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A CLOSER LOOK AT THE SOLUTIONS OF A DEGENERATE STOCHASTIC DIFFERENTIAL EQUATION

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Abstract

In an attempt to better understand the existence and uniqueness of strong solutions of stochastic differential equations, we study a classical example of a degenerate stochastic differential equation, attributed to H. Tanaka, for which the existence and uniqueness of strong solutions fails, but for which we can explicitly describe the set of all (weak) solutions.

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1 Introduction

Consider the stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds, \qquad t \geq 0, \tag{1}$$

where B_t is a 1-dimensional Brownian motion starting at the origin and $\sigma, b : \mathbb{R} \to \mathbb{R}$ are given functions.

A fundamental problem in the study of stochastic process is to determine the necessary and sufficient conditions on σ and b which guarantee the existence and the uniqueness of the solution of the above SDE.

There are mainly two notions of solutions to the above SDE: strong and weak solutions. The main difference between them resides in the measurability of the solution. A strong solution X_t of the above SDE is required to be measurable with respect to the augmented filtration $\mathcal{F}_t = \sigma\left(\mathcal{F}_t^B \cup \mathcal{N}\right)$ of the driving Brownian motion B_t (see [3], pp. 285 for the precise definition), and in the case of a weak solution $(X_t, B_t, (\mathcal{F}_t)_{t\geq 0})$ we just require that X_t and B_t to be measurable with respect to the same filtration $\mathcal{F}_t$ (not necessary the augmented filtration of B_t).

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So, a strong solution X_t is a measurable functional of the path B_s, $0 \leq s \leq t$ (and the initial condition X_0), which shows that given the "input" B_t (and the initial condition X_0) we can determine the "output" X_t from the SDE (1) above. This is the *principle of causality* for dynamical systems, which corresponds to the intuition that if we model a certain system by a SDE like (1) which involves the probabilistic quantity B_t, then the solution X_t can be determined from the information available up to time t, that is the augmented σ-algebra generated by B_s, $0 \leq s \leq t$. In the case of a weak solution this is no longer the case; a weak solution is a triple $\left(X_t, B_t, (\mathcal{F}_t)_{t\geq 0}\right)$ which satisfies (1) and both processes X_t and B_t are measurable with respect to the same σ-algebra $\mathcal{F}_t$. If $\mathcal{F}_t$ is the augmented σ-algebra generated by $\mathcal{F}_t^B$ and $\mathcal{G}_t$, then the σ-algebra $\mathcal{G}_t$ can be interpreted as the supplementary amount of information needed to predict the behaviour of X_t from the behaviour of B_t.

Starting with the pioneering work of K. Itô (see for example [2]), several well-known authors studied the problem of existence and uniqueness of SDEs like (1) above.

In the driftless case $b \equiv 0$, the problem of weak existence and uniqueness was completely solved by Engelbert and Schmidt ([1]), as follows.

Consider the zero set of the function σ defined by

$$Z(\sigma) = \{x \in \mathbb{R} : \sigma(x) = 0\}$$

and the set of non-local integrability of σ^{-2}, defined by

$$I(\sigma) = \left\{x \in \mathbb{R} : \int_{-\varepsilon}^{\varepsilon} \frac{dy}{\sigma^2(x+y)} = \infty, \quad \forall \varepsilon > 0\right\}.$$

The result is the following.

Theorem 1 ([1]). *The stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s, \qquad t \geq 0, \tag{2}$$

has a non-exploding weak solution for every initial distribution of X_0 if and only if

$$I(\sigma) \subseteq Z(\sigma). \tag{3}$$

Moreover, for any initial distribution of X_0 the solution is weakly unique if and only if

$$I(\sigma) = Z(\sigma). \tag{4}$$

So, in the case of weak solutions the problem is completely solved.

In the case of strong solutions, there are several sufficient conditions which guarantee the existence and uniqueness of the strong solutions of (1) or (2), but there is no necessary and sufficient condition for it.

Consider the following hypotheses:

(A) There exists an increasing function $\rho : [0,\infty) \to [0,\infty)$ such that $\int_{0+} \frac{du}{\rho(u)} = +\infty$ and $(\sigma(x) - \sigma(y))^2 \leq \rho(|x-y|)$ for all $x, y \in \mathbb{R}$.

(B) There exists and increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $(\sigma(x) - \sigma(y))^2 \le |f(x) - f(y)|$ for all $x, y \in \mathbb{R}$.

The following result due to LeGall ([4]) is known.

Theorem 2 ([4])**.** *Suppose that σ,$b : \mathbb{R} \to \mathbb{R}$ are bounded measurable function and they satisfy one of the following three hypotheses:*

1. *σ satisfies (A) and b is Lipschitz;*
2. *σ satisfies (A) and there exists $\varepsilon > 0$ such that $|\sigma| \ge \varepsilon$;*
3. *σ satisfies (B) and there exists $\varepsilon > 0$ such that $\sigma \ge \varepsilon$.*

Then strong uniqueness holds for (1).

It is also known that if for example b is bounded σ is Lipschitz and $|\sigma|$ is bounded away from zero, then for any initial distribution of X_0 (independent of B_t) there exists a unique strong solution of (1).

Note that the condition (A) above requires the diffusion coefficient σ to be continuous, and the condition (B) can only be used if the coefficient σ is bounded below away from zero. In the case when σ has discontinuities and it is not bounded below away from zero, it is possible that the solution of (1) has no solution, or that the solution is not unique.

In an attempt to better understand the strong existence and uniqueness of solutions of SDEs, we present an example of a degenerate (discontinuous, not bounded below away from zero dispersion coefficient), for which strong existence and uniqueness fails, but we can describe explicitly the set of all solutions.

The following classical example, attributed to H. Tanaka (see [3], [5]) illustrates the case.

Example 1. *Consider the SDE*

$$X_t = \int_0^t sgn(X_s)\, dB_s, \qquad t \ge 0, \tag{5}$$

where $sgn(x) = \begin{cases} +1, & x \ge 0 \\ -1, & x < 0 \end{cases}$.

Weak existence and uniqueness holds for (5), but strong existence and uniqueness fails.

The first part of the statement follows from Theorem 1, for in this case $Z(\text{sgn}) = I(\text{sgn}) = \varnothing$. This can also be seen directly: consider a Brownian motion X_t and define B_t by $B_t = \int_0^t \text{sgn}(X_s)\, dX_s$. Then B_t is a continuous, square integrable with $\langle B \rangle_t = \int_0^t \text{sgn}^2(X_s)\, d\langle X \rangle_s = \int_0^t 1 ds = t$, so by Lévy's characterization of Brownian motion B_t is a Brownian motion with respect to the augmented filtration $\mathcal{F}_t$ generated by X_t. Note that

$$\int_0^t \text{sgn}(X_s)\, dB_s = \int_0^t \text{sgn}(X_s)\, \text{sgn}(X_s)\, dX_s = \int_0^t \text{sgn}^2(X_s)\, dX_s = X_t, \qquad t \ge 0,$$

so $(X_t, B_t, \mathcal{F}_t)$ is a weak solution of (5). This shows that weak existence holds for (5).

Also, note that if $(X_t, B_t, \mathcal{F}_t)$ is a weak solution of (5), using again Lévy's characterization of Brownian motion and the fact that $\langle X\rangle_t = \int_0^t \mathrm{sgn}^2(X_s)\, d\langle B\rangle_s = t$, it follows that X_t is a Brownian motion, so weak uniqueness (uniqueness in distribution) holds for (5).

Since X_t and $-X_t$ are at the same time solutions of (5) (and they cannot be the identically zero solution by the discussion above), strong uniqueness cannot hold for (5).

Assume now the existence of a strong solution X_t, so X_t is measurable with respect to the augmented filtration of B_t, or $\mathcal{F}_t^X \subset \mathcal{F}_t^B$ for all $t \geq 0$. Applying Tanaka's formula, we obtain

$$|X_t| = \int_0^t \mathrm{sgn}(X_s)\, dX_s + L_t^0(X) = \int_0^t \mathrm{sgn}(X_s)\,\mathrm{sgn}(X_s)\, dB_s + L_t^0(X) = B_t + L_t^0(X),$$

or

$$B_t = |X_t| - L_t^0(X), \qquad t \geq 0.$$

Since by definition $L_t^0(X) = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \mathrm{meas}\{s \in [0,t] : |X_s| < \varepsilon\}$ is $\mathcal{F}^{|X|}$ measurable, it follows that $\mathcal{F}^B \subset \mathcal{F}^{|X|}$, which leads to the contradiction $\mathcal{F}^X \subset \mathcal{F}^{|X|}$ (i.e. the process X_t is determined by the process $|X_t|$). The contradiction shows that there is no strong solution of (5).

In the next section we will examine closer the above example, by finding the explicit form of all (weak) solutions of (5), thus explaining the lack of strong existence and uniqueness of this SDE.

2 Main results

Given a non-negative continuous process Y_t we define a *sign choice* for Y_t as a process U_t taking the values ± 1, such that $U_t Y_t$ is a continuous process.

For example, we can construct a sign choice as follows. On a probability space $(\Omega, \mathcal{F}, P)$ consider a process V_t taking values ± 1 with probabilities $P(V_t = 1) = 1 - P(V_t = -1) = p \in [0,1]$ for all $t \geq 0$. Define the process U_t by $U_t = V_t$ if $Y_t = 0$ and $U_t = V_s$ otherwise, where $s = \sup\{u \leq t : Y_u = 0\}$ is the last visit of Y to 0 before time t. The process $(U_t)_{t\geq 0}$ is readily seen to be a sign choice for the process $(Y_t)_{t\geq 0}$. If moreover the processes Y and V are independent, then the sign choice U_t is independent of Y_t.

With this preparation we can present the main result, as follows.

Theorem 3. *For any Brownian motion B_t starting at the origin, $\left(U_t Y_t, B_t, (\mathcal{F}_t)_{t\geq 0}\right)$ is a weak solution of (5), where Y_t is the reflecting Brownian motion on $[0,\infty)$ with driving Brownian motion B_t, U_t is a sign choice for Y_t which takes the values ± 1 with equal probability, and $(\mathcal{F}_t)_{t\geq 0}$ is the augmented filtration generated by B_t and U_t.*

Conversely, any weak solution $\left(X_t, B_t, (\mathcal{F}_t)_{t\geq 0}\right)$ of (5) has the representation $X_t = U_t Y_t$, where U and Y are as above.

In particular, any solution of (5) is unique up to a sign choice, i.e. if X_t^1 and X_t^2 are solutions of (5), then

$$P\left(\left|X_t^1\right| = \left|X_t^2\right| \text{ for all } t \geq 0\right) = 1.$$

Proof. If X_t satisfies (5), applying the Tanaka-Itô formula to the function $f(x) = |x|$ and to the process X_t we obtain

$$|X_t| = \int_0^t \operatorname{sgn}(X_s)\, dX_s + L_t^0(X) = B_t + L_t^0(X), \tag{6}$$

where $L_t^0(X) = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(X_s)\, d\langle X \rangle_s$ denotes the (symmetric) semimartingale local time of X at the origin.

Consider the process $Y_t = |X_t|$ and note that $1_{(-\varepsilon,\varepsilon)}(Y_s) = 1_{(-\varepsilon,\varepsilon)}(|X_s|) = 1_{(-\varepsilon,\varepsilon)}(X_s)$ for any $\varepsilon > 0$ and $t \geq 0$. By Lévy's characterization of the Brownian motion it follows that the process X_t is a time change of a Brownian motion, with quadratic variation process

$$\langle X \rangle_t = \int_0^t \operatorname{sgn}^2(X_s)\, ds = \int_0^t 1 ds = t,$$

and in particular $d\langle X \rangle_t$ is absolutely continuous with respect to the Lebesgue measure.

It follows that the process X_t spends zero Lebesgue time at the origin, and therefore we obtain

$$\begin{aligned} L_t^0(Y) &= \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(Y_s)\, d\langle Y \rangle_s \\ &= \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(Y_s)\, ds \\ &= \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(X_s)\, d\langle X \rangle_s \\ &= L_t^0(X) \end{aligned}$$

From (6) it follows that the process $Y_t = |X_t|$ satisfies the SDE

$$Y_t = B_t + L_t^0(Y), \qquad t \geq 0,$$

and therefore the process Y_t is the reflecting Brownian motion on $[0, \infty)$ with driving Brownian motion B_t. In particular, the process $Y_t = |X_t|$ is adapted with respect to the filtration $\mathcal{F}^B$ of the Brownian motion B_t and it is pathwise unique.

We now show the existence of a weak solution of (5). Consider a Brownian motion B_t and let Y_t be the reflecting Brownian motion on $[0, \infty)$ with driving Brownian motion B_t, and let U_t be a sign choice for Y_t.

Consider the process $X_t = U_t Y_t$, $t \geq 0$. We will show that $\left(X_t, B_t, (\mathcal{F}_t)_{t \geq 0}\right)$ is a weak solution to (5), where $\mathcal{F}_t$ is the augmented σ-algebra generated by B_s and U_s, $0 \leq s \leq t$, i.e. the completion of the σ-algebra $\sigma(B_s, U_s : 0 \leq s \leq t)$ which contains all the null sets in $\mathcal{F}_\infty^B \cup \mathcal{F}_\infty^U$.

Note that by construction we have $\operatorname{sgn}(X_s) = U_s 1_{\mathbb{R}^*}(X_s) + 1_{\{0\}}(X_s)$, and therefore $\operatorname{sgn}(X_s) = U_s 1_{\mathbb{R}^*}(X_s) + 1_{\{0\}}(X_s)$ for any $s \geq 0$. Since the process Y (and hence X) spends zero Lebesgue time at the origin, we have almost surely

$$\int_0^t \operatorname{sgn}(X_s)\, dB_s = \int_0^t U_s 1_{\mathbb{R}^*}(X_s)\, dB_s$$

for any $t \geq 0$.

Since Y_t is the reflecting Brownian motion with driving Brownian motion B_t, we obtain

$$\begin{aligned}\int_0^t \operatorname{sgn}(X_s)\, dB_s &= \int_0^t U_s 1_{\mathbb{R}^*}(X_s)\, dY_s - \int_0^t U_s 1_{\mathbb{R}^*}(X_s)\, dL_s^0(Y) \\ &= \int_0^t U_s 1_{\mathbb{R}^*}(X_s)\, dY_s,\end{aligned}$$

where the last equality follows since the local time $L_s^0(Y)$ of Y at the origin increases only when Y_s (and hence X_s) is at the origin.

For $\varepsilon > 0$ arbitrarily fixed, consider the stopping times τ_n and σ_n defined by $\tau_0 = 0$ and

$$\sigma_n = \inf\{s \geq \tau_{n-1} : Y_s = \varepsilon\} \quad \text{and} \quad \tau_n = \{s \geq \sigma_n : Y_s = 0\}, \qquad n \geq 1.$$

Considering $D_t(\varepsilon) = \sup\{i \geq 0 : \tau_i < \infty\}$ the number of downcrossing of the interval $[0, \varepsilon]$ by the process Y_t, we obtain

$$\begin{aligned}\int_0^t \operatorname{sgn}(X_s)\, dB_s &= \int_0^t U_s 1_{\mathbb{R}^*}(X_s)\, dY_s \\ &= \sum_{i\geq 1} \int_{\sigma_i\wedge t}^{\tau_i\wedge t} U_s 1_{\mathbb{R}^*}(X_s)\, dY_s + \sum_{i\geq 1} \int_{\tau_{i-1}\wedge t}^{\sigma_i\wedge t} U_s 1_{\mathbb{R}^*}(X_s)\, dY_s \\ &= \sum_{i\geq 1} U_{\sigma_i\wedge t}\left(Y_{\tau_i\wedge t} - Y_{\sigma_i\wedge t}\right) + \sum_{i\geq 1} \int_{\tau_{i-1}\wedge t}^{\sigma_i\wedge t} U_s 1_{\mathbb{R}^*}(X_s)\, dY_s \\ &= -\varepsilon \sum_{i=1}^{D_t(\varepsilon)} U_{\sigma_i} + U_t(Y_t - \varepsilon) \sum_{i\geq 1} 1_{[\sigma_i, \tau_i)}(t) + \sum_{i\geq 1} \int_{\tau_{i-1}\wedge t}^{\sigma_i\wedge t} U_s 1_{\mathbb{R}^*}(X_s)\, dY_s,\end{aligned}$$

and therefore

$$\begin{aligned}\int_0^t \sigma(X_s)\, dB_s - U_t Y_t &= -\varepsilon \sum_{i=1}^{D_t(\varepsilon)} U_{\sigma_i} + U_t Y_t \sum_{i\geq 1} 1_{[\tau_{i-1}, \sigma_i)}(t) \\ &\quad -\varepsilon U_t \sum_{i\geq 1} 1_{[\sigma_i, \tau_i)}(t) + \sum_{i\geq 1} \int_{\tau_{i-1}\wedge t}^{\sigma_i\wedge t} U_s 1_{R^*}(X_s)\, dY_s.\end{aligned} \tag{7}$$

To prove the claim, we will show that the terms on the right of the above equality converge in L^2 to zero as $\varepsilon \searrow 0$.

By construction, $(U_{\sigma_i})_{i\geq 1}$ are independent random variables with mean $EU_{\sigma_i} = 0$ and variance $EU_{\sigma_i}^2 = 1$, so using the Wald's identity we obtain

$$E\left(\varepsilon \sum_{i=1}^{D_t(\varepsilon)} U_{\sigma_i}\right)^2 = \varepsilon^2 ED_t(\varepsilon)\, EU_{\sigma_1}^2 = \varepsilon^2 ED_t(\varepsilon) \to 0$$

as $\varepsilon \searrow 0$, since by Lévy's characterization of the local time we have $\varepsilon D_t(\varepsilon) \to L_t^0(Y)$ a.s. and also in L^2 (see [3], pp. 416). This shows the a.s. convergence to zero as $\varepsilon \searrow 0$ of the first term on the right of (7).

Next, note that if $t \in [\tau_{i-1}, \sigma_i)$ for some $i \geq 1$, by construction we have $Y_t \in [0, \varepsilon)$, so we obtain

$$E\left(U_t Y_t \sum_{i\geq 1} 1_{[\tau_{i-1},\sigma_i)}(t)\right)^2 \leq \varepsilon^2 E \sum_{i\geq 1} 1_{[\tau_{i-1},\sigma_i)}(t) \leq \varepsilon^2 t \to 0$$

as $\varepsilon \searrow 0$. This proves the a.s. convergence to zero as $\varepsilon \searrow 0$ of the second term on the right of (7). For the third term the proof being similar, we omit it.

To conclude the proof of the claim, using again Wald's identity and the fact that the random variables $\sigma_i - \tau_{i-1}$, $i = 1, 2, \ldots$ are independent (see for example Theorem 2.6.16 in [3]) with mean $E(\sigma_1 - \tau_0) = E\sigma_1 = \varepsilon^2$, we obtain

$$\begin{aligned} E\left(\sum_{i\geq 1} \int_{\tau_{i-1}\wedge t}^{\sigma_i \wedge t} U_s 1_{R^*}(X_s)\, dY_s\right)^2 &\leq E\int_0^t \sum_{i\geq 1} 1_{[\tau_{i-1},\sigma_i]}(s)\, d\langle Y\rangle_s \\ &\leq E \sum_{i=1}^{D_t(\varepsilon)+1} (\sigma_i - \tau_{i-1}) \\ &= E(\sigma_1 - \tau_0)\, E(D_t(\varepsilon) + 1) \\ &= \varepsilon^2 E(D_t(\varepsilon) + 1) \\ &\to 0, \end{aligned}$$

and therefore the last term on the right side of (7) also converges to zero as $\varepsilon \searrow 0$.

We have shown that all the terms on the right side of (7) converge to zero as $\varepsilon \searrow 0$. Passing to the limit with $\varepsilon \searrow 0$ in (7) we obtain $\int_0^t \sigma(X_s)\, dB_s = U_t Y_t = X_t$, which concludes the proof of the first claim.

To prove the second, note that by the previous proof a (weak) solution X_t has the representation $X_t = U_t |X_t| = U_t Y_t$, where Y_t is the reflecting Brownian motion on $[0, \infty)$ with driving Brownian motion B_t and U_t represents the sign of X_t. Note that when $X_t = 0$ we can choose in the above equation either $U_t = 1$ or $U_t = -1$, so $U_t = \pm 1$ for all $t \geq 0$.

Since U_t is altered only when $|X_t| = Y_t = 0$, U_t is a sign choice for Y_t, so it remains to show that $P(U_t = \pm 1) = \frac{1}{2}$. By the previous proof X_t is a Brownian motion starting at the origin, so $P(U_t = 1) = P(X_t \geq 0) = \frac{1}{2}$ for any $t \geq 0$, concluding the proof.

We conclude with the remark that the above theorem explains the lack of strong existence and strong uniqueness of (5): a weak solution has the form $X_t = U_t Y_t$, where Y_t is determined by B_t (it is the reflecting Brownian motion on $[0, \infty)$ with driving Brownian motion B_t), but the sign choice U_t is not necessarily determined by B_t. This causes the solution not to be $\mathcal{F}_t^B$ adapted, hence the lack of a strong solution of (5). Different sign choices U_t produce different solution (although they are the same in absolute value), and this causes the lack of uniqueness for (5). □

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