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# BRASS-DURRMEYER TYPE OPERATORS

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**Abstract**

We introduce the Durrmeyer variant of the Brass operators. We prove the property of polynomial degree preservation and, finally, give conditions for uniform approximation of continuous functions.

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## 1 Indroduction

Firstly, let us recall the extension of the Bernstein operators made by H. Brass.

If $P_n : C[0,1] \to C[0,1]$ and $Q_m : C[0,1] \to C[0,1]$, $n, m \in \mathbb{N}$ are two linear operators with equidistant knots, of the form:

$$P_n(f) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}$$

and

$$Q_m(f) = \sum_{j=0}^{m} f\left(\frac{j}{m}\right) q_{m,j},$$

where $p_{n,i} \in C[0,1]$, $(0 \le i \le n)$, $q_{m,j} \in C[0,1]$, $(0 \le j \le m)$, then their convolution product is defined by

$$(P_n \oplus Q_m)(f) := \sum_{i=0}^{n+m} f\left(\frac{i}{n+m}\right) r_{n+m,i},$$

where

$$r_{n+m,i} := \sum_{j=0}^{i} p_{n,j} q_{m,i-j}.$$

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Here we consider $p_{n,j} := 0 =: q_{m,j}$ if $i > n$ and $j > m$. Now let

$$A_n(f) := \sum_{i=0}^{n} f\left(\frac{i}{n}\right) q_{n,i}, \tag{1}$$

where $q_{n,0}(x) = 1 - x$, $q_{n,n}(x) = x$ and for the remaining indices the functions $q_{n,i}$ are zero.

A fundamental **Brass operator** can be defined as follows. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$ define the set $\Lambda_n$ of vectors $\mu = (\mu_1, ..., \mu_n) \in (\mathbb{N}_0)^n$, which satisfies the condition $\mu_1 + 2\mu_2 + ... + n\mu_n = n$. The vectors $\mu \in \Lambda_n$ are named "admissible" vectors.

For any admissible vector $\mu = (\mu_1, ..., \mu_n) \in \Lambda_n$, $n \in \mathbb{N}$, we name a **fundamental Brass operator**, [1], the following operator:

$$H_n^{\mu} := \oplus_{k=1}^{n} \oplus_{i=1}^{\mu_k} A_k. \tag{2}$$

Notice that for the choice $\mu = (n, 0, ..., 0)$ we obtain the Bernstein operator $B_n$. Then a **general Brass operator** is defined as a convex combination of the fundamental Brass operators:

$$L = \sum_{\mu \in \Lambda_n} \alpha(\mu) H_n^{\mu}, \tag{3}$$

where $\alpha(\mu) \geq 0$, for $\mu \in \Lambda_n$ and $\sum_{\mu \in \Lambda_n} \alpha(\mu) = 1$.

Mention that the main properties of the general Brass operators are immediate consequences of the same properties of the fundamentals Brass operators. So that we can restrict ourseves only to the fundamental Brass operators.

A generalization of the fundamental Brass operators was given in R, Păltănea [2] in the following mode. For $m \in \mathbb{N}$ set

$$D_m := \left\{ \tau = (t_{1,...,}t_m) \in \mathbb{R}^m : t_i > 0, \forall i = \overline{1, m}, \sum_{i=1}^{m} t_i = 1 \right\}.$$

For any integers $0 \leq k \leq m$ define $\theta_{m.k}(x) := x^k (1 - x)^{m-k}$. Also, for a set $A$, we will use the notation $|A|$ for the number of elements of the set $A$.

Now, for any vector $\tau = (t_1, \ldots, t_m) \in D_m$ define the operator

$$\begin{aligned} L_\tau(f, x) &:= \sum_{k=0}^{m} \sum_{1 \leq i_1 < ... < i_k \leq m} f\left(t_{i_1} + ... + t_{i_k}\right) \theta_{m,k}(x) \\ &= \sum_{I \subset \{1,...,m\}} f\left(\sum_{i \in I} t_i\right) \theta_{m,|I|}(x). \end{aligned} \tag{4}$$

Notice that for the choice $m = n$ and $\tau = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$ we obtain the Bernstein operator $B_n$. We shall now use the following result given in [2], Lemma 5.1.1:

**Lemma 1.** *For every admissible vector* $\mu = (\mu_1, ..., \mu_n) \in \Lambda_n$*, there is a vector* $\tau \in D_m$*,* *with $m = \mu_1 + ... + \mu_n$, such that $H_n^\mu = L_\tau$. More over $\tau$ is unique excepting a permutation of the components and has $\mu_k$ components equal to $\frac{k}{n}$, for each $k = \overline{1,n}$.*

From this result it follows that the Brass operators $H_n^\mu$ admit an alternative representation of the form (4), where the components $t_i$ are precised in Lemma 1.

## 2 Main results

Fix $m \in \mathbb{N}$ and $\tau = (t_1, \ldots, t_m) \in D_m$, such that there is a number $n \in \mathbb{N}$ for which we can write $t_i = \frac{r_i}{n}$, with $r_i \in \mathbb{N}$, for $1 \le i \le m$. We introduce a **Brass-Durrmeyer** type operator, assigned to the vector $\tau$, as the operator $U_\tau : C[0,1] \to C[0,1]$, given by:

$$\begin{aligned} U_\tau(f,x) &:= \sum_{I \subset \{1,\ldots,m\}} \theta_{m,|I|}(x) \frac{\int_0^1 f(t)\theta_{n,\sum_{j\in I} r_j}(t)dt}{\int_0^1 \theta_{n,\sum_{j\in I} r_j}(t)dt} \\ &= \sum_{i=0}^m \theta_{m,i}(x) \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=i}} \frac{\int_0^1 f(t)\theta_{n,\sum_{j\in I} r_j}(t)dt}{\int_0^1 \theta_{n,\sum_{j\in I} r_j}(t)dt}. \end{aligned} \tag{5}$$

We are interested in computing the moments of the operator $U_\tau$. Obviously, $U_\tau(e_0, x) = 1$. Using the Beta function, denoted by $B$, we have succesively

$$\begin{aligned} U_\tau(e_1,x) &= \sum_{i=0}^m \theta_{m,i}(x) \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=i}} \frac{\int_0^1 t\theta_{n,\sum_{j\in I} r_j}(t)dt}{\int_0^1 \theta_{n,\sum_{j\in I} r_j}(t)dt} \\ &= \sum_{i=0}^m \theta_{m,i}(x) \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=i}} \frac{B\left(\sum_{j\in I} r_j + 2, n - \sum_{j\in I} r_j + 1\right)}{B\left(\sum_{j\in I} r_j + 1, n - \sum_{j\in I} r_j + 1\right)} \\ &= \sum_{i=0}^m \theta_{m,i}(x) \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=i}} \frac{\sum_{j\in I} r_j + 1}{n+2} \\ &= \frac{1}{n+2} \sum_{i=0}^m \theta_{m,i}(x) \left( \sum_{I \subset \{1,\ldots,m\} |I|=i} \sum_{j\in I} r_j + \binom{m}{i} \right) \\ &= \frac{1}{n+2} \sum_{i=0}^m \theta_{m,i}(x) \left( \sum_{j=1}^m r_j \cdot |\{I \subset \{1,...,m\} \,|\, |I| = i \text{ and } j \in I\}| + \binom{m}{i} \right) \\ &= \frac{n}{n+2} \sum_{i=1}^m \theta_{m,i}(x) \binom{m-1}{i-1} + \frac{1}{n+2} \\ &= \frac{n}{n+2} x \sum_{i=0}^{m-1} \theta_{m-1,i}(x) \binom{m-1}{i} + \frac{1}{n+2} = \frac{n}{n+2}x + \frac{1}{n+2}. \end{aligned}$$

We prove now the central result of our paper. Denote by $\Pi_k$ the set of polynomials of degree at most $k$.

**Theorem 1.** *We have* $U_\tau(\Pi_k) \subset \Pi_k, \forall k = \overline{1,m}$. *More exactly, for each* $k = \overline{1,m}$, *we have*

$$U_\tau(e_k, x) =$$
$$\frac{(n+1)!}{(n+k+1)!}\sum_{c=0}^{k}\left(\sum_{p=c}^{k}\sum_{1\le i_1<\ldots<i_{k-p}\le k} i_1\ldots i_{k-p}\sum_{\substack{p_1,\ldots,p_m\ge 0\\ p_1+\ldots+p_m=p\\ |\{p_1,\ldots,p_m\}\cap\mathbb{N}|=c}}\frac{p!}{p_1!\ldots p_m!}r_1^{p_1}\ldots r_m^{p_m}\right)x^c.$$

*Proof.* Let us compute:

$$\begin{aligned}
U_\tau(e_k,x) &= \sum_{i=0}^{m}\theta_{m,i}(x)\sum_{\substack{I\subset\{1,\ldots,m\}\\|I|=i}}\frac{\int_0^1 t^k\theta_{n,\sum_{j\in I}r_j}(t)dt}{\int_0^1\theta_{n,\sum_{j\in I}r_j}(t)dt}\\
&= \sum_{i=0}^{m}\theta_{m,i}(x)\sum_{\substack{I\subset\{1,\ldots,m\}\\|I|=i}}\frac{B\left(\sum_{j\in I}r_j+k+1,n-\sum_{j\in I}r_j+1\right)}{B\left(\sum_{j\in I}r_j+1,n-\sum_{j\in I}r_j+1\right)}\\
&= \sum_{i=0}^{m}\theta_{m,i}(x)\sum_{\substack{I\subset\{1,\ldots,m\}\\|I|=i}}\frac{\left(\sum_{j\in I}r_j+k\right)\ldots\left(\sum_{j\in I}r_j+1\right)}{(n+k+1)\ldots(n+2)}\\
&= \frac{(n+1)!}{(n+k+1)!}\sum_{i=0}^{m}\theta_{m,i}(x)\sum_{\substack{I\subset\{1,\ldots,m\}\\|I|=i}}\left(\sum_{j\in I}r_j+k\right)\ldots\left(\sum_{j\in I}r_j+1\right).
\end{aligned}$$

We shall now use the relations

$$(a+k)\ldots(a+1)=\sum_{p=0}^{k}\left(\sum_{1\le i_1<\ldots<i_{k-p}\le k}i_1\ldots i_{k-p}\right)\cdot a^p$$

and

$$\left(\sum_{j=0}^{i}r_j\right)^p = \sum_{\substack{q_1,\ldots,q_i\ge 0\\ q_1+\ldots+q_i=p}}\frac{p!}{q_1!\ldots q_i!}r_1^{q_1}\ldots r_i^{q_i}$$

to obtain

$$U_\tau(e_k,x)=\frac{(n+1)!}{(n+k+1)!}\sum_{i=0}^{m}\theta_{m,i}(x)\sum_{I=\{j_1,\ldots,j_i\}\subset\{1,\ldots,m\}}$$
$$\sum_{p=0}^{k}\left(\sum_{1\le i_1<\ldots<i_{k-p}\le k} i_1\ldots i_{k-p}\right)\sum_{\substack{q_1,\ldots,q_i\ge 0\\ q_1+\ldots+q_i=p}}\frac{p!}{q_1!\ldots q_i!}r_{j_1}^{q_1}\ldots r_{j_i}^{q_i}.$$

Note that, for any $I=\{j_1,\ldots,j_i\}\subset\{1,\ldots,m\}$ and for any $q_1,\ldots,q_i\in\mathbb{N}_0$, such that $q_1+\ldots+q_i=p$, the product $r_{j_1}^{q_1}\ldots r_{j_i}^{q_i}$ can be written in the form $r_1^{p_1}\ldots r_m^{p_m}$, where $p_j=0$, if $j\notin I$ and $p_j=q_s$, if there is $s\in\{1,\ldots,i\}$, such that $j=j_s$.

Denote

$$\sigma_{k-p}=\sum_{1\le i_1<\ldots<i_{k-p}\le k} i_1\ldots i_{k-p}.$$

Then we can rewrite

$$\begin{aligned}
U_\tau(e_k,x) &= \frac{(n+1)!}{(n+k+1)!}\sum_{i=0}^{m}\theta_{m,i}(x)\sum_{p=0}^{k}\sigma_{k-p}\sum_{\substack{p_1,\ldots,p_m\ge 0\\ p_1+\ldots+p_m=p}}\frac{p!}{p_1!\ldots p_m!}r_1^{p_1}\ldots r_m^{p_m}\\
&\quad\cdot\left|\left\{I\subset\{1,\ldots,m\}\,|\,|I|=i \text{ and } p_j\ge 1\ \Rightarrow j\in I,\forall j=\overline{1,m}\right\}\right|\\
&= \frac{(n+1)!}{(n+k+1)!}\sum_{i=0}^{m}\theta_{m,i}(x)\sum_{p=0}^{k}\sigma_{k-p}\sum_{c=0}^{p}\sum_{\substack{p_1,\ldots,p_m\ge 0\\ p_1+\ldots+p_m=p\\ |\{p_1,\ldots,p_m\}\cap\mathbb{N}|=c}}\frac{p!}{p_1!\ldots p_m!}r_1^{p_1}\ldots r_m^{p_m}\\
&\quad\cdot\left|\left\{I\subset\{1,\ldots,m\}\,|\,|I|=i \text{ and } p_j\ge 1\ \Rightarrow j\in I,\forall j=\overline{1,m}\right\}\right|\\
&= \frac{(n+1)!}{(n+k+1)!}\sum_{i=0}^{m}\theta_{m,i}(x)\sum_{p=0}^{k}\sigma_{k-p}\sum_{c=0}^{p}\sum_{\substack{p_1,\ldots,p_m\ge 0\\ p_1+\ldots+p_m=p\\ |\{p_1,\ldots,p_m\}\cap\mathbb{N}|=c}}\frac{p!}{p_1!\ldots p_m!}r_1^{p_1}\ldots r_m^{p_m}\\
&\quad\cdot\binom{m-c}{i-c}\\
&= \frac{(n+1)!}{(n+k+1)!}\sum_{c=0}^{k}\left(\sum_{p=c}^{k}\sigma_{k-p}\sum_{\substack{p_1,\ldots,p_m\ge 0\\ p_1+\ldots+p_m=p\\ |\{p_1,\ldots,p_m\}\cap\mathbb{N}|=c}}\frac{p!}{p_1!\ldots p_m!}r_1^{p_1}\ldots r_m^{p_m}\right)x^c.
\end{aligned}$$

□

We obtain the following useful:

**Corollary 1.**

$$U_\tau\left(e_2,x\right)=\frac{1}{(n+3)(n+2)}\left[2+\left(3n+\sum_{j=1}^{m}r_j^2\right)\cdot x+\left(2\sum_{1\le j_1<j_2\le m}r_{j_1}r_{j_2}\right)\cdot x^2\right]. \quad (6)$$

Finally, we deduce the conditions for uniform approximation of continuos functions, using sequences of Brass-Durrmeyer type operators $U_\tau$.

**Theorem 2.** *Let* $\tau(n)=(t_1(n),...,t_{m_n}(n))\in D_{m_n}, n\in\mathbb{N}$. *If condition*

$$\sum_{j=1}^{m_n}t_j^2(n)=o(1),\ n\to\infty, \quad (7)$$

*holds, then* $U_{\tau(n)}(f,x)$ *converges uniformly to* $f(x)$ *on interval* $[0,1]$, *as* $n\to\infty$, *for each* $f\in C\,[0,1]$.

*Proof.* We have $U_{\tau(n)}(e_0)=e_0$. Also, from $U_{\tau(n)}(e_1)=\frac{n}{n+2}\cdot e_1+\frac{1}{n+2}\cdot e_0$ we obtain immediately that $U_{\tau(n)}(e_1)$ converges uniformly to $e_1$ on $[0,1]$ when $n\to\infty$.

We have

$$\sum_{1\le i_1<i_2\le m_n}t_{i_1}(n)t_{i_2}(n)=\frac{1}{2}\left(\left(\sum_{i=1}^{m_n}t_i(n)\right)^2-\sum_{i=1}^{m_n}t_i^2(n)\right)=\frac{1}{2}\left(1-\sum_{i=1}^{m_n}t_i^2(n)\right).$$

Then, from (7) it follows

$$\sum_{1\le i_1<i_2\le m_n}t_{i_1}(n)t_{i_2}(n)=\frac{1}{2}+o(1),\ n\to\infty. \quad (8)$$

Since $t_i(n)=\frac{r_i(n)}{n}$, then from relations (7), (8) and (6) we deduce that $U_{\tau(n)}(e_2)$ converges uniformly to $e_2$ on $[0,1]$ when $n\to\infty$. Hence we can apply the theorem of Korovkin. □

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