r, F) + S(r, G). \end{aligned}$$

In view of Lemma 1 we obtain

$$(n + m)T(r, f) \leq (5k + 5m + 7)T(r) + S(r). \quad (3.15)$$

Similarly

$$(n + m)T(r, g) \leq (5k + 5m + 7)T(r) + S(r). \quad (3.16)$$

Combining (3.15) and (3.16) we get

$$(n - 5k - 4m - 7)T(r) \leq S(r),$$

a contradiction since $n > 5k + 4m + 7$.

We now assume that $H \equiv 0$. That is

$$\frac{F^{(k+2)}}{F^{(k+1)}} - \frac{2F^{(k+1)}}{F^{(k)} - 1} \equiv \frac{G^{(k+2)}}{G^{(k+1)}} - \frac{2G^{(k+1)}}{G^{(k)} - 1}.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F^{(k)} - 1} \equiv \frac{BG^{(k)} + A - B}{G^{(k)} - 1}, \quad (3.17)$$

where $A (\neq 0)$ and B are constants. From (3.17) it is clear that $F^{(k)}$ and $G^{(k)}$ share 1 CM and hence $F^{(k)}$ and $G^{(k)}$ share (1, 2). Thus $n > 2k + m + 4$. we now discuss the following three cases separately.

Case I. Let $B \neq 0$ and $A = B$. Then from (3.17) we get

$$\frac{1}{F^{(k)} - 1} = \frac{BG^{(k)}}{G^{(k)} - 1}. \quad (3.18)$$

If $B = -1$, then from (3.18) we obtain

$$F^{(k)}G^{(k)} \equiv 1,$$

i.e.,

$$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv 1. \quad (3.19)$$

So we have

$$[f^n(a_m f^m + \dots + a_0)]^{(k)} [g^n(a_m g^m + \dots + a_0)]^{(k)} \equiv 1,$$

which by the assumptions and Proposition 2 is a contradiction.

If $B \neq -1$, then it follows from (3.18) and the fact that f and g are entire that

$$F^{(k)} - \left(1 + \frac{1}{B}\right) = -\frac{1}{BG^{(k)}} \neq 0.$$

So using Lemma 2 we get

$$\begin{aligned} (n+m)T(r, f) &= T(r, F) + O(1) \\ &\leq N_{k+1}(r, 0; F) + S(r, f) \\ &\leq (k+1)\overline{N}(r, 0; f) + mT(r, f) + S(r, f) \\ &\leq (k+1+m)T(r, f) + S(r, f), \end{aligned}$$

i.e.,

$$[n - (k + 1)]T(r, f) \leq S(r, f),$$

which is a contradiction because $n > 2k + m + 4$.

Case II. Let $B \neq 0$ and $A \neq B$. Then from (3.17) we get

$$G^{(k)} + \frac{A - B}{B} \neq 0.$$

So by Lemma 2 we have

$$(n + m)T(r, g) = T(r, G) + O(1) \leq N_{k+1}(r, 0; G) + S(r, G).$$

Proceeding as case I we obtain

$$[n - (k + 1)]T(r, g) \leq S(r, g),$$

which is also a contradiction.

Case III. Let $B = 0$ and $A \neq 0$. Then from (3.17) we obtain

$$F^{(k)} = \frac{1}{A}G^{(k)} + 1 - \frac{1}{A}, \quad (3.20)$$

i.e.,

$$F = \frac{1}{A}G + \psi(z), \quad (3.21)$$

where $\psi(z)$ is a polynomial of degree at most k . By (3.21) and Lemma 1 we can say that

$$T(r, f) = T(r, g) + S(r, f). \quad (3.22)$$

By the assumptions and Proposition 1, it is clear that either both f and g are transcendental entire functions or both are polynomials.

First we suppose that both f and g are transcendental entire functions. If $\psi(z) \not\equiv 0$, then in view of (3.22), Lemma 1 and the second fundamental theorem of Nevanlinna we obtain

$$\begin{aligned} (n + m)T(r, f) &= T(r, F) + O(1) \\ &\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, \psi(z); F) + S(r, F) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) \\ &\leq 2(m + 1)T(r, f) + S(r, f), \end{aligned}$$

which is impossible. Hence in this case $\psi(z) \equiv 0$.

Now we assume that both f and g are polynomials. We suppose that f and g have γ and δ pairwise distinct zeros respectively. Then f and g are of the form

$$f(z) = c(z - p_1)^{l_1}(z - p_2)^{l_2} \dots (z - p_\gamma)^{l_\gamma},$$

$$g(z) = d(z - q_1)^{m_1}(z - q_2)^{m_2} \dots (z - q_\delta)^{m_\delta},$$

so that

$$f^n(z) = c^n(z - p_1)^{nl_1}(z - p_2)^{nl_2} \dots (z - p_\gamma)^{nl_\gamma}, \quad (3.23)$$

$$g^n(z) = d^n(z - q_1)^{nm_1}(z - q_2)^{nm_2} \dots (z - q_\delta)^{nm_\delta}, \quad (3.24)$$

where c and d are nonzero constants, $nl_i > 2k + m + 4$, $nm_j > 2k + m + 4$, $i = 1, 2, \dots, \gamma$, and $j = 1, 2, \dots, \delta$. Differentiating (3.20) we obtain

$$F^{(k+1)} = \frac{1}{A} G^{(k+1)},$$

i.e.,

$$(a_m f^{n+m})^{(k+1)} + \dots + (a_0 f^n)^{(k+1)} = \frac{1}{A} \left[(a_m g^{n+m})^{(k+1)} + \dots + (a_0 g^n)^{(k+1)} \right]. \quad (3.25)$$

Using (3.23) and (3.24), (3.25) can be written as

$$\begin{aligned} (z - p_1)^{nl_1 - (k+1)} (z - p_2)^{nl_2 - (k+1)} \dots (z - p_\gamma)^{nl_\gamma - (k+1)} \alpha(z) &= (z - q_1)^{nm_1 - (k+1)} \\ &\quad (z - q_2)^{nm_2 - (k+1)} \dots (z - q_\delta)^{nm_\delta - (k+1)} \beta(z), \end{aligned} \quad (3.26)$$

where $\alpha(z)$ and $\beta(z)$ are polynomials such that $\deg \alpha(z) = m \sum_{i=1}^{\gamma} l_i + (\gamma - 1)(k + 1)$

and $\deg \beta(z) = m \sum_{j=1}^{\delta} m_j + (\delta - 1)(k + 1)$. Now

$$\begin{aligned} \sum_{i=1}^{\gamma} [nl_i - (k + 1)] - m \sum_{i=1}^{\gamma} l_i &= \sum_{i=1}^{\gamma} [(n - m)l_i - (k + 1)] \\ &> \gamma(k + 3) \\ &> (\gamma - 1)(k + 1), \end{aligned}$$

i.e.,

$$\sum_{i=1}^{\gamma} [nl_i - (k + 1)] > m \sum_{i=1}^{\gamma} l_i + (\gamma - 1)(k + 1).$$

Similarly,

$$\sum_{j=1}^{\delta} [nm_j - (k + 1)] > m \sum_{j=1}^{\delta} m_j + (\delta - 1)(k + 1).$$

Thus from (3.26) we deduce that there is ζ such that

$$f^n(\zeta) [a_m f^m(\zeta) + \dots + a_1 f(\zeta) + a_0] = g^n(\zeta) [a_m g^m(\zeta) + \dots + a_1 g(\zeta) + a_0] = 0,$$

where ζ has multiplicity greater than $2k + m + 4$. This together with (3.21) implies $\psi(z) = 0$. Thus from (3.20), (3.21) together with the fact that $F^{(k)}$ and $G^{(k)}$ share 1 CM we obtain $A = 1$ and so $F \equiv G$. That is

$$f^n P(f) \equiv g^n P(g). \quad (3.27)$$

Hence

$$f^n [a_m f^m + \dots + a_1 f + a_0] \equiv g^n [a_m g^m + \dots + a_1 g + a_0].$$

Let $h = \frac{f}{g}$. If h is a constant, then by putting $f = gh$ in above we get

$$a_m g^{n+m} (h^{n+m} - 1) + a_{m-1} g^{n+m-1} (h^{n+m-1} - 1) + \dots + a_0 g^n (h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = (n + m, \dots, n + m - i, \dots, n + 1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \dots, m$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, $d = (n + m, \dots, n + m - i, \dots, n + 1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \dots, m$.

If h is not a constant, then from (3.27) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(x, y) = x^n (a_m x^m + a_{m-1} x^{m-1} + \dots + a_0) - y^n (a_m y^m + a_{m-1} y^{m-1} + \dots + a_0).$$

We omit the proof of the case when $P(z) = c_0$, where $c_0 (\neq 0)$ is a complex constant, since using Lemma 4 and proceeding in the same way the proof can be carried out in the line of proof as above. This completes the proof of the theorem. \square

Proof of Corollary 1. By (3.19)

$$[f^n(\mu f^m + \lambda)]^{(k)} [g^n(\mu g^m + \lambda)]^{(k)} \equiv 1. \quad (3.28)$$

We consider following subcases.

Subcase(i) We assume that $\lambda = 0$ and $\mu \neq 0$. Then as $n > k$, by Lemma 4 we obtain $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k \mu^2 (c_1 c_2)^{n+m} [(n+m)c]^{2k} = 1$. Similar result holds for $\lambda \neq 0$ and $\mu = 0$.

Subcase (ii) Let $\lambda\mu \neq 0$. Since f and g are entire functions from above it is clear that

$$f \neq 0 \quad \text{and} \quad g \neq 0. \quad (3.29)$$

Let $f(z) = e^{\alpha(z)}$, where $\alpha(z)$ is an entire function. Then we obtain

$$[\mu f^{n+m}]^{(k)} = t_1(\alpha', \alpha'', \dots, \alpha^{(k)}) e^{(n+m)\alpha(z)} \quad (3.30)$$

and

$$[\lambda f^n]^{(k)} = t_2(\alpha', \alpha'', \dots, \alpha^{(k)}) e^{n\alpha(z)}, \quad (3.31)$$

where $t_i(\alpha', \alpha'', \dots, \alpha^{(k)}) \neq 0$ ($i = 1, 2$) are differential polynomials. Since g is an entire function, we have from (3.28) that

$$[f^n(\mu f^m + \lambda)]^{(k)} \neq 0.$$

So from (3.30) and (3.31) we get

$$t_1(\alpha', \alpha'', \dots, \alpha^{(k)})e^{m\alpha(z)} + t_2(\alpha', \alpha'', \dots, \alpha^{(k)}) \neq 0. \quad (3.32)$$

Since α is an entire function, we have $T(r, \alpha') = S(r, f)$ and

$$T(r, \alpha^{(j)}) \leq T(r, \alpha') + S(r, f) = S(r, f)$$

for $j = 1, 2, \dots, k$. Hence we have

$$T(r, t_i) = S(r, f) \quad (3.33)$$

for $i = 1, 2$. So by (3.32), (3.33), Lemma 1 and Lemma 10 we get

$$\begin{aligned} mT(r, f) &\leq T(r, t_1 e^{m\alpha}) + S(r, f) \\ &\leq \overline{N}(r, 0; t_1 e^{m\alpha}) + \overline{N}(r, 0; t_1 e^{m\alpha} + t_2) + S(r, f) \\ &\leq T\left(r, \frac{1}{t_1}\right) + S(r, f) \\ &= S(r, f), \end{aligned}$$

which is a contradiction.

Again from (3.27) we have

$$f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda). \quad (3.34)$$

If $\lambda\mu = 0$, then from $|\lambda| + |\mu| \neq 0$ we get $f = tg$, where t is a constant such that $t^{n+m} = 1$. Let $\lambda\mu \neq 0$ and $h = \frac{f}{g}$. So from (3.34) we obtain

$$\mu g^m(h^{n+m} - 1) = \lambda(1 - h^n).$$

If $h^{n+m} = 1$, by the above equation we get $h^n = 1$, i.e., $f^n = g^n$ and $f^m = g^m$.

If $h^{n+m} \neq 1$, then substituting $f = gh$ in (3.34) we get

$$g^m = -\frac{\lambda}{\mu} \frac{1 + h + \dots + h^{n-1}}{1 + h + \dots + h^{n+m-1}}.$$

Since g is an entire function, every zero of $h^{n+m} - 1$ is a zero of $h^n - 1$ and hence of $h^m - 1$. Noting that $n > 2k + m + 4$, we obtain h is a constant, which is a contradiction as f and g are nonconstant. Therefore $h^{n+m} \equiv 1$, that is $f^{n+m} \equiv g^{n+m}$. This completes the proof of Corollary 1. \square

Proof of Corollary 2. By (3.19) we have

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1. \quad (3.35)$$

Then we consider following two subcases.

Subcase(I) Let $m = 0$. Then

$$[f^n]^{(k)}[g^n]^{(k)} \equiv 1. \quad (3.36)$$

Since $n > k$ by Lemma 4 we obtain $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k (c_1 c_2)^n (nc)^{2k} = 1.$$

Subcase(II) Let $m \geq 1$. Since f and g are entire functions, we have $f \neq 0$ and $g \neq 0$. Let $f(z) = e^{\alpha(z)}$, where $\alpha(z)$ is a nonconstant entire function. Clearly

$$[f^{n+m}(z)]^{(k)} = s_m(\alpha', \alpha'', \dots, \alpha^{(k)}) e^{(n+m)\alpha(z)}. \quad (3.37)$$

$$(-1)^{m-i} [{}^m C_i f^{n+i}(z)]^{(k)} = s_i(\alpha', \alpha'', \dots, \alpha^{(k)}) e^{(n+i)\alpha(z)}. \quad (3.38)$$

$$(-1)^m [f^n(z)]^{(k)} = s_0(\alpha', \alpha'', \dots, \alpha^{(k)}) e^{n\alpha(z)}. \quad (3.39)$$

where $s_i(\alpha', \alpha'', \dots, \alpha^{(k)})$ ($i = 0, 1, 2, \dots, m$) are differential polynomials. Obviously

$$s_i(\alpha', \alpha'', \dots, \alpha^{(k)}) \neq 0$$

for $i = 0, 1, 2, \dots, m$, and

$$[f^n(f-1)^m]^{(k)} \neq 0.$$

From (3.37) and (3.38) we have

$$s_m(\alpha', \alpha'', \dots, \alpha^{(k)}) e^{m\alpha(z)} + \dots + s_0(\alpha', \alpha'', \dots, \alpha^{(k)}) \neq 0. \quad (3.40)$$

Since $\alpha(z)$ is an entire function, we obtain $T(r, \alpha') = S(r, f)$ and $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2, \dots, k$. Hence $T(r, s_i) = S(r, f)$ for $i = 1, 2, \dots, m$.

So from (3.40), Lemmas 1 and 10 we obtain

$$\begin{aligned} mT(r, f) &= T(r, s_m e^{m\alpha} + \dots + s_1 e^\alpha) + S(r, f) \\ &\leq \overline{N}(r, 0; s_m e^{m\alpha} + \dots + s_1 e^\alpha) + \overline{N}(r, 0; s_m e^{m\alpha} + \dots + s_1 e^\alpha + s_0) + S(r, f) \\ &\leq \overline{N}(r, 0; s_m e^{(m-1)\alpha} + \dots + s_1) + S(r, f) \\ &\leq (m-1)T(r, f) + S(r, f), \end{aligned}$$

which is a contradiction.

Again by (3.27) we have

$$f^n(f-1)^m \equiv g^n(g-1)^m. \quad (3.41)$$

If $m = 0$, then by (3.41) we get $f \equiv tg$ for a constant t such that $t^n = 1$. Now let $m \geq 1$. Then from (3.41) we get

$$f^n[f^m + \dots + (-1)^i {}^m C_{m-i} f^{m-i} + \dots + (-1)^m] = g^n[g^m + \dots + (-1)^i {}^m C_{m-i} g^{m-i} + \dots + (-1)^m]. \quad (3.42)$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = gh$ in (3.42) we get

$$g^{n+m}(h^{n+m} - 1) + \dots + (-1)^i {}^m C_{m-i} g^{n+m-i}(h^{n+m-i} - 1) + \dots + (-1)^m g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = (n + m, \dots, n + m - i, \dots, n + 1, n)$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, $d = (n + m, \dots, n + m - i, \dots, n + 1, n)$.

If h is not a constant, then from (3.41) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^n(x-1)^m - y^n(y-1)^m$. This completes the proof of Corollary 2. \square

References

- [1] Alzahary, T.C. and Yi, H.X., *Weighted value sharing and a question of I. Lahiri*, Complex Var. Theory Appl., **49** (2004), 1063-1078.
- [2] Banerjee, A., *Meromorphic functions sharing one value*, Int. J. Math. Math. Sci., **22** (2005), 3587-3598.
- [3] Banerjee, A., *Uniqueness of certain non-linear differential polynomials sharing the same value*, Int. J. Pure Appl. Math., **48** (2008), 41-56.
- [4] Chen, J.F., Zhang, X.Y., Lin, W.C. and Chen, S.J., *Uniqueness of entire functions that share one value*, Comput. Math. Appl., **56** (2008), 3000-3014.
- [5] Fang, M.L. and Hua, X.H., *Entire functions that share one value*, J. Nanjing Univ. Math. Biquarterly, **13** (1996), 44-48.
- [6] Fang, M.L., *Uniqueness and value sharing of entire functions*, Comput. Math. Appl., **44** (2002), 828-831.
- [7] Hayman, W.K., *Picard values of meromorphic Functions and their derivatives*, Ann. of Math., **70** (1959), 9-42.
- [8] Hayman, W.K., *Meromorphic Functions*, The Clarendon Press, Oxford (1964).
- [9] Lahiri, I., *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sc., **28** (2001), 83-91.
- [10] Lahiri, I., *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161** (2001), 193-206.

- [11] Lahiri, I., *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46** (2001), 241-253.
- [12] Liu, L., *Uniqueness of meromorphic functions and differential polynomials*, Comput. Math. Appl., **56** (2008), 3236-3245.
- [13] Yang, C.C., *On deficiencies of differential polynomials II*, Math. Z., **125** (1972), 107-112.
- [14] Yang, L., *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.
- [15] Yi, H.X., *Meromorphic functions that shares one or two values II*, Kodai Math. J., **22** (1999), 264-272.
- [16] Zhang, Q.C., *Meromorphic function that shares one small function with its derivative*, J. Inequal. Pure Appl. Math., **6** (2005), Art. 116.
- [17] Zhang, T. and Lu, W., *Uniqueness theorems on meromorphic functions sharing one value*, Comput. Math. Appl., **55** (2008), 2981-2992.
- [18] Zhang, X.Y., Chen, J.F. and Lin, W.C., *Entire or meromorphic functions sharing one value*, Comput. Math. Appl., **56** (2008), 1876-1883.
- [19] Zhang, X.Y. and Lin, W.C., *Uniqueness and value sharing of entire functions*, J. Math. Anal. Appl., **343** (2008), 938-950.