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ON THE INDICATRIX OF A COMPLEX CARTAN SPACE

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Abstract

In this paper we study the geometry of the complex indicatrix of a complex Cartan space. We prove that the intrinsec Chern-Cartan complex nonlinear connection of the indicatrix coincides with the induced Chern-Cartan complex nonlinear connection. Also, the induced Chern-Cartan linear connection on the complexified bundle $T_{\mathbb{C}}(S^{\star}M)$ of the complex indicatrix is studied and the existence of an almost hermitian contact structure is proved. The notions are introduced in similar manner with the corresponding results from complex Finsler geometry, see [7, 8].

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1 Preliminaries

Let us begin our study with a short review of complex Cartan geometry and set up the basic notions and terminology. For more, see Ch. 6 from [5].

Let M be an arbitrary n -dimensional complex manifold. Consider the holomorphic cotangent bundle $T^{'*}M$ together with the canonical projection $\pi : T^{'*}M \to M$. Let $(U,(z^k))$, $k = 1,...,n$ be a local chart on M. It is well known that this chart induces canonically another one on $T^{'*}M$ of the form $(\pi^{-1}(U),(z^k,\zeta_k))$, $k = 1,...,n$, where (ζ_k) should be regarded as the components of a point $\zeta \in T^{'*}_z M$ with respect to the canonical base $\{dz^k\}$, where (z^k) are here regarded as the corresponding holomorphic coordinate functions on U. We call $(\pi^{-1}(U),(z^k,\zeta_k))$ a local canonical coordinate system on $T^{'*}M$. If $(\pi^{-1}(U^{'}),(z^{'k},\zeta^{'}_k))$ is an another local chart on $T^{'*}M$ then the transition laws of these coordinates are

$$z^{'k} = z^{'k}(z)\,,\ \zeta^{'}_k = \frac{\partial z^j}{\partial z^{'k}}\zeta_j\,,\ \mathrm{rank}(\frac{\partial z^j}{\partial z^{'k}}) = n. \tag{1}$$

We can define locally, the canonical differential 1-form of Liouville type on $T^{'*}M$ by

$$\omega|_{\pi^{-1}(U)} = \zeta_k dz^k + \overline{\zeta}_k d\overline{z}^k =: \omega^{'} + \omega^{''}. \tag{2}$$

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Relations (1) guarantee that we have defined globally the canonical differential Liouville 1-form ω on $T^{'*}M$. Also, we define the canonical differential 2-form $\Omega = d\omega = (d^{'} + d^{''})\omega$ on $T^{'*}M$ which is clearly closed and nondegenerated. Locally, we have

$$\Omega|_{\pi^{-1}(U)} = dz^k \wedge d\zeta_k + d\overline{z}^k \wedge d\overline{\zeta}_k. \tag{3}$$

A complex Cartan space is a pair (M, C), where $C : T^{'*}M \to \mathbb{R}_+ \cup \{0\}$ is a continuous function satisfying the conditions:

(i) $H := C^2$ is smooth on $T_0^{'*}M := T^{'*}M - \{\text{zero section}\}$;

(ii) $C(z, \zeta) \geq 0$, the equality holds if and only if $\zeta = 0$;

(iii) $C(z, \lambda\zeta) = |\lambda| C(z, \zeta)$ for any $\lambda \in \mathbb{C}$, *the homogeneity condition*;

(iv) the hermitian matrix $(h^{\overline{j}i}(z, \zeta))$ is positive definite, where $h^{\overline{j}i} = \frac{\partial^2 H}{\partial\zeta_i \partial\overline{\zeta}_j}$ is the fundamental metric tensor.

Consequently, by (iii), we have

$$\frac{\partial H}{\partial \zeta_k}\zeta_k = \frac{\partial H}{\partial \overline{\zeta}_k}\overline{\zeta}_k = H\,,\ \frac{\partial h^{\overline{j}i}}{\partial \zeta_k}\zeta_k = \frac{\partial h^{\overline{j}i}}{\partial \overline{\zeta}_k}\overline{\zeta}_k = 0\,,\ H = h^{\overline{j}i}\zeta_i\overline{\zeta}_j \tag{4}$$

see, also Proposition 6.5.1. from [5].

The geometry of a complex Cartan space consist in the study of the geometric objects of complex manifold $T^{'\star}M$ endowed with a hermitian metric structure defined by $h^{\overline{j}i}$. In this sense, the first step is the study of sections of the complexified tangent bundle of $T^{'*}M$ which is decomposed into the direct sum $T_{\mathbb{C}}(T^{'*}M) = T^{'}(T^{'*}M) \oplus T^{''}(T^{'*}M)$. Let $V^{'}(T^{'*}M) = \ker(d\pi)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial\zeta_k}\}$ and let $V^{''}(T^{'*}M) = \mathrm{span}\{\frac{\partial}{\partial\overline{\zeta}_k}\}$ be its conjugate.

A *complex nonlinear connection*, briefly c.n.c., on $T^{'*}M$ is given by a supplementary complex subbundle to $V^{'}(T^{'*}M)$ in $T^{'}(T^{'*}M)$, i.e. $T^{'}(T^{'*}M) = H^{'}(T^{'*}M) \oplus V^{'}(T^{'*}M)$. The horizontal bundle $H^{'}(T^{'*}M)$ is locally spanned by $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} + N_{jk}\frac{\partial}{\partial \zeta_j}$, where $N_{jk}(z, \zeta)$ are the coefficients of the c.n.c., which obey a certain rule of change at the charts changes such that $\frac{\delta}{\delta z^k} = \frac{\partial z^{'j}}{\partial z^k}\frac{\delta}{\delta z^{'j}}$ performs. Obiously, we also have that $\frac{\partial}{\partial \zeta_k} = \frac{\partial z^k}{\partial z^{'j}}\frac{\partial}{\partial \zeta^{'}_j}$. The pair $\{\delta_k := \frac{\delta}{\delta z^k}\,,\ \dot{\partial}^k := \frac{\partial}{\partial\zeta_k}\}$ will be called the adapted frame of the c.n.c. By conjugation an adapted frame $\{\delta_{\overline{k}}\,,\ \dot{\partial}^{\overline{k}}\}$ is obtained on $T^{''}(T^{'*}M)$. The dual adapted bases are given by

$$\{dz^k\,,\ \delta\zeta_k = d\zeta_k - N_{kj}dz^j\,,\ d\overline{z}^k\,,\ \delta\overline{\zeta}_k = d\overline{\zeta}_k - N_{\overline{k}\overline{j}}d\overline{z}^j\}.$$

A c.n.c. related only to the fundamental function of the complex Cartan space (M, H) is the Chern-Cartan c.n.c., (see [5]), locally given by

$$\overset{CC}{N}_{jk} = -h_{j\overline{m}}\frac{\partial h^{\overline{m}l}}{\partial z^k}\zeta_l, \tag{5}$$

where $(h_{j\overline{m}})$ denotes the inverse of $(h^{\overline{m}j})$.

Let us consider the Sasaki type lift of the metric tensor $h^{\bar{j}i}$, locally given by

$$G = h_{i\bar{j}} dz^i \otimes d\overline{z}^j + h^{\bar{j}i} \delta\zeta_i \otimes \delta\overline{\zeta}_j. \tag{6}$$

On the sections of $T_{\mathbb{C}}(T^{'*}M)$ bundle two almost complex structures act. One is the natural complex structure $\overset{*}{J}$ on the complex manifold $T^{'*}M$, given by

$$\overset{*}{J}(\partial_k) = i\partial_k\,,\ \overset{*}{J}(\dot{\partial}^k) = i\,\dot{\partial}^k\,,\ \overset{*}{J}(\partial_{\overline{k}}) = -i\partial_{\overline{k}}\,,\ \overset{*}{J}(\dot{\partial}^{\overline{k}}) = -i\,\dot{\partial}^{\overline{k}},$$

where $\partial_k = \frac{\partial}{\partial z^k}$. With respect to the adapted frames of a c.n.c. it is given by

$$\overset{*}{J}(\delta_k) = i\delta_k\,,\ \overset{*}{J}(\dot{\partial}^k) = i\,\dot{\partial}^k\,,\ \overset{*}{J}(\delta_{\overline{k}}) = -i\delta_{\overline{k}}\,,\ \overset{*}{J}(\dot{\partial}^{\overline{k}}) = -i\,\dot{\partial}^{\overline{k}}\,.$$

The second almost complex structure is

$$\overset{*}{F}(\delta_k) = \dot{\partial}_k\,,\ \overset{*}{F}(\dot{\partial}_k) = -\delta_k,\,,\ \overset{*}{F}(\delta_{\overline{k}}) = \dot{\partial}_{\overline{k}}\,,\ \overset{*}{F}(\dot{\partial}_{\overline{k}}) = -\delta_{\overline{k}},$$

where $\dot{\partial}_k = h_{k\bar{j}}\,\dot{\partial}^{\bar{j}} := \frac{\partial}{\partial \zeta^k}$ and $\dot{\partial}_{\overline{k}} = h_{j\bar{k}}\,\dot{\partial}^{j} := \frac{\partial}{\partial \overline{\zeta}^k}$. A direct calculation shows that

$$G(\overset{*}{F}X, \overset{*}{F}Y) = G(X,Y),\ \forall\, X,Y \in \mathcal{X}(T_0^{'*}M), \tag{7}$$

where $\mathcal{X}(T_0^{'*}M)$ denotes the $F(M)$-module of complex vector fields on $T_0^{'*}M$.

At the end of this introductory section we recall that a hermitian connection D of (1, 0) type, which satisfies in addition $D_{\overset{*}{J}X}Y = \overset{*}{J} D_X Y$, for all horizontal vectors X, is the so called Chern-Cartan linear connection, which is locally given by the following set of coefficients

$$\overset{CC}{H^i_{jk}} = -h_{j\overline{m}}\delta_k(h^{\overline{m}i})\,,\ \overset{CC}{C^{ik}_j} = -h_{j\overline{m}}\,\dot{\partial}^k\,(h^{\overline{m}i})\,,\ \overset{CC}{H^{\bar{i}}_{\bar{j}k}} = \overset{CC}{C^{\bar{i}k}_{\bar{j}}} = 0, \tag{8}$$

where $\overset{CC}{D}_{\delta_k}\ \delta_j = \overset{CC}{H^i_{jk}}\ \delta_i\,,\ \overset{CC}{D}_{\dot{\partial}^k}\dot{\partial}^i = \overset{CC}{C^{ik}_j}\dot{\partial}^j$ etc. (see [5]). Of course, $\overline{\overset{CC}{D}_X\, Y} = \overset{CC}{D}_{\overline{X}}\ \overline{Y}$ is performed. Also, by (4) it follows that $\overset{CC}{C^{ik}_j}\ \zeta_k = \overset{CC}{C^{ik}_j}\ \zeta_i = 0$.

2 On the geometry of the indicatrix bundle of a complex Cartan space

Let us consider $\dim_{\mathbb{C}} M = n+1$ and (z^k, ζ_k), $k = 1, \ldots, n+1$ the complex coordinates on the manifold $T^{'*}M$.

We consider $S^*_z M = \{\zeta \in T^{'*}_z M \,/\, H(z,\zeta) = 1\}$ the indicatrix at z of a complex Cartan space (M, C) and $p : S^*M \to M$ the indicatrix bundle, where $S^*M = \cup_z S^*_z M$ (or according to [9] the unit cotangent bundle).

$S^*M \subset T_0^{'*}M$ is a holomorphic subbundle and as a complex manifold S^*M is compact and strictly connected in 0 hypersurface of $T_0^{'*}M$. If $i : S^*M \to T_0^{'*}M$ is the inclusion map and $(\widetilde{z}^k, \omega_\alpha)$, $\alpha = 1, \ldots, n$ is a parametric representation of the indicatrix hypersurface, then we have the following local representation

$$\widetilde{z}^k = z^k \, , \, \zeta_k = B_k^\alpha(z)\omega_\alpha \, , \, \mathrm{rank}(B_k^\alpha) = n. \tag{9}$$

On $T^{'}(S^*M)$, we have

$$\frac{\partial}{\partial \widetilde{z}^k} = \frac{\partial}{\partial z^k} + B_{jk}^\alpha \omega_\alpha \frac{\partial}{\partial \zeta_j} \, , \, \frac{\partial}{\partial \omega_\alpha} = B_k^\alpha \frac{\partial}{\partial \zeta_k}, \tag{10}$$

where $B_{jk}^\alpha = \frac{\partial B_j^\alpha}{\partial z^k}$. The dual frames are connected by

$$dz^k = d\widetilde{z}^k \, , \, d\zeta_k = B_{kj}^\alpha \omega_\alpha d\widetilde{z}^j + B_k^\alpha d\omega_\alpha. \tag{11}$$

The vertical distribution $V^{'}(S^*M)$ spanned by $\{\dot{\partial}^\alpha = \frac{\partial}{\partial \omega_\alpha}\}$ is a subdistribution of $V^{'}(T^{'*}M)$. On the indicatrix S^*M we have $H(\widetilde{z}^k, \zeta_k(\omega)) = 1$ and by derivation with respect to $\dot{\partial}^\alpha$ it results that $B_k^\alpha \frac{\partial h^{\bar{j}i}}{\partial \zeta_k} \zeta_i \overline{\zeta}_j + h^{\bar{j}i} B_i^\alpha \overline{\zeta}_j = 0$, for any $\alpha = 1, \ldots, n$. In account of (4) homogeneity conditions $\frac{\partial h^{\bar{j}i}}{\partial \zeta_k} \zeta_i = 0$ it follows that $h^{\bar{j}i} B_i^\alpha \overline{\zeta}_j = 0$, which say that the Liouville vector field $\Gamma = \zeta_k \frac{\partial}{\partial \zeta_k}$ is normal to the vertical distribution $V^{'}(S^*M)$. Moreover, Γ is one unitary vector since $G(\Gamma, \Gamma) = 1$ on the indicatrix.

Let us consider the inverse $\mathcal{R} = \{\dot{\partial}^\alpha = B_k^\alpha \, \dot{\partial}^k \, , \, \Gamma = \zeta_k \, \dot{\partial}^k\}$ along $V^{'}(S^*M)$ and $\mathcal{R}^{-1} = \{B_\alpha^k, \zeta^k\}$ the inverse matrices of this frame, where $\zeta^k = h^{\bar{j}k} \overline{\zeta}_j$, that is

$$B_\alpha^k B_k^\beta = \delta_\alpha^\beta \, , \, B_\alpha^k \zeta_k = 0 \, , \, B_k^\alpha \zeta^k = 0 \, , \, B_k^\alpha B_\alpha^j + \zeta_k \zeta^j = \delta_k^j \, , \, \zeta^k \zeta_k = 1. \tag{12}$$

The fundamental function $\widetilde{H}(\widetilde{z}, \omega) = H(z, \zeta(\omega))$ of the complex Cartan space defines a metric tensor on the indicatrix S^*M, namely $h^{\bar{\beta}\alpha} = B_j^\alpha B_{\bar{k}}^{\bar{\beta}} h^{\bar{k}j}$, where $B_{\bar{k}}^{\bar{\beta}} = \overline{B_k^\beta}$. It is easy to check that $h_{\alpha\bar{\beta}} = h_{j\bar{k}} B_\alpha^j B_{\bar{\beta}}^{\bar{k}}$ is the inverse of $h^{\bar{\beta}\alpha}$ and $h_{j\bar{k}} = B_j^\alpha B_{\bar{k}}^{\bar{\beta}} h_{\alpha\bar{\beta}} + \zeta_j \overline{\zeta}_k$. Also, on the indicatrix, from $\zeta^k \zeta_k = 1$ it follows that $\omega^\alpha \omega_\alpha = 1$, where $\omega^\alpha = h^{\bar{\beta}\alpha} \overline{\omega}_\beta$.

On $T^{'}(S^*M)$ we consider the local frame $\{\widetilde{\delta}_k = \widetilde{\partial}_k + \widetilde{N}_{\alpha k} \, \dot{\partial}^\alpha, \dot{\partial}^\alpha\}$ where $\widetilde{\partial}_k = \frac{\partial}{\partial \widetilde{z}^k}$ and its dual frame $\{\widetilde{d}z^k = d\widetilde{z}^k, \delta\omega_\alpha = d\omega_\alpha - \widetilde{N}_{\alpha k} \widetilde{d}z^k\}$, where $\widetilde{N}_{\alpha k}$ will be called the coefficients of the induced c.n.c. iff $\delta\omega_\alpha = B_\alpha^k \delta\zeta_k$, that is $d\omega_\alpha - \widetilde{N}_{\alpha j} d\widetilde{z}^j = B_\alpha^k (d\zeta_k - N_{kj} dz^j)$ and therefore in view of (11) it satisfies

$$\widetilde{N}_{\alpha j} = B_\alpha^k (N_{kj} - B_{kj}^\beta \omega_\beta). \tag{13}$$

Let be $\overset{CC}{N}_{kj}$ be the Chern-Cartan c.n.c. from (5) and $\overset{CC}{N}_{\alpha j} = -h_{\alpha\bar{\beta}} \frac{\partial h^{\bar{\beta}\gamma}}{\partial \widetilde{z}^j} \omega_\gamma$. Then, we have

Proposition 1. *The induced c.n.c. by the Chern-Cartan c.n.c.* $\overset{CC}{N}_{kj}$ *coincides with* $\overset{CC}{N}_{\alpha j}$.

Proof. Using (10) and (12) a straightforward calculation leads to

$$\begin{aligned}
\overset{CC}{N}_{\alpha j} &= -h_{\alpha\overline{\beta}}\frac{\partial h^{\overline{\beta}\gamma}}{\partial\widetilde{z}^j}\omega_\gamma = -h_{\alpha\overline{\beta}}\frac{\partial^2\widetilde{H}}{\partial\widetilde{z}^j\partial\overline{\omega}_\beta} = -h_{\alpha\overline{\beta}}\frac{\partial}{\partial\widetilde{z}^j}\left(B^{\overline{\beta}}_{\overline{k}}\frac{\partial\widetilde{H}}{\partial\overline{\zeta}_k}\right)\\
&= -h_{\alpha\overline{\beta}}B^{\overline{\beta}}_{\overline{k}}\left(\frac{\partial^2 H}{\partial z^j\partial\overline{\zeta}_k} + B^\gamma_{lj}\omega_\gamma h^{\overline{k}l}\right)\\
&= -h_{m\overline{p}}B^m_\alpha B^{\overline{p}}_{\overline{\beta}}B^{\overline{\beta}}_{\overline{k}}\left(\frac{\partial^2 H}{\partial z^j\partial\overline{\zeta}_k} + B^\gamma_{lj}\omega_\gamma h^{\overline{k}l}\right)\\
&= -h_{m\overline{p}}B^m_\alpha(\delta^{\overline{p}}_{\overline{k}} - \overline{\zeta}_k\overline{\zeta}^p)\left(\frac{\partial^2 H}{\partial z^j\partial\overline{\zeta}_k} + B^\gamma_{lj}\omega_\gamma h^{\overline{k}l}\right)\\
&= -B^m_\alpha(h_{m\overline{p}}\frac{\partial^2 H}{\partial z^j\partial\overline{\zeta}_p} + B^\gamma_{mj}\omega_\gamma - \overline{\zeta}_k\zeta_m\frac{\partial^2 H}{\partial z^j\partial\overline{\zeta}_k} - \zeta_m\zeta^l B^\gamma_{lj}\omega_\gamma)\\
&= B^m_\alpha(\overset{CC}{N}_{mj} - B^\gamma_{mj}\omega_\gamma) = \overset{CC}{\widetilde{N}}_{\alpha j},
\end{aligned}$$

where we used $B^m_\alpha\zeta_m = 0$. □

Also, we have $\widetilde{\delta}_k = \widetilde{\partial}_k + \widetilde{N}_{\alpha k}\,\dot{\partial}^\alpha = \widetilde{\partial}_k + \widetilde{N}_{\alpha k}B^\alpha_j\,\dot{\partial}^j = \partial_k + (B^\alpha_{jk}\omega_\alpha + \widetilde{N}_{\alpha k}B^\alpha_j)\,\dot{\partial}^j$ and by using (12) and (13), it follows

$$\widetilde{\delta}_k = \delta_k + H_{0k}\Gamma\,,\ \dot{\partial}^k = B^k_\alpha\,\dot{\partial}^\alpha - \zeta^k\Gamma, \tag{14}$$

where $H_{0k} = (B^\beta_{lk}\omega_\beta - N_{lk})\zeta^l$.

Further, let us consider the dual induced coframes $\{\widetilde{d}z^k = d\widetilde{z}^k, \delta\omega_\alpha = d\omega_\alpha - \widetilde{N}_{\alpha k}\widetilde{d}z^k\}$. The induced frame and coframe on the whole $T_{\mathbb{C}}(S^*M)$ and the induced metric structure are obtained by conjugation anywhere

$$\widetilde{G} = h_{i\overline{j}}\widetilde{d}z^i\otimes\widetilde{d}\overline{z}^j + h^{\overline{\beta}\alpha}\delta\omega_\alpha\otimes\delta\overline{\omega}_\beta, \tag{15}$$

where $h^{\overline{j}i}(\widetilde{z},\zeta(\omega))$ is the metric tensor of the space along the points of the indicatrix and $(h_{i\overline{j}})$ denotes its inverse.

Now, let us proceed to find the induced Chern-Cartan linear connection. For this purpose, we consider the Gauss-Weingarten equations of the hypersurface S^*M with respect to the Chern-Cartan linear connection. Consider for any $X \in \Gamma(T_{\mathbb{C}}(S^*M))$ and $Y \in \Gamma(V_{\mathbb{C}}(S^*M))$ the decomposition

$$D_XY = \widetilde{D}_XY + H(X,Y), \tag{16}$$

where $\widetilde{D}_XY \in \Gamma(V_{\mathbb{C}}(S^*M))$ is the tangential component and $H(X,Y)$ is the normal component.

Let $\overset{CC}{\widetilde{D}}_{\widetilde{\delta}_k}\dot{\partial}^{\beta}=\overset{CC}{\widetilde{H}}{}^{\beta}_{\alpha k}\dot{\partial}^{\alpha}$ and $\overset{CC}{\widetilde{D}}_{\dot{\partial}^{\gamma}}\dot{\partial}^{\beta}=\overset{CC}{\widetilde{C}}{}^{\beta\gamma}_{\alpha}\dot{\partial}^{\alpha}$. Since D preserves the distributions and $V^{'}(S^*M)$ is spanned by $\dot{\partial}^{\alpha}=B^{\alpha}_{k}\,\dot{\partial}^{k}$ and $V^{'\perp}(S^*M)$ is spanned by Γ, then we have

$$\begin{aligned}\overset{CC}{\widetilde{D}}_{\widetilde{\delta}_k}\dot{\partial}^{\beta} &= \overset{CC}{\widetilde{D}}_{\widetilde{\delta}_k}(B^{\beta}_{j}\,\dot{\partial}^{j})=\widetilde{\delta}_k(B^{\beta}_{j})\,\dot{\partial}^{j}+B^{\beta}_{j}\overset{CC}{D}_{\delta_k+H_{0k}\Gamma}\dot{\partial}^{j}\\ &= \{B^{\beta}_{ik}+B^{\beta}_{j}\overset{CC}{H}{}^{j}_{ik}+B^{\beta}_{j}H_{0k}\zeta_l\overset{CC}{C}{}^{jl}_{i}\}\,\dot{\partial}^{i}\end{aligned}$$

and taking into account the homogeneity condition $\zeta_l\overset{CC}{C}{}^{jl}_{i}=0$ of a complex Cartan metric it results

$$\overset{CC}{\widetilde{H}}{}^{\beta}_{\alpha k}=B^{i}_{\alpha}(B^{\beta}_{ik}+B^{\beta}_{j}\overset{CC}{H}{}^{j}_{ik}). \tag{17}$$

Similarly, we find $\overset{CC}{\widetilde{C}}{}^{\beta\gamma}_{\alpha}=B^{\beta}_{j}B^{\gamma}_{k}B^{i}_{\alpha}\overset{CC}{C}{}^{jk}_{i}$ and $\overset{CC}{\widetilde{H}}{}^{\beta}_{\alpha\overline{k}}=\overset{CC}{\widetilde{C}}{}^{\beta\overline{\gamma}}_{\alpha}=0$.

For the normal component we have $G(\overset{CC}{D}_X\dot{\partial}^{\alpha},\Gamma)=G(H(X,\dot{\partial}^{\alpha}),\Gamma)$ and furthermore if we take X to be a vector of the adapted frames $\{\widetilde{\delta}_k,\widetilde{\delta}_{\overline{k}}\}$, it easily follows that

$$H^{\alpha}_{k}=B^{\alpha}_{jk}\zeta^{j}+B^{\alpha}_{j}\overset{CC}{H}{}^{j}_{ik}\zeta^{i}\,,\;H^{\alpha}_{\overline{k}}=0. \tag{18}$$

If X is $\dot{\partial}^{\beta}$ or $\dot{\partial}^{\overline{\beta}}$, then the fundamental form will be

$$H^{\alpha\beta}=B^{\alpha}_{i}B^{\beta}_{j}\overset{CC}{C}{}^{ij}_{k}\zeta^{k}\,,\;H^{\alpha\overline{\beta}}=0. \tag{19}$$

Now, following the general setting from the geometry of hypersurface [3], [6], a linear connection D induces a normal connection $D^{\perp}$ and let $A_{\Gamma}X:=A(X)\in V^{'}(S^*M)$ be the shape operator. Then, we have

$$D_X\Gamma=-A_{\Gamma}X+D^{\perp}_{X}\Gamma. \tag{20}$$

By $G(D_X\Gamma,\dot{\partial}^{k})=-G(A(X),\dot{\partial}^{k})$, where X is taken to be one of the vectors $\widetilde{\delta}_k$, $\widetilde{\delta}_{\overline{k}}$, $\dot{\partial}^{\beta}$ or $\dot{\partial}^{\overline{\beta}}$, we obtain

$$A_{k\alpha}=-B^{j}_{\alpha}(N_{jk}+\zeta_i\overset{CC}{H}{}^{i}_{jk})\,,\;A_{\overline{k}\alpha}=0\,,\;A^{\beta}_{\alpha}=-\delta^{\beta}_{\alpha}\,,\;A^{\overline{\beta}}_{\alpha}=0. \tag{21}$$

By direct calculations we get the normal components of the Chern-Cartan linear connection, namely

$$\overset{CC}{D}{}^{\perp}_{\widetilde{\delta}_k}\Gamma=H_{0k}\Gamma\,,\;\overset{CC}{D}{}^{\perp}_{\widetilde{\delta}_{\overline{k}}}\Gamma=0\,,\;\overset{CC}{D}{}^{\perp}_{\dot{\partial}^{\beta}}\Gamma=-B^{\beta}_{j}\zeta^{j}\Gamma\,,\;\overset{CC}{D}{}^{\perp}_{\dot{\partial}^{\overline{\beta}}}\Gamma=0. \tag{22}$$

3 An almost hermitian contact structure on the indicatrix

Let $i_* : T_{\mathbb{C}}(S^*M) \to T_{\mathbb{C}}(T^{'*}M)$ be the extension of the tangent inclusion map to the complexified bundles. As in the case when (M,F) is a complex Finsler space (see [7]), it results a unique vector field $\xi \in \Gamma(T_{\mathbb{C}}(S^*M))$ such that $\overset{*}{F}(i_*\xi) = \Gamma$. By the fact that $\Gamma \in V'^{\perp}(S^*M)$ it follows that $\xi \in \Gamma(T^{''}(S^*M))$ and applying $\overset{*}{F}$ to the above formula, we have

$$i_*(\xi) = -\overset{*}{F}(\Gamma) = -\overline{\zeta}^k \overset{*}{F}(\dot{\partial}_{\overline{k}}) = \overline{\zeta}^k \delta_{\overline{k}} = \overline{\zeta}^k(\omega)(\widetilde{\delta}_{\overline{k}} - H_{\overline{0k}}\overline{\Gamma}). \tag{23}$$

Taking the tangential component, it follows that $\xi = \overline{\zeta}^k(\omega)\widetilde{\delta}_{\overline{k}}$. By conjugation we obtain its expression to $T_{\mathbb{C}}(S^*M)$, namely $\xi = \zeta^k(\omega)\widetilde{\delta}_k + \overline{\zeta}^k(\omega)\widetilde{\delta}_{\overline{k}}$.

Now, separating the tangential and normal components of $\overset{*}{F}(i_*X) = i_*\Phi(X) + \eta(X)\Gamma$, by applying again $\overset{*}{F}$, it results

$$-i_*(X) = i_*\Phi^2(X) + \eta(\Phi(X))\Gamma - \eta(X)i_*\xi. \tag{24}$$

Because Φ is a $(1,1)$-complex tensor on S^*M and $\Gamma \notin \mathcal{X}(S^*M)$ we conclude that

$$\Phi^2 = -I + \eta \otimes \xi\,,\ \eta \circ \Phi = 0\,,\ \Phi \circ \xi = 0. \tag{25}$$

On the other hand, by $G(i_*\Phi(X),\Gamma) = 0$ it follows that $\eta(X) = G(\overset{*}{F}(X),\Gamma)$ and taking X to be either $\widetilde{\delta}_{\overline{k}}$ or $\dot{\partial}^{\overline{\alpha}}$ we obtain $\eta(X) = \omega^{''}(X)$ for any $X \in \Gamma(T^{''}(S^*M))$, where $\omega^{''} = \overline{\zeta}_k d\overline{z}^k$. By conjugation it results $\eta(\overline{X}) = \overline{\eta(X)} = \omega^{'}(X)$ the action of η on $T^{'}(S^*M)$, where $\omega^{'} = \zeta_k dz^k$. Thus, on $T_{\mathbb{C}}(S^*M)$ the contact form is $\eta = \zeta_k(\omega)\widetilde{d}z^k + \overline{\zeta}_k(\omega)\widetilde{d\overline{z}}^k$. It is obvious that $\eta(\xi) = 1$ on $T_{\mathbb{C}}(S^*M)$. Now, by definitions of G and $\widetilde{G}$ structures, we obtain

$$\widetilde{G}(\Phi X, \Phi Y) = \widetilde{G}(X,Y) - \eta(X)\overline{\eta(Y)} \tag{26}$$

that mean $(\widetilde{G},\Phi,\xi,\eta)$ is an *almost hermitian contact structure* on $T_{\mathbb{C}}(S^*M)$.

Finally, let us obtain the explicit action of the Φ tensor under the adapted frames. Using $i_*\widetilde{\delta}_k = \delta_k + H_{0k}\Gamma$ and $\dot{\partial}^k = B^k_\alpha\,\dot{\partial}^\alpha - \zeta^k\Gamma$, from $i_*\Phi(X) = \overset{*}{F}(i_*X) - \eta(X)\Gamma$ by direct calculations, one gets

$$i_*\Phi\widetilde{\delta}_k = h_{k\overline{j}}B^{\overline{j}}_{\overline{\alpha}}\,\dot{\partial}^{\overline{\alpha}} - H_{0k}\overline{\zeta}^l\widetilde{\delta}_{\overline{l}} + (H_{0k}\overline{\zeta}^l H_{\overline{0l}} - \zeta_k)\overline{\Gamma} - \zeta_k\Gamma.$$

Because Φ is the tangential component of $i_*\Phi$, it follows that

$$\Phi\widetilde{\delta}_k = h_{k\overline{j}}B^{\overline{j}}_{\overline{\alpha}}\,\dot{\partial}^{\overline{\alpha}} - H_{0k}\overline{\zeta}^l\widetilde{\delta}_{\overline{l}}.$$

Similarly, we deduce

$$\Phi\,\dot{\partial}^\alpha = -h^{\overline{j}k}B^\alpha_k\widetilde{\delta}_{\overline{j}}\,,\ \Phi\widetilde{\delta}_{\overline{k}} = h_{j\overline{k}}B^j_\alpha\,\dot{\partial}^\alpha - H_{\overline{0k}}\zeta^l\widetilde{\delta}_l\,,\ \Phi\,\dot{\partial}^{\overline{\alpha}} = -h^{\overline{k}j}B^{\overline{\alpha}}_{\overline{k}}\widetilde{\delta}_j$$

and, so

$$\begin{aligned}\Phi &= h_{k\overline{j}}B^{\overline{j}}_{\overline{\alpha}}\,\dot{\partial}^{\overline{\alpha}} \otimes \widetilde{d}z^k - H_{0k}\overline{\zeta}^l\widetilde{\delta}_{\overline{l}} \otimes \widetilde{d}z^k + h_{j\overline{k}}B^j_\alpha\,\dot{\partial}^\alpha \otimes \widetilde{d\overline{z}}^k - \\ &- H_{\overline{0k}}\zeta^l\widetilde{\delta}_l \otimes \widetilde{d\overline{z}}^k - h^{\overline{j}k}B^\alpha_k\widetilde{\delta}_{\overline{j}} \otimes \delta\omega_\alpha - h^{\overline{k}j}B^{\overline{\alpha}}_{\overline{k}}\widetilde{\delta}_j \otimes \delta\overline{\omega}_\alpha.\end{aligned}$$

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