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SOME VARIATIONAL CONSIDERATIONS ON FIRST BOUNDARY VALUE PROBLEM IN NON-CLASSICAL ELASTICITY

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Abstract

We shall transform the first boundary value problem of Elasticity for micropolar bodies in a variational form. For the operator of Elasticity, built in this context, we prove that it is positive definite. This property ensures the existence of a solution for the first boundary value problem and the possibility to approximate it by variational methods.

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1. Introduction

There are many studies dedicated to the boundary value problems in psysical mathematics. In the context of classical Elasticity many such boundary value problems are approached by variational methods. Few such studies are reviewed in the book [3]. In the paper [2] the authors give some solutions for elliptic equations defined on certain Sobolev spaces, using a variational method. Some similar results are obtained in paper [1] in which the authors obtain positive solutions for singular p-Laplacian problems with sign changing nonlinearities using variational methods.

In the present study, we shall consider some similar considerations in the contex of non-classical Elasticity. Some of the previous results on variational formulations for the boundary value problem, including the boundary value problems of non-classical Elasticity, are presented in the book [10].

Also, study [5] deals with some new results regarding the applications of calculus of variations in the study of boundary value problems. The paper is concerned with the linear theory of micromorphic elastic solids. The authors present a minimum principle and existence results in the equilibrium theory. In the study [9] there are considered two cases of the theory of the heat conduction model with three-phase-lag. For each one the authors propose a suitable Lyapunov function. These functions are relevant tools which allow the study of several qualitative properties.

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In our study [7] we generalize the previous results on minimum principles in order to cover the dipolar elastic materials with stretch. For the boundary value problem considerd in this context, we prove an extension of the principle of minimum potential energy and, as a consequence, a generalized existence result for the above mentioned boundary value problem.

2. Basic equations

We shall consider an elastic homogeneous body which occupies a properly regular region Ω of the three-dimensional Euclidian space, bounded by the piece-wise smooth surface Σ and we denote the closure of Ω by $\bar{\Omega}$.

We refer the motion of the body to a fix system of rectangular Cartesian axes $Ox_i,\ i = 1,2,3$ and adopt the Cartesian tensor notation. Points in Ω are denoted by $x_j,\ j = 1,2,3$. Throughout this work the Einstein summation convention over repeated indices is used. The subscript j after the comma indicates partial differentiation with respect to the spatial argument x_j. All Latin subscripts are understood as ranging over the integers $(1,2,3)$, while the Greek indices have the range $(1,2)$.

Also, the spatial argument of a function will be omitted when there is no likelihood of confusion.

We restricte our considerations only to the case of Elastostatics. Thus we consider the equilibrium equations in the well known form

$$\begin{aligned}
&(\mu+\alpha)u_{i,\,kk} + (\lambda+\mu-\alpha)u_{k,\,ki} + 2\alpha\varepsilon_{ijk}\varphi_{k,\,j} + F_i = 0,\\
&(\gamma+\varepsilon)\varphi_{i,\,kk} + (\beta+\gamma-\varepsilon)\varphi_{k,\,ki} + 2\alpha\varepsilon_{ijk}u_{k,\,j} - 4\alpha\varphi_i + G_i = 0.
\end{aligned} \tag{1}$$

The geometric equations have the form

$$e_{ij} = u_{j,\,i} + \varepsilon_{ijk}\varphi_k,\ \eta_{ij} = \varphi_{j,\,i} \tag{2}$$

The basic equations of Elastostatics of micropolar bodies are complete if we also consider, the constitutive equations

$$\begin{aligned}
t_{ij} &= \lambda e_{kk} + (\mu+\alpha)e_{ij} + (\mu-\alpha)e_{ji},\\
\mu_{ij} &= \beta\varphi_{kk}\delta_{ij} + (\gamma+\varepsilon)\eta_{ij} + (\gamma-\varepsilon)\eta_{ji},
\end{aligned} \tag{3}$$

In the above equations we have used the following notations

- u_i and φ_i the components of the displacement vector and the components of the microrotation vector, respectively;

- F_i and G_i the components of the mass forces and the components of the couple mass forces, respectively;

- e_{ij} and η_{ij} the components of deformation tensor and the components of the couple strain tensor, respectively;

- t_{ij} and μ_{ij} the components of the stress tensor and the components of the couple stress tensor, respectively;
- α, β, γ, ε and μ the characterisctic constants of the body which describe the elastic properties of the material;
- ε_{ijk} is permutation tensor of Ricci and δij is the tensor of Kronecker.

As usual, we denote by n_i the components (the direction cosines) of the outward unit normal to the surface $\partial\Omega$ and we shall use the notations

$$t_i = t_{ij}\ n_j,\ \mu_i = \mu_{ij}\ n_j, \tag{4}$$

where t_i and μ_i represent the stress and the couple stress on unit area of the surface $\partial\Omega$ in direction of the normal.
From (3) and (4) we deduce:

$$\begin{aligned} t_j &= \lambda u_{k,\,k}\ n_j + 2\mu u_{j,\,i}\ n_i + (\mu-\alpha)\,(u_{j,\,i}+u_{i,\,j})\,n_i + 2\alpha\varepsilon_{kji}\varphi_k n_i,\\ \mu_j &= \beta\varphi_{k,\,k}\ n_j + 2\gamma\varphi_{j,\,i}n_i + (\gamma-\varepsilon)\,(\varphi_{i,\,j}-\varphi_{j,\,i})\,n_i \end{aligned} \tag{5}$$

3. Main results

For the first boundary value problem, we must add, to the above basic equations, the following null boundary conditions on the surface $\partial\Omega$:

$$u_i(x) = 0,\ \varphi_i(x) = 0\ \ for\ \ \forall\, x \in \partial\Omega. \tag{6}$$

Thus, the first fundamental boundary value problem of Elasticity for micropolar bodies consits of determination of the functions u_i and φ_i which satisfy the equilibrium equations (1) and the boundary conditions (6).
Let us introduce the vectorial notations

$$\begin{aligned} \vec{u} &= (u_1,\ u_2,\ u_3,\ \varphi_1,\ \varphi_2,\ \varphi_3)\\ \vec{F} &= (F_1,\ F_2,\ F_3,\ G_1,\ G_2,\ G_3) \end{aligned}$$

Also, we introduce the operators

$$\begin{aligned} L_i(u,\ \varphi) &= (\mu+\alpha)u_{i,\,kk} + (\lambda+\mu-\alpha)u_{k,\,ki} + 2\alpha\varepsilon_{ijk}\varphi_{k,\,j}\\ M_i(u,\ \varphi) &= (\gamma+\varepsilon)\varphi_{i,\,kk} + (\beta+\gamma-\varepsilon)\varphi_{k,\,ki} + 2\alpha\varepsilon_{ijk}u_{k,\,j} - 4\alpha\varphi_i \end{aligned} \tag{7}$$

We must outline that with the help of the operators (7), the equilibrium equations (1) received the simpler form

$$\begin{aligned} L_i(u,\ \varphi) + F_i &= 0,\\ M_i(u,\ \varphi) + G_i &= 0. \end{aligned} \tag{8}$$

If we introduce the vectorial operator A by

$$A\vec{u} = (-L_1\vec{u},\ -L_2\vec{u},\ -L_3\vec{u},\ -M_1\vec{u},\ -M_2\vec{u},\ -M_3\vec{u})$$

The operator A is called the operator of elasticity for first fundamental boundary value problem of micropolar bodies .

Taking into account the significance of the vector fields $\vec{u}$ and $\vec{F}$, with the help of the operator A we can write the equilibrium equations in the form

$$A\vec{u} = \vec{F}.$$

Also, the boundary conditions become

$$\vec{u} = 0 \ \ on \ \ \partial\Omega.$$

Therefore, our fundamental problem become

$$\begin{cases} A\vec{u} = \vec{F} \\ \vec{u} = 0 \ \ on \ \ \partial\Omega. \end{cases} \tag{9}$$

For this form of the problem, it is easy to prove the uniqueness of the solution. For this aim, we consider the bilinear form

$$\begin{aligned} 2E(\vec{u}, \ \vec{v}) = \lambda e_{mm}(\vec{u})\, e_{nn}(\vec{v}) + (\mu+\alpha) e_{ij}(\vec{u})\, e_{ij}(\vec{v}) + \\ +(\mu-\alpha) e_{ji}(\vec{u})\, e_{ji}(\vec{v}) + \beta\varphi_{mm}(\vec{u})\, \varphi_{nn}(\vec{v}) + (\gamma+\varepsilon)\varphi_{ij}(\vec{u})\, \varphi_{ij}(\vec{v}) + \\ +(\gamma-\varepsilon)\varphi_{ji}(\vec{u})\, \varphi_{ji}(\vec{v}) \end{aligned} \tag{10}$$

It is easy to see that if we substitute $\vec{v}$ by $\vec{u}$ the bilinear form $E(\vec{u}, \ \vec{v})$ from (10) becomes the quadratic form $E(\vec{u})$ which is the density of the internal energy of the micropolar elastic body.

Eringen proved in [5] that the quadratic form $E(\vec{u})$ is positive definite if and only if the elastic coefficients satisfy the following inequalities

$$\begin{aligned} 3\lambda + 2\mu > 0, \ \mu > 0, \ \alpha > 0, \ \mu - \alpha > 0 \\ 3\beta + 2\gamma > 0, \ \gamma > 0, \ \varepsilon > 0, \ \gamma - \varepsilon > 0 \end{aligned} \tag{11}$$

Also, based on the fact that the internal energy is positive definite it is easy to obtain the uniqueness of the solution of the boundary value problem (9).

By using the same procedure as in the classical elasticiy, from (1), (5) and (10) we can deduce the following reciprocal relations

$$\int_\Omega \vec{v}A\vec{u}\, dv = 2\int_\Omega E(\vec{u}, \ \vec{v})\, dv - \int_{\partial\Omega} \vec{v}S(\vec{u})\, d\sigma \tag{12}$$

$$\int_\Omega \vec{u}A\vec{u}\, dv = 2\int_\Omega E(\vec{u}, \ \vec{u})\, dv - \int_{\partial\Omega} \vec{u}S(\vec{u})\, d\sigma \tag{13}$$

$$\int_\Omega [\vec{v}A\vec{u} - \vec{u}A\vec{v}]\, dv = \int_{\partial\Omega} [\vec{u}S(\vec{v}) - \vec{v}S(\vec{u})]\, d\sigma \tag{14}$$

In the above relations the operator S is defined by

$$\vec{u}S\left(\vec{v}\right) = u_i t_i\left(\vec{v}\right) + \varphi_i\,\mu_i\left(\vec{v}\right) \tag{15}$$

With the help of the reciprocal relations (12)-(14) we can obtain different kind of displacements and microrotations.

For instance, in the case of an infinite domain Ω, we consider two points $X\left(x_i\right)$ and $Y\left(y_i\right)$ belong to such a domain. We suppose that in the point $Y\left(y_i\right)$ act a system of concentrated forces $\vec{Q}^{(k)}$ of the form

$$\vec{Q}^{(k)} = \delta_{kj}\vec{e}_j$$

where $\vec{e}_j$ are the unit vectors of the coordinate axes.
Then in the infinite domain Ω we obtain the following displacements and microrotations

$$\begin{aligned} u_n^{(k)}(X;\ Y) = {} & \frac{3-\nu}{16\pi\mu(1-\nu)}\frac{\delta_{nk}}{r} + \\ & + \frac{1}{16\pi\mu(1-\nu)}\frac{(x_n-y_n)(x_k-y_k)}{r^3} - \\ & - \frac{\gamma+\varepsilon}{16\pi\mu^2}\varepsilon_{nim}\varepsilon_{mjk}\varrho,_{ij} \\ \varphi_n^{(k)}(X;\ Y) = {} & \frac{1}{8\pi\mu}\varepsilon_{nik}\varrho,_{i} \end{aligned} \tag{16}$$

In the above relations we have used the notations

$$r = \left[\sum_{i=1}^{3}\left(x_i-y_i\right)^2\right]^{1/2}$$

$$\varrho(r,l) = \frac{1}{r}\frac{1}{(1-e)^{r/l}}$$

Now, we suppose that in the point $Y\left(y_i\right)$ of the infinite domain Ω act a system of concentrated mass moments $\vec{M}^{(k)}$ of the form

$$\vec{M}^{(k)} = \delta_{kj}\vec{e}_j$$

where δ_{ij} is the Kronecker symbol.
Then in the infinite domain Ω we obtain the following displacements and microrotations

$$\begin{aligned} u_n^{(k)}(X;\ Y) &= \frac{1}{8\pi\mu}\varepsilon_{nik}\varrho,_{i} \\ \varphi_n^{(k)}(X;\ Y) &= -\frac{1}{16\pi\alpha}\chi,_{nk} + \frac{\mu+\alpha}{16\pi\mu\alpha}\varepsilon_{nim}\varepsilon_{mjk}\varrho,_{ij} \end{aligned} \tag{17}$$

In the last above relation we used the notation

$$\chi(r,\ h) = \frac{1}{r}\left(1-\frac{1}{e^{r/h}}\right)$$

Also, in the above representations of the displacements and of the microrotations, the coefficients ν, l and h are constants which characterize the mechanical properties of the micropolar material and have the expressions

$$\nu = \frac{\lambda}{2(\lambda+\mu)},$$
$$l^2 = \frac{(\gamma+\varepsilon)(\mu+\alpha)}{4\mu\alpha}$$
$$h^2 = \frac{\beta+2\gamma}{4\alpha}$$

In the Literature on the subject, ν is the Poisson′s coefficient of transverse contraction.

Let us consider a particular micropolar elastic body, namely a material for which the characteristic coefficients satisfy the equalities

$$\begin{aligned} &\lambda = \beta = 0\\ &\mu = \alpha = \gamma = \varepsilon = \frac{1}{2}\\ &l = 1,\ h = \frac{1}{\sqrt{2}} \end{aligned} \tag{18}$$

It is easy to see that these coefficients satisfy the restrictions (11).

We shall introduce (18) in (16) and (17) and we denote the correspondent displacements and microrotations by $U_{np}^{(k)}$ and $\Phi_{np}^{(k)}$, $p = 1, 2$.
By direct calculations we can prove that

$$\begin{aligned} A_0\vec{U}_1^{(k)} &= \vec{Q}^{(k)}\delta(X-Y),\\ A_0\vec{U}_2^{(k)} &= \vec{M}^{(k)}\delta(X-Y), \end{aligned} \tag{19}$$

where we used the notation

$$\vec{U}_p^{(k)} = \left(U_{1p}^{(k)},\ U_{2p}^{(k)},\ U_{3p}^{(k)},\ \Phi_{1p}^{(k)},\ \Phi_{2p}^{(k)},\ \Phi_{3p}^{(k)}\right),\ p = 1, 2$$

Also, as usual, δ is the well known Dirac's distribution. In (19) the operator A_0 is obtained from the operator A by substituting the general coefficients by the particular case from (18). Moreover, taking into account (18) from the constitutive equations (3) we obtain

$$t_{ij} = e_{ij},\ \mu_{ij} = \eta_{ij}. \tag{20}$$

The relations (5), written for a sphere having the center in point $Y(y_i)$ and the radius $r = |X-Y|$, lead to

$$\begin{aligned} t_{sp}^{(k)} &= \left(U_{sp,q}^{(k)} + \varepsilon_{rsq}\Phi_{rp}^{(k)}\right) n_q,\\ \mu_{sp}^{(k)} &= \Phi_{sp,\,q}^{(k)} n_q\\ n_q &= \frac{y_q - x_q}{r} \end{aligned} \tag{21}$$

For the above functions $\varrho(r,l)$ and $\chi(r,h)$ we introduce the notations

$$\varrho^0(r) = \varrho(r,1),\ \chi^0(r) = \chi(r,\frac{1}{2})$$

For sufficiently small r, the derivatives of the functions $\varrho^0(r)$ and $\chi^0(r)$ have the following behavior

$$\begin{aligned}
&\varrho^0_{,\,j} n_q = A_{jq}(r) = \mathcal{O}(1)\\
&\varrho^0_{,\,ij} n_q = B_{ijq}(r) = \mathcal{O}(\frac{1}{r})\\
&\chi^0_{,\,sk} n_q = C_{skq}(r) = \mathcal{O}(\frac{1}{r})\\
&\varrho^0_{,\,jq} n_q = A_j(r) = \mathcal{O}(1)\\
&\quad\varrho^0_{,\,ijq} n_q = -\frac{\delta_{ij}}{2r^2} + \frac{(x_s-y_s)(x_k-y_k)}{2r^4} + B_{ij}(r),\ B_{ij}(r) = \mathcal{O}(1)\\
&\quad\chi^0_{,\,skq} n_q = -\frac{\delta_{sk}}{r^2} - \frac{(x_s-y_s)(x_k-y_k)}{r^4} + C_{sk}(r),\ C_{sk}(r) = \mathcal{O}(1)
\end{aligned} \tag{22}$$

where $\mathcal{O}$ is the well known Lambdau symbol.

Taking into account the relations (21) and (22), the stresses and the couple stresses, on an element of the surface of the sphere above considered, receive the form

$$\begin{aligned}
t^{(k)}_{s1} &= \frac{1}{8\pi}\left[\frac{3\delta_{sk}}{r^2} - \frac{(x_s-y_s)(x_k-y_k)}{r^4}\right] +\\
&\quad + \frac{1}{8\pi}\varepsilon_{sir}\varepsilon_{rjk}\left[\frac{\delta_{ij}}{r^2} - \frac{(x_s-y_s)(x_k-y_k)}{r^4}\right] +\\
&\quad + \frac{1}{4\pi}\left(\varepsilon_{sir}\varepsilon_{rjk}A_{ij} - \varepsilon_{sir}\varepsilon_{rjk}B_{ij}\right)\\
\mu^{(k)}_{s1} &= \frac{1}{4\pi}\varepsilon_{sjk}A_j\\
t^{(k)}_{s2} &= \frac{1}{8\pi}\left(\varepsilon_{sjk}A_j - \varepsilon_{rsm}C_{rkm} - \varepsilon_{rsp}\varepsilon_{riq}\varepsilon_{qjk}B_{ijp}\right)\\
\mu^{(k)}_{s2} &= \frac{1}{8\pi}\left[\frac{\delta_{sk}}{r^2} - \frac{(x_s-y_s)(x_k-y_k)}{r^4}\right] -\\
&\quad - \frac{1}{8\pi}\varepsilon_{sir}\varepsilon_{rjk}\left[\frac{\delta_{ij}}{r^2} - \frac{(x_i-y_i)(x_j-y_j)}{r^4}\right] -\\
&\quad - \frac{1}{8\pi}C_{sk} - \frac{1}{4\pi}\varepsilon_{sir}\varepsilon_{rjk}B_{ij}
\end{aligned} \tag{23}$$

Now, we want to prove that the above definite operator A is positive definite, as in the classical elasticity.

As usual, we denote by $L_2(\Omega)$ the Hilbert space of vector functions f such that

$$\int_\Omega \|f\|^2 dv < \infty$$

the norm has been obtained from the scalar product

$$(\vec{u},\ \vec{v}) = \int_{\Omega} \vec{u}.\vec{v}\, dv = \int_{\Omega} \sum_{i=1}^{3} u_i v_i\, dv$$

Let us denote by $\mathcal{C}_0$ a subset of the Hilbert space $L_2(\Omega)$, which consists of all vectorial functions f with the properties:

i) f is continuous on $\bar{\Omega} = \Omega \cup \partial\Omega$;

ii) f is twice derivable in Ω;

iii) $f = 0$ on the boundary $\partial\Omega$.

Taking into account the relation (14) we deduce that A is a linear symmetrical operator for each $\vec{u} \in \mathcal{C}_0$. Also, it is easy to observe, taking into account the relation (13) that A is a positive operator on the set $\mathcal{C}_0$.

Now, we can complete the properties of the operator A.

Theorem 1. *The operator of the elasticity of micropolar bodies is positive definite on the set* $\mathcal{C}_0$.

Proof. Taking into account that A is a symmetrical operator and, also, A is positive, we must prove that

$$(A\vec{u},\ \vec{u}) \geq C_1^2 \|\vec{u}\|,\ \forall\, \vec{u} \in \mathcal{C}_0, \tag{24}$$

where

$$\|\vec{u}\| = (\vec{u},\ \vec{u})\,,\ C_1^2 = const.,\ C_1^2 > 0$$

As it is suggested in [8], we consider a ball $S(Y,\ R)$ having the center in the point $Y(y_i)$ and the radius R. We denote by Σ_R the boundary of the ball and by Ω_R the domain obtained by subtracting the ball from the domain Ω, that is

$$\Omega_R = \Omega \setminus S(Y,\ R)$$

We write the reciprocal relation (12) for the domain Ω_R, where $\vec{u} \in \mathcal{C}_0$. Next, we shall write the reciprocal relation (12) for some particular solutions, for instance, as in (16) or (17), $\vec{v} = \vec{U}_p^{(k)}$.

By taking into account the fact that on the domain Ω_R we have

$$A_0 \vec{U}_p^{(k)} = 0$$

where A_0 is the operator A in the particular case that we consider. This way, (12) becomes

$$2\int_{\Omega_R} E\left(\vec{u},\ \vec{U}_p^{(k)}\right) dv - \int_{\Sigma_R} \vec{u} S\left(\vec{U}_p^{(k)}\right) d\sigma = 0,\ p = 1, 2. \tag{25}$$

where, in view of (15), we have

$$\int_{\Sigma_R} \vec{u} S\left(\vec{U}_p^{(k)}\right) d\sigma = \int_{\Sigma_R} \left[u_i t_i\left(\vec{U}_p^{(k)}\right) + \varphi_i \mu_i\left(\vec{U}_p^{(k)}\right)\right] d\sigma \tag{26}$$

In (25) and (26) the functions t_s and μ_s are calculated by using the relations (23) where r is replaced by R.
Now, we pass to the limit in (26), as $R \to 0$, and taking into account the order of magnitude of the expressions in (23), we deduce

$$\lim_{R\to 0} \int_{\Sigma_R} \vec{u} S\left(\vec{U}_1^{(k)}\right) d\sigma = u_k(Y)$$
$$\lim_{R\to 0} \int_{\Sigma_R} \vec{u} S\left(\vec{U}_2^{(k)}\right) d\sigma = \varphi_k(Y)$$
$$\lim_{R\to 0} \int_{\Omega_R} E\left(\vec{u},\, \vec{U}_p^{(k)}\right) dv = \int_{\Omega} E\left(\vec{u},\, \vec{U}_p^{(k)}\right) dv$$

If we substitute the particular coefficients (18) in the bilinear form E from (10) and taking into account the previous equalities, the relation (25) leads to

$$u_k(Y) = \sum_{i,j=1}^{3} \int_{\Omega} \left[t_{ij}\left(\vec{u}\right)\ t_{ij}\left(\vec{U}_1^{(k)}\right) + \eta_{ij}\left(\vec{u}\right)\ \eta_{ij}\left(\vec{U}_1^{(k)}\right)\right] dv$$
$$\varphi_k(Y) = \sum_{i,j=1}^{3} \int_{\Omega} \left[t_{ij}\left(\vec{u}\right)\ t_{ij}\left(\vec{U}_2^{(k)}\right) + \eta_{ij}\left(\vec{u}\right)\ \eta_{ij}\left(\vec{U}_2^{(k)}\right)\right] dv \quad (27)$$

Hence, with the help of the relations (26) we can compute the displacements and the microrotations in the point Y by using some operators applied to the functions t_{ij} and μ_{ij}. From the above considerations we deduce that

$$t_{ij},\ \eta_{ij} \in C^1(\Omega),$$
$$\left\{\left|t_{ij}\left(\vec{U}_1^{(k)}\right)\right|,\ \left|\eta_{ij}\left(\vec{U}_1^{(k)}\right)\right|\right\} \leq \frac{const.}{r^2}$$

that is, the kerns of the integrals from (27) have, in point $Y(y_i)$ a weak (eliminable) singularity. According to a known result, the above mentioned operators are continuous. By using a result from book [8], the functions of the form

$$\chi_{ij}^{(k)}(Y) = \int_{\Omega} \Psi_{ij}\left(\vec{u}\right) \Psi_{ij}\left(\vec{U}^{(k)}\right) dv$$

where

$$\Psi_{ij}\left(\vec{u}\right) = t_{ij}\left(\vec{u}\right), ...,$$

which appear in (27), possess a finite norm on the Hilbert space $L_2(\Omega)$ and verify an inequality of the form

$$\left\|\chi_{ij}^{(k)}\right\| \leq C_k \left\|\Psi_{ij}\right\|, \quad (28)$$

where the constants C_k depend only on the domain Ω and

$$\left\|\Psi_{ij}\right\|^2 = \int_{\Omega} \Psi_{ij}^2 dv$$

By using the visible inequality

$$\left\|\sum_k a_k\right\|^2 \leq 2\sum_k \|a_k\|^2$$

and the relations (27) one obtains

$$\|\vec{u}\| = \int_\Omega \sum_{i=1}^3 \left(u_i^2 + \varphi_i^2\right) dv \leq C_1 \sum_{i,j,k=1}^3 \left\|\chi_{ij}^{(k)}\right\|^2$$

Then, by taking into account (28), we deduce

$$\|\vec{u}\| \leq C_2 \sum_{i,j=1}^3 \left(\|t_{ij}\|^2 + \|\varphi_{ij}\|^2\right) \tag{29}$$

where C_2 is a positive constant.
On the other hand, the quadratic form $E(\vec{u})$ is positive definite. Thus, we deduce that there exists a positive constant μ_0 such that

$$2E(\vec{u}) \geq \mu_0 \sum_{i,j=1}^3 \left(t_{ij}^2 + \varphi_{ij}^2\right)$$

Now, we integrate the last inequality on the domain Ω and take into account that, according to (13), for $\vec{u} \in \mathcal{C}_0$ we have

$$2\int_\Omega E(\vec{u})\, dv = (A\vec{u},\ \vec{u})$$

Thus, we obtain

$$(A\vec{u},\ \vec{u}) \geq \mu_0 \sum_{i,j=1}^3 \left(\|t_{ij}\|^2 + \|\varphi_{ij}\|^2\right) \tag{30}$$

By comparing (30) and (29), we deduce

$$(A\vec{u},\ \vec{u}) \geq \gamma_1^2 \|\vec{u}\|^2$$

where

$$\gamma_1^2 = \frac{\mu_0}{C_2}$$

and the demonstration of the theorem is concluded. □

Corollary. *The representations of the displacements and of the microrotations from Theorem 1 satisfy a Korn-type inequality.*

Proof. Taking into account relations (27) we deduce that the functions $u_{k,\,i}$ and $\varphi_{k,\,i}$ are expressed by means of certain singular operators, which are applied to the functions t_{ij} and η_{ij}. But in point $Y\left(y_i\right)$ these operators have only singularities of order $\mathcal{O}\left(\frac{1}{r^3}\right)$. According to [8], the functions $u_{k,\,i}$ and $\varphi_{k,\,i}$ satisfy some inequalities of the form

$$\int_\Omega \left(u_{k,\,i}^2 + \varphi_{k,\,i}^2\right) dv \leq B \int_\Omega \sum_{r,s=1}^{3} \left(t_{rs}^2 + \eta_{rs}^2\right) dv,\ k,\ i = 1, 2, 3$$

where B is a positive constants.
Now, by summimg up the inequalities above by all the values of the indices k and i, we are lead to

$$\sum_{k,i=1}^{3} \int_\Omega \left(u_{k,\,i}^2 + \varphi_{k,\,i}^2\right) dv \leq B_1 \int_\Omega \sum_{r,s=1}^{3} \left(t_{rs}^2 + \eta_{rs}^2\right) dv,$$

where

$$B_1 = 9B$$

Finally, by taking into account (30), the last inequality above it leads to

$$\int_\Omega \sum_{k,i=1}^{3} \left(u_{k,\,i}^2 + \varphi_{k,\,i}^2\right) dv \leq D \int_\Omega E\left(\vec{u}\right) dv,$$

where

$$D = \frac{2B_1}{\mu_0}$$

This is a inequality of Korn-type inequality for the first boundary value problem in the elasticity of micropolar bodies. □

Observation. From the property (24) of the operator A we deduce the existence of the solution of the boundary value problem (9). Also, there is the possibility of approximating this solution by mean of variational methods. We denote by $\mathcal{H}$ the space obtained by the closing of the set $\mathcal{C}_0$ with regard to the norm defined by the metric

$$[\vec{u},\ \vec{v}] = (A\vec{u},\ \vec{v})$$

By using the same procedure as in [8] we can prove the results included in the following theorem.

Theorem 2.
i) *There is a single vector* $\vec{u}_0 \in \mathcal{H}$ *which achieves the minimum of the functional*

$$\mathcal{F}\left(\vec{u}\right) = E\left(\vec{u}\right) - \sum_{i=1}^{3} \left(F_i u_i + G_i \varphi_i\right)$$

ii) The string of the Ritz approximation of the solutions of the boundary value problem (9) converges, on the average, to $\vec{u}_0 \in \mathcal{H}$.

The function $\vec{u}_0$ defined in this theorem is the solution of the first boundary value problem of the elasticity of micropolar bodies and generalizes the solution defined in [8] for the first boundary value problem of the classical elasticity.

Conclusion. The operator A built in the context of elasticity of micropolar bodies is positive definite and, as a consequence, based on this property, we have proved the existence of the solution of the first boundary value problem of the elasticity of micropolar bodies.

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