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SZÁSZ - INVERSE BETA OPERATORS REVISITED

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Abstract

In this paper we deal with some properties of a mixed sequence of summation integral type operators, namely Szász - Inverse Beta operators. These properties are refering to the monotonic convergence under convexity, preservation of Lipschitz constants, using probabilistic methods.

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1 Introduction

The classical Szász - Mirakjan operators are defined as :

$$S_t(f;x) = \sum_{k=0}^{\infty} s_{t,k}(x) f\left(\frac{k}{t}\right) \tag{1}$$

with

$$s_{t,k}(x) = e^{-tx}\frac{(tx)^k}{k!},\ t > 0,\ x \geq 0,\ k \in \mathbb{N} \cup \{0\}\,. \tag{2}$$

They can be represented, for each $t > 0$ and for any real measurable function f on $[0, +\infty)$ for which the mean value
$E\left[\left|f\left(\frac{Y_{tx}}{t}\right)\right|\right] < \infty$ exists, as:

$$S_t(f;x) = E\left[f\left(\frac{Y_{tx}}{t}\right)\right],\ t > 0,\ x \geq 0. \tag{3}$$

The random variable Y_{tx} has the Poisson distribution and takes the value k with a probability defined by (2). It is well known that, these operators preserve the affine functions because of

$$\begin{cases} S_t\left(e_0;x\right) = e_0(x) = 1, \\ S_t\left(e_1;x\right) = e_1(x) = x, \\ S_t\left(e_2;x\right) = x^2 + \dfrac{x}{t},\ t > 0,\ x \geq 0 \quad . \end{cases} \tag{4}$$

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These operators are endowed with the property of monotonic convergence under convexity and preserve the convexity too.

Others well known operators, are Inverse - Beta operators or Stancu operators of second type [12] defined as

$$\begin{cases} T_t(f;x) = \dfrac{1}{B(tx,t+1)} \int\limits_0^\infty \dfrac{u^{tx-1}}{(1+u)^{tx+t+1}} f(u)du \\ \qquad\qquad = \int\limits_0^\infty f(u) b_{tx,t+1}(u)du,\ t>0,\ x>0, \\ T_t(f;0) = f(0), \end{cases} \tag{5}$$

with

$$B(tx,t+1) = \int\limits_0^\infty \frac{u^{tx-1}}{(1+u)^{tx+t+1}} du,\ t>0,\ x>0, \tag{6}$$

Inverse - Beta function. These operators can be represented in a probabilistic manner as well, for each $t>0$ and for any real measurable function f on $[0,+\infty)$, if the mean value $E\left[\left|f\left(W_{tx,t+1}\right)\right|\right] < \infty$ exists for $t>0$, $x>0$:

$$\begin{cases} T_t(f;x) = E\left[f\left(W_{tx,t+1}\right)\right],\ t>0,\ x>0, \\ T_t(f;0) = f(0). \end{cases} \tag{7}$$

The random variable $W_{tx,t+1}$ has the Inverse - Beta distribution with the probability density function:

$$b_{tx,t+1}(u) = \frac{1}{B(tx,t+1)} \cdot \frac{u^{tx-1}}{(1+u)^{tx+t+1}},\ t>0,\ x>0,\ u>0. \tag{8}$$

It is known [see 9.IV.10.(3)] that, if we consider two independent random variables U_{tx}, V_{t+1} having Gamma distributions with probability density function

$$d_\alpha(u) = \begin{cases} \dfrac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)} & ,\ \alpha>0,\ u>0, \\ 0 & ,\ u=0 \end{cases} \tag{9}$$

for $\alpha = tx$ respectively for $\alpha = t+1$, then the probability density function of the ratio $\dfrac{U_{tx}}{V_{t+1}}$ is $b_{tx,t+1}(u) = \int\limits_0^\infty y d_{U_{tx}}(uy) d_{V_{t+1}}(y)dy$ defined as (8). So,

$$\begin{cases} T_t(f;x) = E\left[f\left(W_{tx,t+1}\right)\right] = E\left[f\left(\dfrac{U_{tx}}{V_{t+1}}\right)\right],\ t>0,\ x>0, \\ T_t(f;0) = f(0). \end{cases} \tag{10}$$

Inverse- Beta operators preserve the affine functions on $[0,+\infty)$

$$\begin{cases} T_t\left(e_0;x\right) = e_0(x) = 1, \\ T_t\left(e_1;x\right) = e_1(x) = x, \\ T_t\left(e_2;x\right) = x^2 + \dfrac{x(x+1)}{t-1},\ t>1, \end{cases} \tag{11}$$

and they benefit from the monotonic convergence under convexity.

A variant of these operators, which do not preserve the affine functions, was investigated with respect to the monotonic convergence under convexity and the preservation of Lipschitz constants by Adell J. A., De la Cal J., Miguel S. M., [1].

Recently, Gupta V., Noor M. A., [6] considered the following operators:

$$\begin{aligned} L_t(f;x) &= e^{-tx}f(0)+\sum_{k=1}^{\infty}s_{t,k}(x)\int_0^{\infty}b_{t,k}(u)f(u)du \\ &= \int_0^{\infty}J_t(u;x)f(u)du,\ x\geq 0 \end{aligned} \tag{12}$$

with $s_{t,k}(x)$ as (2) and

$$b_{t,k}(u)=\frac{1}{B(k,t+1)}\cdot\frac{u^{k-1}}{(1+u)^{t+k+1}},\ t>0,\ u>0, \tag{13}$$

$B(k,t+1)$ being Inverse - Beta function as (6),

$$J_t(u;x)=e^{-tx}\delta(u)+\sum_{k=1}^{\infty}s_{t,k}(x)b_{t,k}(u), \tag{14}$$

$\delta(u)$ being the Dirac's delta function, for which $\int\limits_0^{\infty}\delta(u)f(u)du=f(0)$.

The iterative constructions of these operators were studied recently by Finta Z., Govil N. K., Gupta V. [5].

We remark that, all these operators were defined for $t=n$ positive integers numbers but they remain valid when t is a continuous positive number.

In the next section, we propose a probabilistic representation of Szász - Inverse Beta operators (12) - (14).

2 A probabilistic representation of Szász - Inverse Beta operators

Mazhar S. M., Totik V. [8] defined and studied Durrmeyer - Szász's operators:

$$\begin{aligned} M_t(f;x) &= \int_0^{\infty}H_t(u;x)f(u)du,\ x\geq 0,\ t>0, \\ H_t(u;x) &= t\sum_{k=1}^{\infty}s_{t,k}(x)s_{t,k}(u) \end{aligned}$$

with $s_{t,k}(x)$ as in (2). Adell J. A., De la Cal J., [2]gave an interesting probabilistic representation of these operators by means of the mean value $M_t(f;x)=E\left[f\left(\frac{U_{N(tx)+1}}{t}\right)\right]$, $t>$

0, $x \geq 0$, where $\{N(t) : t \geq 0\}$ is a standard Poisson process and $\{U_t : t \geq 0\}$ is a Gamma standard process independent of the former. Using the same ideea as Adell J. A., De la Cal J., [2] we give the following probabilistic representation for Szász - Inverse Beta operators (12) - (14).

Let $\{N(t) : t \geq 0\}$ be a standard Poisson process and let $\{U_t : t \geq 0\}$, $\{V_t : t \geq 0\}$ be two mutually independent Gamma processes defined all on the same probability space. Note that, the Poisson process is a stochastic process starting at the origin, having stationary independent increments with probability

$$P\left(N(t) = k\right) = \frac{e^{-t}t^k}{k!},\ t \geq 0,\ k \in \mathbb{N} \cup \{0\}$$

and the Gamma process is a stochastic process starting at the origin ($U_0 = 0$), having stationary independent increments and so, that for $t > 0$, U_t has the Gamma probability density function $d_t(u)$ as (9) and without loss of generality [10] we can to assume that $\{U_t : t \geq 0\}$, $\{V_t : t \geq 0\}$ for each $t > 0$ have a.s. non- decreasing right-continuous paths.

Szász - Inverse Beta operators (12) - (14) can be represented as the mean value :

$$L_t(f;x) = E\left[f\left(Z_{tx}\right)\right] = E\left[f\left(\frac{U_{N(tx)}}{V_{t+1}}\right)\right],\ t > 0,\ x \geq 0, \tag{15}$$

where the random variable $\dfrac{U_{N(tx)}}{V_{t+1}}$ has the probability density function $J_t\left(\cdot;x\right)$ defined as (14).

Indeed,

$$\begin{aligned}
E\left[f\left(\frac{U_{N(tx)}}{V_{t+1}}\right)\right] &= \int_0^\infty f(u)\left(\int_0^\infty y d_{U_{N(tx)}}(yu) d_{V_{t+1}}(y)dy\right)du \\
&= \int_0^\infty f(u)\left(\int_0^\infty y\sum_{k=0}^\infty \frac{e^{-tx}(tx)^k}{k!} d_{U_k}(yu) d_{V_{t+1}}(y)dy\right)du \\
&= e^{-tx}f(0) + \sum_{k=1}^\infty s_{t,k}(x)\int_0^\infty f(u)\left(\int_0^\infty \frac{y^{k+t}}{\Gamma(k)}\cdot\frac{u^{k-1}}{\Gamma(t+1)}e^{-y(u+1)}dy\right)du \\
&= e^{-tx}f(0) + \sum_{k=1}^\infty s_{t,k}(x)\int_0^\infty f(u)\left(\int_0^\infty \frac{\left(\dfrac{v}{u+1}\right)^{k+t}}{\Gamma(k)}\cdot\frac{u^{k-1}}{\Gamma(t+1)}e^{-v}\frac{dv}{u+1}\right)du \\
&= e^{-tx}f(0) + \sum_{k=1}^\infty s_{t,k}(x)\int_0^\infty f(u)\frac{b_{t,k}(u)}{\Gamma(k+t+1)}\left(\int_0^\infty v^{k+t}e^{-v}dv\right)du \\
&= e^{-tx}f(0) + \sum_{k=1}^\infty s_{t,k}(x)\int_0^\infty f(u)b_{t,k}(u)du = L_t(f;x).
\end{aligned}$$

On the other hand, Szasz - Inverse Beta operators can be represented as a combination between Szasz operators (1) - (2) and Inverse- Beta operators (5):

$$L_t(f;x) = (S_t \circ T_t)(f;x) = S_t(T_t)(f;x),\, t>0,\, x\geq 0. \tag{16}$$

3 Some properties of Szasz - Inverse Beta operators

In view of (16), because a part of the properties of Szász - Inverse Beta operators depend on the same properties of Szász - Mirakjan operators (1) - (2) and of Inverse - Beta operators (5), in the following we shall study the property of monotonic convergence under convexity for Inverse - Beta operators and others properties (5).

Lemma 1. If $(U_{tx})_{t>0,\,x\geq 0}$, $(V_{t+1})_{t>0}$ are two independent Gamma processes defined on the same probability space, then for all $1<r\leq s$ and $x>0$ we have

$$E\left(\frac{U_{rx}}{V_{r+1}} \,|\, \frac{U_{sx}}{V_{s+1}}\right) = \frac{U_{sx}}{V_{s+1}}\, a.\, s. \tag{17}$$

Proof. In the same line of argumentation as in [1], since the random vectors (U_{rx}, U_{sx}) and (V_{r+1}, V_{s+1}) are independent, together with the properties of the conditional mathematical expectation, we have :

$$E\left(\frac{U_{rx}}{V_{r+1}} \,|\, U_{sx},\, V_{s+1}\right) = E\left(U_{rx} \,|\, U_{sx}\right)\cdot E\left(\frac{1}{V_{r+1}} \,|\, V_{s+1}\right).$$

But with [1, Lemma 1] we obtain

$$E\left(U_{rx} \,|\, U_{sx}\right) = \frac{r}{s} U_{sx}\, a.\, s.$$

and

$$E\left(\frac{U_{rx}}{V_{r+1}} \,|\, U_{sx},\, V_{s+1}\right) = \frac{r}{s} U_{sx} E\left(\frac{1}{V_{r+1}} \,|\, V_{s+1}\right). \tag{18}$$

Because the random variables $\dfrac{V_{s+1}}{V_{r+1}}$ and V_{s+1} are independent [7] we have

$$\begin{aligned} E\left(\frac{1}{V_{r+1}} \,|\, V_{s+1}\right) &= \frac{1}{V_{s+1}} E\left(\frac{V_{s+1}}{V_{r+1}} \,|\, V_{s+1}\right) = \frac{1}{V_{s+1}} E\left(\frac{V_{s+1}}{V_{r+1}}\right) \\ &= \frac{1}{V_{s+1}}\left[1 + E\left(\frac{V_{s+1}-V_{r+1}}{V_{r+1}}\right)\right] \\ &= \frac{1}{V_{s+1}}\left[1 + E\left(V_{s+1}-V_{r+1}\right)\cdot E\left(\frac{1}{V_{r+1}}\right)\right]. \end{aligned}$$

With $E\left(V_{s+1}\right) = s+1$ and $E\left(\dfrac{1}{V_{r+1}}\right) = \dfrac{1}{r}$ we obtain

$$E\left(\frac{1}{V_{r+1}} \,|\, V_{s+1}\right) = \frac{s}{r} V_{s+1}\, a.\, s. \tag{19}$$

So, from (18) with (19) we have (17). □

Lemma 2. Let $t > 1$ be set. For the Inverse - Beta operators (5) with (10) it follows that:

1. If f is a real convex function on $(0, +\infty)$ then $T_t f$ is convex too.
2. If f is a non-decreasing and convex function on $(0, +\infty)$and $1 < r < s$ then $T_r f \geq T_s f \geq f$.
3. If $f \in Lip_{(0,+\infty)}(C, \alpha)$, $\alpha \in (0, 1]$ then $T_t f \in Lip_{(0,+\infty)}(C, \alpha)$, $\alpha \in (0, 1]$

Proof. 1. If f is a convex function on $(0, +\infty)$ then for $0 < x < z < y$ we have:

$$\frac{U_{tz}}{V_{t+1}} = \frac{U_{ty} - U_{tz}}{U_{ty} - U_{tx}} \cdot \frac{U_{tx}}{V_{t+1}} + \frac{U_{tz} - U_{tx}}{U_{ty} - U_{tx}} \cdot \frac{U_{ty}}{V_{t+1}} \; a.\, s.$$

$$f\left(\frac{U_{tz}}{V_{t+1}}\right) \leq \frac{U_{ty} - U_{tz}}{U_{ty} - U_{tx}} \cdot f\left(\frac{U_{tx}}{V_{t+1}}\right) + \frac{U_{tz} - U_{tx}}{U_{ty} - U_{tx}} \cdot f\left(\frac{U_{ty}}{V_{t+1}}\right).$$

From [4, Appendix, Lemma 3]when U_{tx}, $t > 0$, $x \geq 0$ is a Gamma process and $0 \leq tx < tz < ty$, we obtain

$$E\left(U_{tz} - U_{tx} \,|\, U_{ty} - U_{tx}\right) = \frac{z - x}{y - x}\left(U_{ty} - U_{tx}\right) \; a.\, s.$$

$$E\left[f\left(\frac{U_{tz}}{V_{t+1}} \,|\, U_{ty} - U_{tx}\right)\right] \leq \frac{y - z}{y - x} \cdot f\left(\frac{U_{tx}}{V_{t+1}}\right) + \frac{z - x}{y - x} \cdot f\left(\frac{U_{ty}}{V_{t+1}}\right).$$

Having in view the mean value

$$T_t f(z) \leq \frac{y - z}{y - x} \cdot T_t f(x) + \frac{z - x}{y - x} \cdot T_t f(y).$$

2. For $1 < r < s$ and $x > 0$, using the probabilistic version of Jensen's inequality when f is a convex function on $(0, +\infty)$ as well as the properties of conditional mean value, it follows that:

$$\begin{aligned} T_r(f; x) &= E\left[f\left(\frac{U_{rx}}{V_{r+1}}\right)\right] = E\left[E\left[f\left(\frac{U_{rx}}{V_{r+1}}\right) \,|\, \frac{U_{sx}}{V_{s+1}}\right]\right] \\ &\geq E\left[f\left(E\left(\frac{U_{rx}}{V_{r+1}} \,|\, \frac{U_{sx}}{V_{s+1}}\right)\right)\right] = E\left[f\left(\frac{U_{sx}}{V_{s+1}}\right)\right] = T_s(f; x). \end{aligned}$$

Certainly, f convex on $(0, +\infty)$ with (11) results in

$$T_r(f; x) = E\left[f\left(\frac{U_{rx}}{V_{r+1}}\right)\right] \geq f\left(E\left[\frac{U_{rx}}{V_{r+1}}\right]\right) = f\left(T_r\right)(e_1; x) = f(x).$$

3. Let $T_t(f; x) = E\left[f\left(W_{tx,t+1}\right)\right] = E\left[f\left(\frac{U_{tx}}{V_{t+1}}\right)\right]$, $t > 0$, $x > 0$ be Inverse - Beta operators (5) with (10), where $W_{tx,t+1}$ has a Inverse - Beta distribution with probability

density function (8). It is known that, if the random vector $(W_{tx,t+1}, W_{ty-tx,t+1})$ has the probability density function

$$h_t(u,v;x,y) = \frac{\Gamma(tx+t+1)}{\Gamma(tx)+\Gamma(ty-tx)\Gamma(t+1)} \cdot \frac{u^{tx-1}v^{ty-tx-1}}{(1+u+v)^{ty+t+1}}$$

$u,v>0$, $y>x>0$, then $W_{tx,t+1}$ has an Inverse - Beta distribution with parameters tx, t+1 and the probability density function

$$b_{tx,t+1}(u) = \int_0^\infty h_t(u,v;x,y)dv$$

as (8) and $W_{ty-tx,t+1}$ has an Inverse - Beta distribution with parameters $ty-tx$, $t+1$ and the probability density function

$$b_{ty-tx,t+1}(v) = \int_0^\infty h_t(u,v;x,y)du.$$

It is easy to show, that $W_{tx,t+1} + W_{ty-tx,t+1}$ has an Inverse - Beta distribution with parameters ty, $t+1$.

If $f \in Lip_{(0,+\infty)}(C,\alpha)$, $\alpha \in (0,1]$ then $(\exists) C > 0$ so, that $|f(y) - f(x)| \leq C\,|y-x|^\alpha$ and

$$\begin{aligned}
|T_t(f;x) - T_t(f;y)| &= |E[f(W_{tx,t+1})] - E[f(W_{ty,t+1})]| \\
&= |E[f(W_{tx,t+1})] - E[f(W_{tx,t+1} + W_{ty-tx,t+1})]| \\
&\leq E\,|f(W_{tx,t+1}) - f(W_{tx,t+1} + W_{ty-tx,t+1})| \\
&\leq E\left[C\,|W_{ty-tx,t+1}|^\alpha\right] \\
&\leq C\,(E[W_{ty-tx,t+1}])^\alpha = C\,|y-x|^\alpha
\end{aligned}$$

because

$$\begin{aligned}
E[f(W_{ty-tx,t+1})] &= \frac{1}{B(t(y-x),t+1)} \int_0^\infty \frac{u^{ty-tx}}{(1+u)^{ty-tx+t+1}}du \\
&= \frac{B(t(y-x)+1,t)}{B(t(y-x),t+1)} = y-x.
\end{aligned}$$

□

Theorem 1. *Let $t>1$ be set. For Szász - Inverse Beta operators (12) - (14), it follows that:*

1. $L_t(e_i;x) = e_i(x)$, $i = \overline{0,1}$;
2. $L_t(e_2;x) = \dfrac{t}{t-1}x^2 + \dfrac{2}{t-1}x$;

3. *If f is a convex function on $[0,+\infty)$ then $L_t f$ is also convex and if, in addition, f is non-decreasing then for $1 < r < s$, $L_r f \geq L_s f \geq f$;*

4. *If $f \in Lip_{(0,+\infty)}(C,\alpha)$, $\alpha \in (0,1]$ then $L_t f \in Lip_{(0,+\infty)}(C,\alpha)$, $\alpha \in (0,1]$.*

Proof. The proof is immediately deducible, using the representation (16) of $L_t f$, the linearity of $S_t f$, lemma 2 and the proper properties of Szász - Mirakjan and Inverse Beta operators. □

Remark 1. We have with (15)

$$\begin{aligned} L_t\left(e_1;x\right) &= E\left[\frac{U_{N(tx)}}{V_{t+1}}\right] = x \\ L_t\left(e_2 - x^2;x\right) &= L_t\left(\left(e_1 - x\right)^2;x\right) = D^2\left[\frac{U_{N(tx)}}{V_{t+1}}\right] \\ &= E\left[\left(\frac{U_{N(tx)}}{V_{t+1}} - x\right)^2\right] = \frac{x(2+x)}{t-1},\ t>1,\ x\geq 0. \end{aligned}$$

Remark 2. An interesting result which was obtained by De la Cal J., Carcamo J., [4] for the Bernstein - type operators which preserves the affine functions, namely "centered Bernstein-type operators", can be used for Szasz - Inverse Beta operators (12) - (14) :

Theorem 2 (De la Cal J., Carcamo J., [4])**.** *If $L_1 = L_2 \circ L_3$, where L_1, L_2, L_3 are centered Bernstein-type operators ($Lf(x) = E\left[f\left(Y_x\right)\right],\ x \in I \subset \mathbb{R},\ L_1(x) = E\left[Y_x\right] = x$), in the same interval I and if $\mathbf{L}_{cx}$ is the set of all convex functions in the domain of the three operators, then $L_1 f \geq L_2 f$, $f \in \mathbf{L}_{cx}$.*

If, in addition L_3 preserves convexity, then $L_1 f \geq L_2 f \vee L_3 f$, $f \in \mathbf{L}_{cx}$ where $f \vee g$ denotes the maximum of f and g.

In view of this result and using the representation (16) for Szasz-Inverse Beta operators, we have $L_t f \geq S_t f$, $f \in \mathbf{L}_{cx}[0,+\infty)$ and $L_t f \geq S_t f \vee T_t f$, $f \in \mathbf{L}_{cx}[0,+\infty)$, where S_t are Szasz-Mirakjan operators (1)-(2), T_t are Inverse Beta operators (5) and L_t are Szasz-Inverse Beta operators (12) (14).

Theorem 3. *For any function $f \in \mathbf{C}_B[0,+\infty)$ and for any compact set $K \subset [0,+\infty)$ we have $\lim\limits_{t\to\infty} L_t(f) = f$ uniform on K.*

Proof. It follows from the Bohmann-Korovkin's theorem and from theorem 1. □

Theorem 4. *1. If $f \in \mathbf{C}_B[0,+\infty)$, then for every $x \in [0,+\infty)$ we have:*

$$|L_t(f;x) - f(x)| \leq \left(1 + \sqrt{x(2+x)}\right)\omega\left(f;\frac{1}{\sqrt{t-1}}\right),\ t>1.$$

2. If $f' \in \mathbf{C}_B[0,+\infty)$, then for every $x \in [0,+\infty)$

$$|L_t(f;x) - f(x)| \leq \sqrt{\frac{x(2+x)}{t-1}}\left(1 + \sqrt{x(2+x)}\right)\omega\left(f';\frac{1}{\sqrt{t-1}}\right),\ t>1.$$

3. If f is a bounded function on $[0,+\infty)$, differentiable in some neighborhood of x and has the second derivative f'' for any $x \in [0,+\infty)$, then for $t > 1$

$$\lim_{t\to\infty}(t-1)\left[L_t(f;x)-f(x)\right]=\frac{x(2+x)}{2}f''(x).$$

If $f \in \mathbf{C}^2[0,+\infty)$, then the convergence is uniform on any compact $K \subset [0,+\infty)$.

Proof. (1) and (2) were obtained from a result of Shisha O., Mond B., [11] using Remark 1 for Szász - Inverse Beta operators (12)-(14) and for (3) see Cismaşiu C. [3]. □

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