

UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE THREE SETS III

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Abstract

In the paper we prove some results related to the uniqueness of meromorphic as well as entire functions. Our results will improve and supplement several known results .

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1 Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We shall use the standard notations of value distribution theory :

$$T(r, f), m(r, f), N(r, \infty; f), \overline{N}(r, \infty; f), \dots$$

(see [8]). For any constant a , we define

$$\Theta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \rightarrow \infty$, outside a possible exceptional set of finite linear measure.

We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

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Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_m(a; f)$ the set of all a -points of f with multiplicities not exceeding m , where an a -point is counted according to its multiplicity. If $E_\infty(a; f) = E_\infty(a; g)$ for some $a \in \mathbb{C} \cup \{\infty\}$, we say that f, g share the value a CM. For a set S of distinct elements of \mathbb{C} we define $E_m(S, f) = \bigcup_{a \in S} E_m(a, f)$.

In 1976 F. Gross [7] asked the following question:

Question A *Can one find two finite sets S_j ($j = 1, 2$) such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

For meromorphic function it is natural to ask the following question .

Question B[20] *Can one find three finite sets S_j ($j = 1, 2, 3$) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical ?*

The above questions created a ripple among the researchers to find the smallest cardinalities of the range sets under weaker hypothesis and naturally in the last couple of years or so several investigations have been done by many authors. {cf.[1]-[6], [13], [16], [18], [20], [22] [24]}.

A recent increment to uniqueness theory has been to considering weighted sharing instead of sharing IM/CM which implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001 in [10, 11] and since then this notion played a vital role as far as the uniqueness of meromorphic or entire functions sharing sets are concerned. Below we are giving the definition.

Definition 1. [10, 11] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .*

The definition implies that if f, g share a value a with weight k then z_0 is an a -point of f with multiplicity m ($\leq k$) if and only if it is an a -point of g with multiplicity m ($\leq k$) and z_0 is an a -point of f with multiplicity m ($> k$) if and only if it is an a -point of g with multiplicity n ($> k$), where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for any integer p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2. [10] *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞ . We denote by $E_f(S, k)$ the set $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$.*

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

In 2007 improving the result of Yi-Lin [24] the present first author proved the following result in the direction of *Question B*.

Theorem A. [1] Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and $n (\geq 3)$ is an integer. If for two non-constant meromorphic functions f and g , $E_f(S_1, 6) = E_g(S_1, 6)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 1$ then $f \equiv g$.

Recently the present first author has further improved and supplemented the above theorem as follows.

Theorem B. [3] Let $S_i, i = 1, 2, 3$ be defined as in Theorem A. If for two non-constant meromorphic functions f and g , $E_f(S_1, 5) = E_g(S_1, 5)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 1$ then $f \equiv g$.

Theorem C. [3] Let $S_i, i = 1, 2, 3$ be defined as in Theorem A. If for two non-constant meromorphic functions f and g , $E_f(S_1, 4) = E_g(S_1, 4)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 1$ then $f \equiv g$.

The following example shows that in Theorems A-C $a \neq 0$ is necessary.

Example 1. Let $f(z) = e^z$ and $g(z) = e^{-z}$ and $S_1 = \{z : z^3 - 1 = 0\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$. Since $f - \omega^l = g - \omega^{3-l}$, where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $0 \leq l \leq 2$, clearly $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2, 3$ but $f \not\equiv g$.

The following two examples establishes the sharpness of the lower bound of n in Theorems A-C.

Example 2. Let $f(z) = \alpha e^z$ and $g(z) = \beta e^{-z}$ and $S_1 = \{\alpha, \beta\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$, where α and β be two arbitrary non zero constants such that $\alpha + \beta = -a$ and $\alpha\beta = b$. Clearly $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2, 3$ but $f \not\equiv g$.

Example 3. Let $f(z) = \sqrt{\alpha}\sqrt{\beta} e^z$ and $g(z) = \sqrt{\alpha}\sqrt{\beta} e^{-z}$ and $S_1 = \{\alpha, \beta\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$, where α and β be defined as in Exmple 2. Clearly $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2, 3$ but $f \not\equiv g$.

Regarding Theorems A-C following example establishes the fact that the set S_1 can not be replaced by any arbitrary set containing three distinct elements.

Example 4. Let $f(z) = \sqrt{\alpha\beta} e^{h(z)}$ and $g(z) = \sqrt{\alpha\beta} e^{-h(z)}$ and $S_1 = \{\alpha, \beta, \sqrt{\alpha\beta}\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$, where α and β are nonzero complex numbers and $h(z)$ is a non-constant entire function. Clearly $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$ but f and g do not satisfy the conclusions of Theorems A-C.

Now it is quite natural to ask the following question.

i) What happens in Theorem B if $\Theta(\infty; f) + \Theta(\infty; g) < 1$?

In fact in [2] we have taken up this problem and provided a partial solution in this respect. But to do this the lower bound of n which corresponds to the cardinality of the set S_1 in the above mentioned theorems were increased. In the direction of the above question some investigations have already been carried out by Lu and Wang [17] in the following theorem.

Theorem D. Let $S_1 = \{z : z^2(z - a) - b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants. If for two non-constant meromorphic functions f and g , $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$ and $\delta_2(a, f) + \delta_2(a, g) + 2\Theta(\infty; f) > 1$ then $f \equiv g$ or $f = \frac{e^{\gamma+1}}{e^{2\gamma+e^{\gamma+1}}}$, $g = \frac{e^{\gamma}(e^{\gamma+1})}{e^{2\gamma+e^{\gamma+1}}}$

Remark 1. In the proof of the Theorem D it has been assumed that $a = b = 1$.

In 1998 to deal with the question of Gross Yi [22], proved the following theorem corresponding to entire functions.

Theorem E. Let $S_1 = \{z : z^n(z + a) - b = 0\}$ and $S_2 = \{0\}$, where a, b are nonzero constants such that $z^n(z + a) - b = 0$ has no repeated root and $n (\geq 2)$ is an integer. If for two non-constant entire functions f and g , $E_f(S_1, \infty) = E_g(S_1, \infty)$ and $E_f(S_2, 0) = E_g(S_2, 0)$ then $f \equiv g$.

We now state the following theorems which are the main results of the paper.

Theorem 1. Let $S_1 = \{z : z^n(z - a) - b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n(z - a) - b = 0$ has no repeated roots and $n (\geq 2)$ is an integer. If for two non-constant meromorphic functions f and g , $E_f(S_1, 5) = E_g(S_1, 5)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\delta_2(a, f) + \delta_2(a, g) + \Theta(\infty; f) + \Theta(\infty; g) > 3 - n$ then $f \equiv g$ or $f = \frac{-ae^{\gamma}(e^{n\gamma}-1)}{e^{(n+1)\gamma-1}}$, $g = \frac{-a(e^{n\gamma}-1)}{e^{(n+1)\gamma-1}}$, where γ is a non-constant entire function.

Theorem 2. Let S_1, S_2 and S_3 be defined as in Theorem 1. If for two non-constant meromorphic functions f and g , $E_f(S_1, 4) = E_g(S_1, 4)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\delta_2(a, f) + \delta_2(a, g) + \Theta(\infty; f) + \Theta(\infty; g) > 3 - n$ then $f \equiv g$ or $f = \frac{-ae^{\gamma}(e^{n\gamma}-1)}{e^{(n+1)\gamma-1}}$, $g = \frac{-a(e^{n\gamma}-1)}{e^{(n+1)\gamma-1}}$, where γ is a non-constant entire function.

Remark 2. From Theorem 1 (Theorem 2) we see that if $\Theta(\infty; f) + \Theta(\infty; g) > 1$ we can obtain Theorem B (Theorem C). For if $f \not\equiv g$ then we see that $\Theta(\infty; f) = \Theta(\infty; g) =$

$1 - \limsup_{r \rightarrow \infty} \frac{\sum_{k=1}^n \overline{N}(r, u_k; e^{\gamma})}{nT(r, e^{\gamma})} = 0$, where $u_k = \exp\left(\frac{2k\pi i}{n+1}\right)$ for $k = 1, 2, \dots, n$ which leads to a contradiction. So $f \equiv g$.

We prove the following theorems also which show that *Theorem E* can be improved in two ways by relaxing the nature of sharing the set S_1 .

Theorem 3. Let S_i $i = 1, 2$ be defined as in Theorem E. If for two non-constant entire functions f and g $E_f(S_1, 3) = E_g(S_1, 3)$ and $E_f(S_2, 0) = E_g(S_2, 0)$ then $f \equiv g$.

Theorem 4. Let S_i $i = 1, 2$ be defined as in Theorem E. If for two non-constant entire functions f and g $E_5(S_1, f) = E_5(S_1, g)$ and $E_f(S_2, 0) = E_g(S_2, 0)$ then $f \equiv g$.

Remark 3. Let $S_1 = \{2, -3, -6\}$ and $S_2 = \{0\}$. It is easy to see that $S_1 = \{z : z^2(z + 7) + 36 = 0\}$. From Theorems 3-4 we immediately obtain that if for two non-constant entire functions f and g , $E_f(S_1, 3) = E_g(S_1, 3)$ or $E_5(S_1, f) = E_5(S_1, g)$ and $E_f(S_2, 0) = E_g(S_2, 0)$ then $f \equiv g$.

Though for the standard definitions and notations of the value distribution theory we refer to [8], we now explain some notations which are used in the paper.

Definition 3. [9] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f | = 1)$ the counting function of simple a points of f . For a positive integer m we denote by $N(r, a; f | \leq m)$ ($N(r, a; f | \geq m)$) the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity.

$\bar{N}(r, a; f | \leq m)$ ($\bar{N}(r, a; f | \geq m)$) are defined similarly, where in counting the a -points of f we ignore the multiplicities.

Also $N(r, a; f | < m)$, $N(r, a; f | > m)$, $\bar{N}(r, a; f | < m)$ and $\bar{N}(r, a; f | > m)$ are defined analogously.

Definition 4. We denote by $\bar{N}(r, a; f | = k)$ the reduced counting function of those a -points of f whose multiplicities is exactly k , where $k \geq 2$ is an integer.

Definition 5. Let f and g be two non-constant meromorphic functions such that f and g share (a, k) where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be a a -point of f with multiplicity p , a a -point of g with multiplicity q . We denote by $\bar{N}_L(r, a; f)$ the counting function of those a -points of f and g where $p > q$, by $\bar{N}_E^{(k+1)}(r, a; f)$ the counting function of those a -points of f and g where $p = q \geq k + 1$; each point in these counting functions is counted only once. In the same way we can define $\bar{N}_L(r, a; g)$ and $\bar{N}_E^{(k+1)}(r, a; g)$.

Definition 6. [4] Let f and g be two non-constant meromorphic functions and m be a positive integer such that $E_m(a; f) = E_m(a; g)$ where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a -point of f with multiplicity $p > 0$, an a -point of g with multiplicity $q > 0$. We denote by $\bar{N}_L^{(m)}(r, a; f)$ ($\bar{N}_L^{(m)}(r, a; g)$) the counting function of those a -points of f and g where $p > q$ ($q > p$), each a -point is counted only once.

Definition 7. For a positive integer p we denote $N_p(r, a; f) = \bar{N}(r, a; f) + \bar{N}(r, a; f | \geq 2) + \dots + \bar{N}(r, a; f | \geq p)$. Clearly $\bar{N}(r, a; f) = N_1(r, a; f)$.

Definition 8. [4] Let m be a positive integer. Also let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$ of multiplicity q . We denote by $\bar{N}_{f \geq m+1}(r, a; f | g \neq a)$ ($\bar{N}_{g \geq m+1}(r, a; g | f \neq a)$) the reduced counting functions of those a -points of f and g for which $p \geq m + 1$ and $q = 0$ ($q \geq m + 1$ and $p = 0$).

Definition 9. [10, 11] Let f, g share a value a IM. We denote by $\bar{N}_*(r, a; f, g)$ the reduced counting function of those a -points of f whose multiplicities differ from the multiplicities of the corresponding a -points of g .

Clearly $\bar{N}_*(r, a; f, g) \equiv \bar{N}_*(r, a; g, f)$ and $\bar{N}_*(r, a; f, g) = \bar{N}_L(r, a; f) + \bar{N}_L(r, a; g)$.

Remark 4. If $E_m(a; f) = E_m(a; g)$, then $\bar{N}_*(r, a; f, g) = \bar{N}_L^{(m)}(r, a; f) + \bar{N}_L^{(m)}(r, a; g) + \bar{N}_{f \geq m+1}(r, a; f | g \neq a) + \bar{N}_{g \geq m+1}(r, a; g | f \neq a)$.

Definition 10. [14] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f | g = b)$ the counting function of those a -points of f , counted according to multiplicity, which are b -points of g .

Definition 11. [14] Let $a, b_1, b_2, \dots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b_1, b_2, \dots, b_q)$ the counting function of those a -points of f , counted according to multiplicity, which are not the b_i -points of g for $i = 1, 2, \dots, q$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows.

$$F = \frac{f^n(f-a)}{b}, \quad G = \frac{g^n(g-a)}{b}. \quad (2.1)$$

Henceforth we shall denote by H and Φ the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right)$$

and

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}.$$

Lemma 1. Let F, G share $(1, 1)$ and $H \not\equiv 0$. Then

$$N(r, 1; F \mid = 1) = N(r, 1; G \mid = 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Proof. The lemma can be proved in the line of proof of Lemma 1 [11]. \square

Lemma 2. Let S_1, S_2 and S_3 be defined as in Theorem 1 and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_f(S_1, 0) = E_g(S_1, 0)$, $E_f(S_2, 0) = E_g(S_2, 0)$, $E_f(S_3, 0) = E_g(S_3, 0)$ and $H \not\equiv 0$ then

$$N(r, H) \leq \bar{N}_*(r, 0, f, g) + \bar{N}(r, a; f \mid \geq 2) + \bar{N}(r, a; g \mid \geq 2) + \bar{N}_*(r, 1; F, G) \\ + \bar{N}_*(r, \infty; f, g) + \bar{N}_0(r, 0; F') + \bar{N}_0(r, 0; G'),$$

where $\bar{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\bar{N}_0(r, 0; G')$ is similarly defined.

Proof. The lemma can be proved in the line of proof of Lemma 2.2 [2]. \square

Lemma 3. [15] Let f be a nonconstant meromorphic function and let

$$R(f) = \frac{\sum_{k=0}^n a_k f^k}{\sum_{j=0}^m b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 4. *Let F and G be given by (2.1), $n \geq 2$ an integer and $F \not\equiv G$. If F, G share $(1, m)$, f, g share $(0, p)$, (∞, k) , where $0 \leq p < \infty$ then*

$$[np + n - 1] \overline{N}(r, 0; f | \geq p + 1) \leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + S(r, f) + S(r, g).$$

Proof. Suppose 0 is an e.v.P. (Picard exceptional value) of f and g then the lemma follows immediately.

Next suppose 0 is not an e.v.P. of f and g . If $\Phi \equiv 0$, then by integration we obtain

$$F - 1 \equiv C(G - 1).$$

It is clear that if z_0 is a zero of f then it is a zero of g . So it follows that $F(z_0) = G(z_0) = 0$. So $C = 1$ which contradicts $F \not\equiv G$. So $\Phi \not\equiv 0$. Since f, g share $(0, p)$ it follows that a common zero of f and g of order $r \leq p$ is a zero of Φ of order exactly $nr - 1$ where as a common zero of f and g of order $r > p$ is a zero of Φ of order at least $np + n - 1$. Let z_0 is a zero of f with multiplicity q and a zero of g with multiplicity t . From (2.1) we know that z_0 is a zero of F with multiplicity nq and a zero of G with multiplicity nt . So from the definition of Φ it is clear that

$$\begin{aligned} & [np + n - 1] \overline{N}(r, 0; f | \geq p + 1) \\ &= [np + n - 1] \overline{N}(r, 0; g | \geq p + 1) \\ &= [np + n - 1] \overline{N}(r, 0; F | \geq n(p + 1)) \\ &\leq N(r, 0; \Phi) \\ &\leq N(r, \infty; \Phi) + S(r, f) + S(r, g) \\ &\leq \overline{N}_*(r, \infty; F, G) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \end{aligned}$$

The lemma follows from above. \square

Lemma 5. *Let F and G be given by (2.1), $n \geq 2$ an integer and $F \not\equiv G$. If $E_m(1; F) = E_m(1; G)$, f, g share $(0, p)$, (∞, k) , where $0 \leq p < \infty$ then*

$$\begin{aligned} & [np + n - 1] \overline{N}(r, 0; f | \geq p + 1) \\ &\leq \overline{N}_{F \geq m+1}(r, 1; F | G \neq 1) + \overline{N}_L^{(m)}(r, 1; F) + \overline{N}_{G \geq m+1}(r, 1; G | F \neq 1) \\ &\quad + \overline{N}_L^{(m)}(r, 1; G) + \overline{N}_*(r, \infty; F, G) + S(r, f) + S(r, g). \end{aligned}$$

Proof. In view of Remark 4 the proof is obvious. \square

Lemma 6. [2] *Let F, G be given by (2.1) and they share $(1, m)$. If f, g share $(0, p)$, (∞, k) where $2 \leq m < \infty$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}_*(r, 0; f, g) + N_2(r, a; f) + N_2(r, a; g) \\ &\quad + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) - m(r, 1; G) - \overline{N}(r, 1; F | = 3) \\ &\quad - \dots - (m - 2) \overline{N}(r, 1; F | = m) - (m - 2) \overline{N}_L(r, 1; F) - (m - 1) \overline{N}_L(r, 1; G) \\ &\quad - (m - 1) \overline{N}_E^{(m+1)}(r, 1; F) + S(r, F) + S(r, G) \end{aligned}$$

Lemma 7. [4] Let F and G be two meromorphic functions such that $E_m(1; F) = E_m(1; G)$, where $1 \leq m < \infty$. Then

$$\begin{aligned} & \bar{N}(r, 1; F) + \bar{N}(r, 1; G) - N(r, 1; F | = 1) + \left(\frac{m}{2} - \frac{1}{2}\right) \{\bar{N}_{F \geq m+1}(r, 1; F | G \neq 1) \\ & + \bar{N}_{G \geq m+1}(r, 1; G | F \neq 1)\} + \left(m - \frac{1}{2}\right) \{\bar{N}_L^m(r, 1; F) + \bar{N}_L^m(r, 1; G)\} \\ & \leq \frac{1}{2} [N(r, 1; F) + N(r, 1; G)]. \end{aligned}$$

Lemma 8. Let F, G be given by (2.1) and $H \neq 0$. If $E_m(1; F) = E_m(1; G)$, f, g share $(\infty, k), (0, p)$, where $1 \leq m < \infty$. Then

$$\begin{aligned} & \left(\frac{n+1}{2} - 1\right) \{T(r, f) + T(r, g)\} \\ & \leq \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) + \bar{N}(r, 0; g) + \bar{N}(r, \infty; g) + \bar{N}_*(r, 0; f, g) + \bar{N}_*(r, \infty; f, g) \\ & - \left(\frac{m}{2} - \frac{3}{2}\right) \{\bar{N}_{F \geq m+1}(r, 1; F | G \neq 1) + \bar{N}_{G \geq m+1}(r, 1; G | F \neq 1)\} \\ & - \left(m - \frac{3}{2}\right) \{\bar{N}_L^m(r, 1; F) + \bar{N}_L^m(r, 1; G)\} + S(r, f) + S(r, g). \end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned} & T(r, F) + T(r, G) \tag{2.2} \\ & \leq \bar{N}(r, 1; F) + \bar{N}(r, 0; F) + \bar{N}(r, \infty; F) + \bar{N}(r, 1; G) + \bar{N}(r, 0; G) \\ & + \bar{N}(r, \infty; G) - N_0(r, 0; F') - N_0(r, 0; G') + S(r, F) + S(r, G). \end{aligned}$$

Using Lemmas 1, 2, 3 and 7 we see that

$$\begin{aligned} & \bar{N}(r, 1; F) + \bar{N}(r, 1; G) \tag{2.3} \\ & \leq \frac{1}{2} [N(r, 1; F) + N(r, 1; G)] + N(r, 1; F | = 1) \\ & - \left(\frac{m}{2} - \frac{1}{2}\right) \{\bar{N}_{F \geq m+1}(r, 1; F | G \neq 1) + \bar{N}_{G \geq m+1}(r, 1; G | F \neq 1)\} \\ & - \left(m - \frac{1}{2}\right) \{\bar{N}_L^m(r, 1; F) + \bar{N}_L^m(r, 1; G)\} \\ & \leq \frac{n+1}{2} \{T(r, f) + T(r, g)\} + \bar{N}_*(r, 0; f, g) + \bar{N}_*(r, \infty; f, g) + \bar{N}(r, a; f | \geq 2) \\ & + \bar{N}(r, a; g | \geq 2) + \bar{N}_*(r, 1; F, G) - \left(\frac{m}{2} - \frac{1}{2}\right) \{\bar{N}_{F \geq m+1}(r, 1; F | g \neq 1) \\ & + \bar{N}_{G \geq m+1}(r, 1; G | F \neq 1)\} - \left(m - \frac{1}{2}\right) \{\bar{N}_L^m(r, 1; F) + \bar{N}_L^m(r, 1; G)\} \\ & + \bar{N}_0(r, 0; F') + \bar{N}_0(r, 0; G') + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{n+1}{2} \{T(r, f) + T(r, g)\} + \bar{N}_*(r, 0; f, g) + \bar{N}_*(r, \infty; f, g) \\
&\quad + \bar{N}(r, a; f | \geq 2) + \bar{N}(r, a; g | \geq 2) - \left(\frac{m}{2} - \frac{3}{2}\right) \left\{ \bar{N}_{F \geq m+1}(r, 1; F | G \neq 1) \right. \\
&\quad \left. + \bar{N}_{G \geq m+1}(r, 1; G | F \neq 1) \right\} - \left(m - \frac{3}{2}\right) \left\{ \bar{N}_L^m(r, 1; F) + \bar{N}_L^m(r, 1; G) \right\} \\
&\quad + \bar{N}_0(r, 0; F') + \bar{N}_0(r, 0; G') + S(r, f) + S(r, g).
\end{aligned}$$

Using (2.3) in (2.2) the lemma follows. \square

Lemma 9. ([21], Lemma 6) *If $H \equiv 0$, then F, G share $(1, \infty)$. If further F, G share $(\infty, 0)$ then F, G share (∞, ∞) .*

Lemma 10. *If two nonconstant meromorphic functions f, g share $(\infty, 0)$ then for $n \geq 1$*

$$f^n(f-a)g^n(g-a) \neq b^2,$$

where a, b are finite nonzero constants.

Proof. We omit the proof since it can be carried in the line of proof of Lemma 5 [12]. \square

Lemma 11. [21] *Let F, G be two nonconstant meromorphic functions sharing $(1, \infty)$ and (∞, ∞) . If*

$$N_2(r, 0; F) + N_2(r, 0; G) + 2\bar{N}(r, \infty; F) < \lambda T_1(r) + S_1(r),$$

where $\lambda < 1$ and $T_1(r) = \max\{T(r, F), T(r, G)\}$ and $S_1(r) = o(T_1(r))$, $r \rightarrow \infty$, outside a possible exceptional set of finite linear measure, then $F \equiv G$ or $FG \equiv 1$.

Lemma 12. *Let F, G be given by (2.1), F, G share $(1, m)$, $0 \leq m < \infty$ and $\omega_1, \omega_2 \dots \omega_{n+1}$ are the distinct roots of the equation $z^n(z-a) - b = 0$ and $n \geq 2$. Then*

$$\bar{N}_L(r, 1; F) \leq \frac{1}{m+1} \left[\bar{N}(r, 0; f) + \bar{N}(r, \infty; f) - N_{\otimes}(r, 0; f') \right] + S(r, f),$$

where $N_{\otimes}(r, 0; f') = N(r, 0; f' | f \neq 0, \omega_1, \omega_2 \dots \omega_{n+1})$

Proof. We omit the proof since the proof can be carried out in the line of proof of Lemma 2.14 [2]. \square

Lemma 13. *Let F, G be given by (2.1). If $E_m(1; F) = E_m(1; G)$, $1 \leq m < \infty$, ω_i 's are defined as in Lemma 12 and $n \geq 2$. Then*

$$\bar{N}_{F \geq m+1}(r, 1; F | G \neq 1) + \bar{N}_L^m(r, 1; F) \leq \frac{1}{m} \left[\bar{N}(r, 0; f) + \bar{N}(r, \infty; f) - N_{\otimes}(r, 0; f') \right] + S(r, f),$$

Proof. Since

$$\bar{N}_{F \geq m+1}(r, 1; F | G \neq 1) + \bar{N}_L^m(r, 1; F) \leq \bar{N}(r, 1; F | \geq m+1) \leq \frac{1}{m} (N(r, 1; F) - \bar{N}(r, 1; F)),$$

the rest of the proof can be carried out in the line of proof of Lemma 12. \square

3 Proofs of the theorems

Proof of Theorem 1. Let F, G be given by (2.1). Then F and G share $(1, 5), (\infty; \infty)$. We consider the following cases.

Case 1. Let $H \neq 0$. Then $F \neq G$. Noting that f and g share $(0, 0)$ and $(\infty; \infty)$ implies $\overline{N}_*(r, 0; f, g) \leq \overline{N}(r, 0; f) = \overline{N}(r, 0; g)$ and $\overline{N}_*(r, \infty; f, g) \equiv 0$ from *Lemma 3* and *Lemma 6* we get for $\varepsilon > 0$

$$\begin{aligned}
& (n+1)T(r, f) \tag{3.1} \\
& \leq 3\overline{N}(r, 0; f) + N_2(r, a; f) + N_2(r, a; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) - 3\overline{N}_L(r, 1; F) \\
& \quad - 4\overline{N}_L(r, 1; G) + S(r, f) + S(r, g) \\
& \leq \frac{3}{n-1} [\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G)] + [4 - \delta_2(a; f) - \delta_2(a; g) - \Theta(\infty; f) \\
& \quad - \Theta(\infty; g) + \varepsilon] T(r) - 3\overline{N}_L(r, 1; F) - 4\overline{N}_L(r, 1; G) + S(r). \\
& \leq [4 - \delta_2(a; f) - \delta_2(a; g) - \Theta(\infty; f) - \Theta(\infty; g) + \varepsilon] T(r) + S(r).
\end{aligned}$$

In the same way we can obtain

$$(n+1)T(r, g) \leq [4 - \delta_2(a; f) - \delta_2(a; g) - \Theta(\infty; f) - \Theta(\infty; g) + \varepsilon] T(r) + S(r) \tag{3.2}$$

From (3.1) and (3.2) we see that

$$[n - 3 + \delta_2(a; f) + \delta_2(a; g) + \Theta(\infty; f) + \Theta(\infty; g) - \varepsilon] T(r) \leq S(r),$$

which leads to a contradiction for $0 < \varepsilon < n - 3 + \delta_2(a; f) + \delta_2(a; g) + \Theta(\infty; f) + \Theta(\infty; g)$.

Case 2. Since $H \equiv 0$ we get from *Lemma 9* F and G share $(1, \infty)$ and (∞, ∞) . If possible let us suppose $F \neq G$. Then from *Lemma 4* we have

$$\overline{N}(r, 0; f) = \overline{N}(r, 0; g) = S(r).$$

Therefore we see that

$$\begin{aligned}
& N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \tag{3.3} \\
& \leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, 0; g) + N_2(r, a; f) + N_2(r, a; g) + 2\overline{N}(r, \infty; f) \\
& \leq N_2(r, a; f) + N_2(r, a; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r) \\
& \leq [4 - \delta_2(a; f) - \delta_2(a; g) - \Theta(\infty; f) - \Theta(\infty; g) + \varepsilon] T(r) + S(r)
\end{aligned}$$

Using *Lemma 3* we obtain

$$T_1(r) = (n+1) \max\{T(r, f), T(r, g)\} + O(1) = (n+1) T(r) + O(1). \tag{3.4}$$

So again using *Lemma 3* we get from (3.3) and (3.4)

$$\begin{aligned}
& N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\
& \leq \frac{[4 - \delta_2(a; f) - \delta_2(a; g) - \Theta(\infty; f) - \Theta(\infty; g) + \varepsilon] T_1(r) + S_1(r)}{n+1}.
\end{aligned}$$

Noting that $4 - \delta_2(a; f) - \delta_2(a; g) - \Theta(\infty; f) - \Theta(\infty; g) < n + 1$ and $\varepsilon > 0$ be arbitrary by *Lemma 11* we have $FG \equiv 1$, which is impossible by *Lemma 10*. Hence $F \equiv G$ i.e. $f^{n+1} - g^{n+1} \equiv af^n - ag^n$. That is $f^n(f-a) \equiv g^n(g-a)$. This together with the assumption that f and g share $(0, 0)$ implies that f and g share $(0, \infty)$. Also f and g share (∞, ∞) . Suppose $f \not\equiv g$. Let us put $h = \frac{f}{g}$, then $h \neq 1$. Also clearly $0, \infty$ are Picard exceptional values of h and hence we may put $h = e^\gamma$, where γ is an entire function. If γ is constant, then f and g are also constants which contradicts the hypothesis of the theorem. So γ is non-constant entire function. So from $F \equiv G$ we have $g(e^{(n+1)\gamma} - 1) = a(e^{n\gamma} - 1)$ and $f = e^\gamma g$. So the theorem follows. \square

Proof of Theorem 2. We omit the proof since it can be carried out in the line of proof of *Theorem 1* \square

Proof of Theorem 3. Let F, G be given by (2.1). Then F and G share $(1, 3)$. We consider the following cases.

Case 1. Let $H \neq 0$. Then $F \neq G$. Noting that f and g are entire functions and they share $(0, 0)$ from *Lemmas 3, 4, 6* and *12* we get

$$\begin{aligned}
& (n-1)\{T(r, f) + T(r, g)\} \\
\leq & 6\overline{N}(r, 0; f) - 3\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq & \frac{9-3n}{(n-1)}[\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G)] + S(r, f) + S(r, g) \\
\leq & \frac{9-3n}{4(n-1)}[\overline{N}(r, 0; f) + \overline{N}(r, 0; g)] + S(r, f) + S(r, g) \\
\leq & \frac{9-3n}{4(n-1)}[T(r, f) + T(r, g)] + S(r, f) + S(r, g),
\end{aligned} \tag{3.5}$$

which gives a contradiction for $n \geq 2$.

Case 2. Since $H \equiv 0$ we get from *Lemma 9* F and G share $(1, \infty)$. That is $E_f(S_1, \infty) = E_g(S_1, \infty)$. So the theorem follows from *Theorem E*. \square

Proof of Theorem 4. Let F, G be given by (2.1). Then $E_5(1; F) = E_5(1; G)$. We consider the following cases.

Case 1. Let $H \neq 0$. Then $F \neq G$. *Lemmas 3, 5, 8* and *13* and *Remark 4* we get

$$\begin{aligned}
& \left(\frac{n+1}{2} - 1\right)\{T(r, f) + T(r, g)\} \\
\leq & 3\overline{N}(r, 0; f) - \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq & \frac{4-n}{5(n-1)}[T(r, f) + T(r, g)] + S(r, f) + S(r, g),
\end{aligned} \tag{3.6}$$

which gives a contradiction for $n \geq 2$.

Case 2. Since $H \equiv 0$ we get from *Lemma 9* F and G share $(1, \infty)$. That is $E_f(S_1, \infty) = E_g(S_1, \infty)$. So the theorem follows from *Theorem E*. \square

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