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UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract

In this work we obtain sufficient conditions for the univalence of the integral operator $J_{\gamma_1,...,\gamma_n,\beta,n}$.

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1 Introduction

We consider the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of the functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$ and $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting in all starlike functions in $\mathcal{U}$.

We consider the integral operator H_γ for $f \in \mathcal{A}$ and γ, be a complex number, which is given by

$$H_\gamma(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{-1} \left(f(u)\right)^{\frac{1}{\gamma}} du \right\}^\gamma . \tag{1}$$

Miller and Mocanu [2] have studied that the integral operator H_γ is in the class $\mathcal{S}$ for $f \in \mathcal{S}^*$ and $\gamma > 0$.

We introduce a new integral operator

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$$J_{\gamma_1,\ldots,\gamma_n,\beta,n}(z) =$$

$$= \left\{ \left(\sum_{j=1}^{n} \frac{1}{\gamma_j} \right) \int_0^z u^{-\beta} \left(f_1(u)\right)^{\frac{1}{\gamma_1}+\frac{\beta-1}{n}} \ldots \left(f_n(u)\right)^{\frac{1}{\gamma_n}+\frac{\beta-1}{n}} du \right\}^{\frac{1}{\sum_{j=1}^{n} \frac{1}{\gamma_j}}}, \tag{2}$$

for $f_j \in \mathcal{A}$ and complex numbers β, γ_j $(\gamma_j \neq 0)$, $j = \overline{1,n}$.

For $\beta = 1$, from (2) we obtain the integral operator $J_{\gamma_1,\ldots,\gamma_n}$ defined in [4].

For $n = 1, f_1 = f$ and $\gamma_1 = \gamma$, from (2) we get the integral operator $J_{\gamma,\beta}$ defined in [5].

For $n = 1, \beta = 1, \gamma_1 = \gamma, f_1 = f$, from (2) we obtain the integral operator H_γ given by (1).

2 Preliminary results

We need the following lemmas.

Lemma 1. *[3]. Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{1}$$

for all $z \in \mathcal{U}$, then the integral operator F_α defined by

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}} \tag{2}$$

is in the class S.

Lemma 2. *(Schwarz [1]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m,\ (z \in \mathcal{U}_R), \tag{3}$$

the equality (in the inequality (3) for $z \neq 0$) can hold if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

3 Main results

Theorem 1. *Let γ_j, β be complex numbers, $Re\ \gamma_j \neq 0$, M_j real positive numbers, $j = \overline{1,n}$, $p = \sum_{j=1}^{n} Re\ \frac{1}{\gamma_j} > 0$ and $f_j \in \mathcal{A}$,*
$f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + ...,\ j = \overline{1,n}$.

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq M_j, \quad \left(z \in \mathcal{U};\ j = \overline{1,n}\right) \tag{1}$$

and

$$\sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\beta - 1|}{n}\right] \leq \frac{(2p+1)^{\frac{2p+1}{2p}}}{2}, \tag{2}$$

then the integral operator $J_{\gamma_1,\gamma_2,...,\gamma_n,\beta,n}$ given by (2) is in the class $\mathcal{S}$.

Proof. We observe that

$$J_{\gamma_1,\gamma_2,...,\gamma_n,\beta,n}(z) =$$

$$= \left\{\left(\sum_{j=1}^{n}\frac{1}{\gamma_j}\right)\int_0^z u^{\sum_{j=1}^{n}\frac{1}{\gamma_j}-1}\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}+\frac{\beta-1}{n}}\cdots\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}+\frac{\beta-1}{n}} du\right\}^{\frac{1}{\sum_{j=1}^{n}\frac{1}{\gamma_j}}} \tag{3}$$

We consider the function

$$g(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}+\frac{\beta-1}{n}}\cdots\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}+\frac{\beta-1}{n}} du, \tag{4}$$

for $f_j \in \mathcal{A},\ j = \overline{1,n}$. The function g is regular in $\mathcal{U}$.

We define the function h by

$$h(z) = \frac{zg''(z)}{g'(z)}, \quad (z \in \mathcal{U}). \tag{5}$$

We have $h(0) = 0$ and from (4) and (5) we get

$$|h(z)| \leq \sum_{j=1}^{n}\left[\frac{1}{|\gamma_j|} + \frac{|\beta - 1|}{n}\right]\left|\frac{zf_j'(z)}{f_j(z)} - 1\right|, \quad (z \in \mathcal{U}). \tag{6}$$

From (1) and (6) we obtain

$$|h(z)| \leq \sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\beta - 1|}{n}\right], \tag{7}$$

for all $z \in \mathcal{U}$.

Applying Lemma 2 we have

$$\left|\frac{zg''(z)}{g'(z)}\right| \leq \sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\beta-1|}{n}\right] |z|, \tag{8}$$

for all $z \in \mathcal{U}$ and hence, we obtain

$$\frac{1-|z|^{2p}}{p}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2p}}{p}|z| \sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\beta-1|}{n}\right], \quad (z \in \mathcal{U}). \tag{9}$$

Since

$$\max_{|z|\leq 1} \frac{1-|z|^{2p}}{p}|z| = \frac{2}{(2p+1)^{\frac{2p+1}{2p}}}\,, \tag{10}$$

from (2) and (9) we have

$$\frac{1-|z|^{2p}}{p}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1 \tag{11}$$

for all $z \in \mathcal{U}$.

From (4) we obtain $g'(z) = \left(\frac{f_1(z)}{z}\right)^{\frac{1}{\gamma_1}+\frac{\beta-1}{n}} \ldots \left(\frac{f_n(z)}{z}\right)^{\frac{1}{\gamma_n}+\frac{\beta-1}{n}}$ and using (11), by Lemma 1, it results that the integral operator $J_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta,n}$, given by (2), is in the class $\mathcal{S}$. □

Corollary 1. *Let γ be a complex number, $Re\,\gamma \neq 0$, $Re\,\frac{1}{\gamma} > 0$ and $f \in \mathcal{A}$,*

$$f(z) = z + a_{21}z^2 + a_{31}z^3 + \ldots$$

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq |\gamma| \frac{\left(2Re\,\frac{1}{\gamma}+1\right)^{\frac{2Re\,\frac{1}{\gamma}+1}{2Re\,\frac{1}{\gamma}}}}{2}, \quad (z \in \mathcal{U})\,, \tag{12}$$

then the integral operator H_γ is in the class $\mathcal{S}$.

Proof. For $n = 1,\ \beta = 1,\ \gamma_1 = \gamma,\ f_1 = f$ and $p = Re\,\frac{1}{\gamma}$, from Theorem 1 we obtain Corollary 1. □

Corollary 2. *Let the function $f \in \mathcal{A},\ f(z) = z + a_{21}z^2 + a_{31}z^3 + \ldots$*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2}, \quad (z \in \mathcal{U}), \tag{13}$$

then the integral operator Alexander, given by

$$G(z) = \int_0^z \frac{f(u)}{u}du$$

is in the class $\mathcal{S}$.

Proof. For $n = 1,\ \beta = 1,\ \gamma_1 = 1,\ f_1 = f$ from Theorem 1 we have Corollary 2. □

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