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ESTIMATES OF APPROXIMATION IN TERMS OF A WEIGHTED MODULUS OF CONTINUITY

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Abstract

Consider modulus $\omega_\varphi(f,h) = \sup\left\{|f(x)-f(y)|\Big|\, x \geq 0,\ y \geq 0,\ |x-y| \leq h\varphi\left(\frac{x+y}{2}\right)\right\}$, where $\varphi(x) = \frac{\sqrt{x}}{1+x^m}$, $x \in [0,\infty)$, $m \in \mathbb{N}$, $m \geq 2$. We give a characterization of the class of functions f, for which $\omega_\varphi(f,h) \to \infty$, $(h \to 0)$ and then we obtain quantitative estimates of the approximation of the functions of this class by means of the Szász-Mirakjan operators and by the Baskakov operators.

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1 Introduction

In the approximation of functions defined on a non compact interval by sequences of positive linear operators, weighted moduli of continuity are useful tools. In References we give a part of the contributions in this direction. In our paper [13], starting from a class of "admissible" functions, we built a general weighted modulus and we obtain an estimate applicable to general positive linear operators. It was pointed out that for a particular example of weighted function this modulus allows the description of the class of functions, discovered by Totik [14], which can be uniformly approximated by the Szász-Mirajan operators.

In this paper we give a characterization of a class of functions which can be uniformly approximated on compact intervals, in terms of another particular case of the modulus given in [13].

Consider the weight function $\varphi(x) = \frac{\sqrt{x}}{1+x^m}$, $x \in [0,\infty)$, where $m \in \mathbb{N}$, $m \geq 2$ is a given parameter. Correspondingly we consider the following weighted modulus of continuity

$$\omega_\varphi(f,h) = \sup\left\{|f(x)-f(y)|\Big|\, x \geq 0,\ y \geq 0,\ |x-y| \leq h\varphi\left(\frac{x+y}{2}\right)\right\}. \qquad (1)$$

Denote by $\mathcal{F}([0,\infty))$ the linear space of real functions defined on interval $[0,\infty)$. For a real number $p > 0$ consider the function $e_p \in \mathcal{F}[0,\infty)$, $e_p(t) = t^p$. We denote by Π_k the set of polynomials of degree at most k.

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Theorem 1. ([13]) *Let $W \subset \mathcal{F}([0,\infty))$ be a linear subspace, such that $\Pi_{2m+2} \in W$. If $L : W \to \mathcal{F}([0,\infty))$ is a positive linear operator, then for any $f \in W$, any $x \in (0,\infty)$ and any $h > 0$ we have*

$$\begin{aligned} &|L(f,x) - f(x)| \leq |f(x)| \cdot |L(e_0,x) - 1| \\ &+\Big(L(e_0,x) + \frac{4}{h^2 x} L((e_1 - xe_0)^2(2e_0 + x^{2m}e_0 + e_{2m}), x)\Big)\omega_\varphi(f,h). \end{aligned} \tag{2}$$

2 An equivalence theorem

Theorem 2. *For a function $f \in \mathcal{F}([0,\infty))$ the following are equivalent:*

i) $\lim_{h\to+0} \omega_\varphi(f,h) = 0,$

ii) the function $f \circ e_2$ is uniformly continuous on the interval $[0,1]$ and the function $f \circ e_\nu$, $\nu = \frac{2}{2m+1}$, is uniformly continuous on the interval $[1,\infty)$.

Proof. i)⇒ ii) Let $\varepsilon > 0$ be arbitrarily chosen. There is $h_\varepsilon > 0$, such that for any $0 < h < h_\varepsilon$ we have $\omega_\varphi(f,h) < \varepsilon$. It follows that the implication below holds true:

$$|y - x| < h_\varepsilon \varphi\left(\frac{x+y}{2}\right)\ x, y \in [0,\infty) \Rightarrow\ |f(y) - f(x)| < \varepsilon. \tag{3}$$

First we show that if we take $\delta^1_\varepsilon = \frac{h_\varepsilon}{4}$, we have the implication

$$|v - u| < \delta^1_\varepsilon\ u, v \in [0,1] \Rightarrow\ |v^2 - u^2| < h_\varepsilon \varphi\left(\frac{u^2+v^2}{2}\right). \tag{4}$$

Indeed using the inequality $|u+v| \leq \sqrt{2(u^2+v^2)}$ it results that if $|v-u| < \delta^1_\varepsilon$, then

$$\begin{aligned} |v^2 - u^2| &\leq 2|v-u|\sqrt{\frac{u^2+v^2}{2}} \\ &< \frac{1}{2} \cdot h_\varepsilon \sqrt{\frac{u^2+v^2}{2}} \\ &\leq h_\varepsilon \sqrt{\frac{u^2+v^2}{2}} \left(1 + \left(\frac{u^2+v^2}{2}\right)^m\right)^{-1} \\ &= h_\varepsilon \varphi\left(\frac{u^2+v^2}{2}\right). \end{aligned}$$

If in relation (3) we take $x = u^2$, $y = v^2$, $u \in [0,1]$, $v \in [0,1]$ then we obtain the implication

$$|v - u| < \delta^1_\varepsilon,\ u, v \in [0,1] \Rightarrow\ |f(v^2) - f(u^2)| < \varepsilon, \tag{5}$$

i.e. function $f \circ e_2$ is uniformly continuous on the interval $[0,1]$.

Now we show that for $\delta_\varepsilon^2 = \frac{1}{2} \cdot h_\varepsilon$, we have the implication

$$|u - v| < \delta_\varepsilon^2,\ u, v \in [1, \infty) \Rightarrow\ |u^\nu - v^\nu| < h_\varepsilon \varphi\left(\frac{u^\nu + v^\nu}{2}\right). \tag{6}$$

Indeed, let $u, v \in [1, \infty)$, $|v - u| < \delta_\varepsilon^2$. Suppose, for instance that $u \geq v$. Set $t = \frac{v}{u}$. The following inequality

$$\nu \int_1^t x^{\nu-1} dx \leq (t-1)t^{\nu-1} \tag{7}$$

is immediately true, for any number $t \geq 1$. Then, by taking into account this inequality and the relation $\nu - 1 = \nu\left(\frac{1}{2} - m\right)$ we have successively

$$\begin{aligned}
|v^\nu - u^\nu| &= \nu \int_u^v s^{\nu-1} ds = \nu u^\nu \int_1^t x^{\nu-1} dx \\
&\leq u^\nu (t-1) t^{\nu-1} = (v - u) v^{\nu-1} \\
&< \frac{h_\varepsilon}{2} \cdot v^{\nu-1} = \frac{h_\varepsilon}{2} (v^\nu)^{\frac{1}{2} - m} \\
&\leq \frac{h_\varepsilon}{2} \left(\frac{u^\nu + v^\nu}{2}\right)^{\frac{1}{2} - m} \leq h_\varepsilon \varphi\left(\frac{u^\nu + v^\nu}{2}\right).
\end{aligned}$$

Therefore the implication given in (6) is true. Now if we apply relation (3), taking $x = u^\nu$ and $y = v^\nu$, then we obtain the implication

$$|v - u| < \delta_\varepsilon^2,\ u, v \geq 1 \Rightarrow\ |f(v^\nu) - f(u^\nu)| < \varepsilon, \tag{8}$$

i.e. function $f \circ e_\nu$ is uniformly continuous on the interval $[1, \infty)$.

ii)$\Rightarrow$ i) Let $\varepsilon > 0$ be arbitrarily chosen.

First, since function $f \circ e_2$ is uniformly continuous on the interval $[0, 1]$ there is $\delta_\varepsilon^1 > 0$ for which the implication (5) is true. If we take in (5), $u = \sqrt{x}$, $v = \sqrt{y}$, for $x, y \in [0, 1]$ we obtain the implication

$$|\sqrt{y} - \sqrt{x}| < \delta_\varepsilon^1,\ x, y \in [0, 1] \Rightarrow\ |f(y) - f(x)| < \varepsilon. \tag{9}$$

Take $h_\varepsilon^1 = \sqrt{2} \cdot \delta_\varepsilon^1$. For $x, y \in [0, 1]$, $|y - x| < h_\varepsilon^1 \varphi\left(\frac{x+y}{2}\right)$ we have successively

$$\begin{aligned}
|\sqrt{y} - \sqrt{x}| &< \frac{h_\varepsilon^1}{\sqrt{x} + \sqrt{y}} \cdot \varphi\left(\frac{x+y}{2}\right) \\
&\leq \frac{h_\varepsilon^1}{\sqrt{x+y}} \cdot \varphi\left(\frac{x+y}{2}\right) \\
&= \frac{h_\varepsilon^1}{\sqrt{2}} \left(1 + \left(\frac{x+y}{2}\right)^m\right)^{-1} \\
&\leq \delta_\varepsilon^1.
\end{aligned}$$

Hence, taking into account relation (9) we proved the implication:

$$|y-x|<h_\varepsilon^1\varphi\left(\frac{x+y}{2}\right),\ x,y\in[0,1]\Rightarrow\ |f(y)-f(x)|<\varepsilon. \tag{10}$$

Second, since function $f\circ e_\nu$ is uniformly continuous on the interval $[1,\infty)$ there is $\delta_\varepsilon^2\in(0,1)$ for which the implication (8) is true. If we choose in (8) $u=x^{1/\nu}$, $v=y^{1/\nu}$, for $x,y\in[1,\infty)$ we obtain the implication

$$|y^{m+\frac{1}{2}}-x^{m+\frac{1}{2}}|<\delta_\varepsilon^2,\ x,y\in[1,\infty)\Rightarrow\ |f(y)-f(x)|<\varepsilon. \tag{11}$$

Take $h_\varepsilon^2=2^{1-2m}\delta_\varepsilon^2$. For $x,y\in[1,\infty)$, $|y-x|<h_\varepsilon^2\varphi\left(\frac{x+y}{2}\right)$ we have successively

$$\begin{aligned}
|y^{m+\frac{1}{2}}-x^{m+\frac{1}{2}}| &= \frac{|y-x|}{\sqrt{x}+\sqrt{y}}\cdot\sum_{k=0}^{2m}(\sqrt{x})^k(\sqrt{y})^{2m-k}\\
&\le |y-x|(\sqrt{x}+\sqrt{y})^{2m-1}\\
&\le |y-x|(2(x+y))^{m-\frac{1}{2}}\\
&< h_\varepsilon^2\varphi\left(\frac{x+y}{2}\right)2^{2m-1}\left(\frac{x+y}{2}\right)^{m-\frac{1}{2}}\\
&= \delta_\varepsilon^2\sqrt{\frac{x+y}{2}}\left(1+\left(\frac{x+y}{2}\right)^m\right)^{-1}\left(\frac{x+y}{2}\right)^{m-\frac{1}{2}}\\
&\le \delta_\varepsilon^2.
\end{aligned}$$

Hence, taking into account relation (11) we obtained the implication:

$$|y-x|<h_\varepsilon^2\varphi\left(\frac{x+y}{2}\right),\ x,y\in[1,\infty)\Rightarrow\ |f(y)-f(y)|<\varepsilon. \tag{12}$$

Define

$$M:=\max_{t\in[0,\infty)}\varphi(t),\ q:=\min_{t\in[1/2,5/4]}\varphi(t),\ h_\varepsilon:=\frac{q}{M}\cdot\min\{h_\varepsilon^1,h_\varepsilon^2\}.$$

Let $x\in[0,\infty)$, $y\in[0,\infty)$ be such that $|y-x|\le h_\varepsilon\varphi\left(\frac{x+y}{2}\right)$ and $x\le y$. If x,y are both in interval $[0,1]$, or if they are both in interval $[1,\infty)$, then applying relation (10) or relation (12), respectively we deduce that $|f(y)-f(x)|<\varepsilon$. If $x\in[0,1]$ and $y\in[1,\infty)$ we have

$$|1-x|\le|y-x|\le\frac{q}{M}h_\varepsilon^1\varphi\left(\frac{x+y}{2}\right)\le qh_\varepsilon^1\le h_\varepsilon^1\varphi\left(\frac{x+1}{2}\right).$$

Also, since $q\le\varphi(1)=\frac{1}{2}$ and $h_\varepsilon^2<1$ we have

$$|y-1|\le|y-x|\le\frac{q}{M}h_\varepsilon^2\varphi\left(\frac{x+y}{2}\right)\le qh_\varepsilon^2<\frac{1}{2}.$$

It follows $y\le\frac{3}{2}$. Then, using the definition of q we get $q\le\varphi\left(\frac{1+y}{2}\right)$. Consequently we obtain $|y-1|\le h_\varepsilon^2\varphi\left(\frac{1+y}{2}\right)$.

From relations (10) and (12) it follows $|f(1) - f(x)| < \varepsilon$ and $|f(y) - f(1)| < \varepsilon$. So we derived the following implication

$$|y - x| \le h_\varepsilon \varphi\left(\frac{x+y}{2}\right),\ x, y \in [0, \infty) \Rightarrow\ |f(y) - f(x)| < 2\varepsilon. \tag{13}$$

Since $\varepsilon > 0$ was chosen arbitrarily we obtain $\lim_{h \to +0} \omega_\varphi(f, h) = 0$. □

3 Applications

1. Szász-Mirakjan operators

These operators are defined by:

$$S_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!}, \tag{14}$$

$x \in [0, \infty)$, $n \in \mathbb{N}$ and $f \in W$, where $W \subset \mathcal{F}([0, \infty))$ is the linear subspace of the functions f for which the series above is convergent.

Theorem 3. *For any compact interval $[a, b] \subset [0, \infty)$ and for any integer $m \ge 2$ there is a constant $C > 0$, depending only on a, b and m such that, for $f \in W$, $x \in [a, b]$ and $n \in \mathbb{N}$ we have*

$$|S_n(f, x) - f(x)| \le \left(9 + 8x^{2m} + \frac{C}{n}\right) \omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right). \tag{15}$$

Particularly, if $f \circ e_2$ is uniformly continuous on interval $[0, 1]$ and $f \circ e_\nu$, $\nu = \frac{2}{2m+1}$, is uniformly continuous on interval $[1, \infty)$, then

$$\lim_{n \to \infty} \|S_n(f) - f\|_{[a,b]} = 0. \tag{16}$$

Proof. For $j = 0, 1, \ldots, n \in \mathbb{N}$ and $x \in [a, b]$, set $M_n^j(x) := S_n(e_j, x)$. We have $M_n^0(x) = 1$ and the following recurrence relation

$$M_n^{j+1}(x) = xM_n^j(x) + \frac{x}{n} \cdot (M_n^j(x))'$$

holds. From this we deduce by induction that there are the coefficients $a_{j,k}$, such that

$$M_n^j(x) = \sum_{k=0}^{j-1} \frac{a_{j,k}}{n^k} \cdot x^{j-k},\ j \ge 1$$

Since we have the recurrence relation $a_{j+1,k} = (j + 1 - k)a_{j,k-1} + a_{j,k}$, $0 \le k \le j$, (where

$a_{j,-1} = 0$), we obtain $a_{j,0} = 1$ and $a_{j,1} = \frac{j(j-1)}{2}$. It follows

$$\begin{aligned}
&S_n((e_1 - xe_0)^2(2e_0 + x^{2m}e_0 + e_{2m}), x) \\
&= (2 + x^{2m})(M_n^2(x) - 2xM_n^1(x) + x^2M_n^0(x)) + M_n^{2m+2}(x) - 2xM_n^{2m+1}(x) + x^2M_n^{2m}(x) \\
&= (2 + x^{2m})\frac{x}{n} + \left(x^{2m+2} + \frac{(2m+2)(2m+1)}{2n} \cdot x^{2m+1} + \sum_{k=2}^{2m+1} \frac{a_{2m+2,k}}{n^k} \cdot x^{2m+2-k}\right) \\
&-2x\left(x^{2m+1} + \frac{(2m+1)2m}{2n} \cdot x^{2m} + \sum_{k=2}^{2m} \frac{a_{2m+1,k}}{n^k} \cdot x^{2m+1-k}\right) \\
&+x^2\left(x^{2m} + \frac{(2m-1)2m}{2n} \cdot x^{2m-1} + \sum_{k=2}^{2m-1} \frac{a_{2m,k}}{n^k} \cdot x^{2m-k}\right) \\
&= 2(1 + x^{2m}) \cdot \frac{x}{n} + O\left(\frac{1}{n^2}\right),
\end{aligned}$$

whith $O\left(\frac{1}{n^2}\right)$, uniform with regard to $x \in [a, b]$. Then we apply Theorem 1 for $h = \frac{1}{\sqrt{n}}$. The particular case follows from Theorem 2. □

2. Baskakov operators

These operators are defined by:

$$V_n(f, x) = (1 + x)^{-n} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k. \tag{17}$$

$x \in [0, \infty)$, $n \in \mathbb{N}$ and $f \in W_1$, where $W_1 \subset \mathcal{F}([0, \infty))$ is the linear subspace of the functions f for which the series above is convergent.

Theorem 4. *For any compact interval $[a, b] \subset [0, \infty)$ and for any integer $m \geq 2$ there is a constant $C > 0$, depending only on a, b and m such that, for $f \in W$, $x \in [a, b]$ and $n \in \mathbb{N}$ we have*

$$|V_n(f, x) - f(x)| \leq \left(1 + 8(1 + x^{2m})(1 + x) + \frac{C}{n}\right) \omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right). \tag{18}$$

Particularly, if $f \circ e_2$ is uniformly continuous on interval $[0, 1]$ and $f \circ e_\nu$, $\nu = \frac{2}{2m+1}$, is uniformly continuous on interval $[1, \infty)$, then

$$\lim_{n\to\infty} \|V_n(f) - f\|_{[a,b]} = 0. \tag{19}$$

Proof. For $j = 0, 1, \ldots$, $n \in \mathbb{N}$ and $x \in [a, b]$, set $M_n^j(x) := V_n(e_j, x)$. We have $M_n^0(x) = 1$ and the following recurrence relation

$$M_n^{j+1}(x) = xM_n^j(x) + \frac{x(1+x)}{n} \cdot (M_n^j(x))'$$

holds. From this we deduce by induction that there are polynomials $Q_{j,k}$, $1 \le k \le j-1$, of degree j, such that

$$V_n(f,x) = x^j + \sum_{k=1}^{j-1} \frac{Q_{j,k}(x)}{n^k},\ j \ge 1.$$

Also we have the recurrence $Q_{j+1,1}(x) = xQ_{j,1} + jx^j(x+1)$, from which we obtain that $Q_{j,1}(x) = \frac{j(j-1)}{2} \cdot x^{j-1}(1+x)$. Then it follows

$$\begin{aligned}
&V_n((e_1 - xe_0)^2(2e_0 + x^{2m}e_0 + e_{2m}), x) \\
&= (2+x^{2m})(M_n^2(x) - 2xM_n^1(x) + x^2M_n^0(x)) + M_n^{2m+2}(x) - 2xM_n^{2m+1}(x) + x^2M_n^{2m}(x) \\
&= (2+x^{2m})\frac{x(1+x)}{n} \\
&+ \left(x^{2m+2} + \frac{(2m+2)(2m+1)}{2n} \cdot x^{2m+1}(1+x)\ + \sum_{k=2}^{2m+1} \frac{Q_{2m+2,k}(x)}{n^k}\right) \\
&-2x\left(x^{2m+1} + \frac{(2m+1)2m}{2n} \cdot x^{2m}(1+x) + \sum_{k=2}^{2m} \frac{Q_{2m+1,k}(x)}{n^k}\right) \\
&+x^2\left(x^{2m} + \frac{(2m-1)2m}{2n} \cdot x^{2m-1}(1+x) + \sum_{k=2}^{2m-1} \frac{Q_{2m,k}(x)}{n^k}\right) \\
&= 2(1+x)(1+x^{2m}) \cdot \frac{x}{n} + O\left(\frac{1}{n^2}\right),
\end{aligned}$$

whith $O\left(\frac{1}{n^2}\right)$, uniform with regard to $x \in [a,b]$. Then we apply Theorem 1 for $h = \frac{1}{\sqrt{n}}$. The particular case follows from Theorem 2.

□

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