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CONNECTIONS ON TANGENT BUNDLE, GRAVITO-ELECTROMAGNETIC ANALOGIES AND A UNIFIED DESCRIPTION OF GRAVITY AND ELECTROMAGNETISM

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Abstract

In two previous papers, we proposed a new unified mathematical description of the main equations of gravity and electromagnetism, based on Finslerian connections on the tangent bundle of the space-time manifold and on tidal tensors. In the present paper, we present these results and point out the relations between the obtained geometric objects and KCC invariants for a special 1-parameter family of sprays.

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1 Introduction

Starting from a recent gravito-electromagnetic analogy, [6], [7], based on tidal tensors, we proposed in [14], [15], a new, common geometric model for the two physical fields.

The analogy in [6], [7] overcomes the limitations of the classical ones, i.e., the linearized approach, which is only valid in the case of a weak gravitational field and the one based on Weyl tensors, which compares tensors of different ranks. But, in defining the central notion used in this analogy – the one of tidal tensor – it is imposed a restriction upon worldline deviation equations.

In [14], [15], we showed that, passing to the tangent bundle of the space-time manifold and using conveniently chosen connections, we have at least three advantages:

- any restriction upon worldline deviation equations becomes unnecessary;
- out of the 4 sets of equations proposed in [6], [7], only two are needed in order to express Einstein and Maxwell equations;
- the obtained equations are valid not only in the case when one of the two physical fields (gravitational or electromagnetic) is present, but also when both are present. In

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other words, these equations can underlie more than an analogy – a common geometric language for the two fields.

The geometric structures involved in this description are Ehresmann connections and Berwald linear connections attached to a certain 1-parameter family of sprays. These connections have the property that worldlines of charged particles are autoparallel and worldline deviation equations have the simplest possible invariant expressions. Maxwell and Einstein equations can be expressed, [14], [15], directly in terms of tidal tensors arising from these connections.

The idea we use here – the metric tensor contains information about gravity only, while connections contain information about both physical fields – was first proposed by R. Miron and collaborators[2] in [11], [9], [12] (and we adopted it here as it leads to simpler computations than other approaches and to elegant equations); still, we use different connections, meant to offer a simple expression for worldline deviation equations.

The paper is organized as follows. In Section 2, we present results regarding geodesics and geodesic deviations for Ehresmann connections and, in particular, for spray connections. In Section 3, we apply these results for a special 1-parameter family of sprays. Section 4 presents in brief the geometric expressions (obtained in [14], [15]) of Einstein and Maxwell equations in terms of tidal tensors. In the last section, we point out the arising KCC invariants and their interpretation.

2 Ehresmann connections and geodesic deviation

2.1 General connections

Consider a 4-dimensional Lorentzian manifold (M, g), regarded as space-time manifold, with local coordinates $x = (x^i)_{i=\overline{0,3}}$ and Levi-Civita connection ∇; on the tangent bundle (TM, π, M), we denote the local coordinates by $(x \circ \pi, y) =: (x^i, y^i)_{i=\overline{0,3}}$ and by $_{,i}$ and $\cdot_i$, partial differentiation with respect to x^i and y^i respectively. A Finslerian tensor field on TM, [5], is a tensor field on TM, whose local components transform by the same rule as the components of a tensor field on M.

An Ehresmann (nonlinear) connection N on TM, [1], [9], gives rise to the adapted basis

$$(\delta_i = \frac{\partial}{\partial x^i} - N^j_{\ i}(x,y)\frac{\partial}{\partial y^j}, \qquad \dot{\delta}_i = \frac{\partial}{\partial y^i}), \tag{1}$$

and to its dual $(dx^i, \delta y^i = dy^i + N^i_{\ j}dx^j)$. For a vector field $X = X^i\delta_i + \tilde{X}^i\dot{\delta}_i$ on TM, the horizontal component $hX = X^i\delta_i$ and the vertical component $vX = \tilde{X}^i\dot{\delta}_i$, are Finslerian tensor fields.

[2]Other attempts of unifying gravity and electromagnetism, based on tangent bundle geometry, try to include also information regarding electromagnetism in the metric tensor – thus getting Finslerian (Randers-type, Beil-type etc.) metrics, [3], [4]. Also, recently, Wanas, Youssef and Sid-Ahmed produced another description, [16], based on teleparallelism on TM. Another version, using complex Lagrange geometry, is proposed by G. Munteanu, [13].

Consider a curve $c : t \mapsto (x^i(t))$ on the base manifold M. For any vector field $X = X^i(t)\dfrac{\partial}{\partial x^i}$ along c, its complete lift $X^C = X^i\dfrac{\partial}{\partial x^i} + \dfrac{dX^i}{dt}\dfrac{\partial}{\partial y^i}$ to TM is written in the adapted basis as:

$$X^C = X^i\delta_i + \frac{\delta X^i}{dt}\dot{\delta}_i,$$

where the *adapted derivative* $\dfrac{\delta X^i}{dt}$ is:

$$\frac{\delta X^i}{dt} = \frac{dX^i}{dt} + N^i{}_j(x,\dot{x})X^j. \tag{2}$$

The curve c is called a *geodesic* (an autoparallel curve) of N if:

$$\frac{\delta y^i}{dt} = 0, \qquad y^i = \frac{dx^i}{dt}; \tag{3}$$

Considering variations through geodesics of a given geodesic (3), the components w^i of the deviation vector field obey the relations:

$$\frac{\delta^2 w^i}{dt^2} = R^i{}_{jk}(x,y)y^k w^j + \mathbb{T}^i{}_j(x,y)\frac{\delta w^j}{dt}, \tag{4}$$

where

$$R^i_{jk} = \delta_k N^i{}_j - \delta_j N^i{}_k \tag{5}$$

are the components of the curvature of N and

$$\mathbb{T}^i{}_j = y^k N^i{}_{k\cdot j} - N^i{}_j,$$

those of the *strong torsion* of N, [10].

It follows that the simplest form of the geodesic deviation equations is obtained in the case when $\mathbb{T}^i{}_j = 0$.

2.2 Spray connections

A *semispray* is a vector field on TM of the form:

$$S = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}; \tag{6}$$

in particular, if the coefficients $G^i = G^i(x,y)$ are homogeneous of degree 2 in y, then S is called a *spray.*

A *path* or a *geodesic* of the semispray S is, [1], [5], a curve $c : t \mapsto (x^i(t))$ on M, with the property that its lift $c' : t \mapsto (x^i(t), \dot{x}^i(t))$ to TM is an integral curve of S, i.e., :

$$\frac{dy^i}{dt} + 2G^i(x,y) = 0, \quad y^i = \dot{x}^i. \tag{7}$$

Conversely, for any ODE system of the form (7) which is globally defined, the functions G^i define a semispray on TM.

Any semispray on TM gives rise to an Ehresmann connection, called the *semispray connection* and an affine connection on TM, called the *Berwald connection.*

The semispray connection N has the local coefficients

$$G^i{}_j := G^i{}_{\cdot j}.$$

The adapted basis is generally nonholonomic, the nonvanishing Lie brackets of the basis vectors are:

$$[\delta_j, \delta_k] = R^i{}_{jk}\dot{\delta}_i, \quad [\delta_j, \dot{\delta}_k] = G^i{}_{jk}\dot{\delta}_i,$$

where $G^i{}_{jk} = G^i{}_{\cdot jk}$.

Generally, paths of a semispray and geodesics of the attached Ehresmann connection do not coincide.

Particular case. If S is a spray, then:

i) $2G^i = N^i{}_j y^j$, i.e., geodesics of S coincide with geodesics of the spray connection N.

ii) The spray S is expressed in the adapted basis as: $S = y^i\delta_i$.

iii) ([15]), The spray connection N has identically vanishing strong torsion. Conversely, if the strong torsion of an Ehresmann connection N on TM is identically zero, then N is a spray connection.

The *Berwald connection* of a semispray S is locally given by:

$$D_{\delta_k}\delta_j = G^i_{jk}\delta_i, \quad D_{\delta_k}\dot{\delta}_j = G^i_{jk}\dot{\delta}_i, \quad D_{\dot{\delta}_k}\delta_j = D_{\dot{\delta}_k}\dot{\delta}_j = 0. \tag{8}$$

The curvature of N becomes then the only nonvanishing torsion component for D :

$$\delta y^i(\mathbb{T}(\delta_k, \delta_j)) = R^i{}_{jk}.$$

The curvature of D is:

$$\begin{aligned} \mathbb{R} \;=\; & R_j{}^i{}_{kl}\delta_i \otimes dx^j \otimes dx^k \otimes dx^l + \; R_j{}^i{}_{kl}\dot{\delta}_i \otimes \delta y^j \otimes dx^k \otimes dx^l + \\ & + G_j{}^i{}_{kl}\delta_i \otimes dx^j \otimes dx^k \otimes \delta y^l, \end{aligned} \tag{9}$$

where

$$R_j{}^i{}_{kl} = R^i{}_{kl\cdot j}, \quad G_j{}^i{}_{kl} = G^i{}_{jk\cdot l}.$$

For a Finsler vector field $X = X^i(x,y)\delta_i$ on TM, it makes then sense the *Berwald covariant derivative* $DX := D_S X$, which is[3], [2], [5]:

$$DX = (DX^i)\delta_i, \quad DX^i = S(X^i) + G^i{}_{jk}(x,y)y^j X^k. \tag{10}$$

[3]In particular, for spray connections, DX^i coincides with the *dynamical covariant derivative* $\nabla X^i = S(X^i) + G^i{}_j(x,y)X^k$.

In the following, we will always assume that S is a spray. In this case, geodesics of S and N coincide.

If X is a vector field along a geodesic $c : t \mapsto x^i(t)$ of S, then:

$$\frac{\delta X^i}{dt} = DX^i.$$

In particular, a curve $c : t \mapsto x^i(t)$ on M is a geodesic of S if and only if $DS(c') = 0$, i.e.,

$$Dy^i = 0, \qquad y^i = \dot{x}^i. \tag{11}$$

Geodesic deviation equations are then written as:

$$D^2 w^i = E^i{}_j(x, y) w^j, \quad y = \dot{x}, \tag{12}$$

i.e., $D^2 w = E(w)$ where $w = w^i(t)\delta_i$ and

$$E^i{}_j = R^i_{jk} y^k = R_l{}^i{}_{jk} y^l y^k \tag{13}$$

define the *tidal tensor* $E = E^i{}_j \delta_i \otimes dx^j$ attached to the spray connection[4] N.

3 A special family of spray connections

Consider the following 1-parameter family of Lagrangians depending on α :

$$\overset{\alpha}{L} = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} + \alpha A_i \dot{x}^i; \tag{14}$$

where, g_{ij} is as above, $A = A_i(x)dx^i$ is a 1-form on M and $\alpha \in \mathbb{R}$ is a parameter.

Extremal curves $x = x(t)$ (with $t = const \cdot s$) for the action $\int \overset{\alpha}{L} dt$ are given by:

$$\frac{dy^i}{dt} + \gamma^i{}_{jk} y^j y^k - \alpha \|y\| F^i{}_j y^j = 0, \quad y^i = \dot{x}^i, \tag{15}$$

where

$$F^i{}_j := g^{ih}(A_{j,h} - A_{h,j}), \qquad \|y\| = \sqrt{g_{ij} y^i y^j}; \tag{16}$$

we get a 1-parameter family of sprays on TM, with coefficients $G^i = \overset{\alpha}{G^i}$:

$$2\overset{\alpha}{G^i}(x, y) = \gamma^i{}_{jk} y^j y^k + 2\overset{\alpha}{B^i}, \tag{17}$$

where[5]

$$2\overset{\alpha}{B^i} = -\alpha \|y\| F^i{}_j y^j. \tag{18}$$

[4] The idea of tidal tensor (accordingly, of Jacobi endomorphism, see [2], [5]) and Section 5) makes sense in the more general case of semispray connections, in this case, in (13), there appears an extra term.

[5] Indices are raised here by means of the pseudo-Riemannian metric $g_{ij}(x)$ and not by the Finslerian metric tensor induced by the Randers fundamental function $\overset{\alpha}{L}$. This makes our approach different from classical Randers geometry and similar to Ingarden geometry, [12].

We denote: $F^i = F^i{}_j y^j$, i.e., $2\overset{\alpha}{B}{}^i := -\alpha \|y\| F^i$. If there is no risk of confusion, we will use the simpler notations G^i, B^i, $G^i{}_j, B^i{}_j, ...$ instead of $\overset{\alpha}{G}{}^i, \overset{\alpha}{B}{}^i$ $\overset{\alpha}{G}{}^i{}_j, \overset{\alpha}{B}{}^i{}_j$ etc.

The 2-homogeneous functions B^i, (18), are the components of a (horizontal) Finslerian vector field $B = B^i \delta_i$ on TM. With $l_i := \dfrac{y_i}{\|y\|} = \dfrac{g_{ij}y^j}{\|y\|}$, we have:

$$B^i{}_j = B^i{}_{\cdot j} = -\frac{\alpha}{2}(F^i l_j + \|y\| F^i{}_j),$$
$$B^i{}_{jk} := B^i{}_{\cdot jk} = -\frac{\alpha}{2}(l_{\cdot jk}F^i + l_j F^i{}_k + l_k F^i{}_j). \tag{19}$$

The spray connection coefficients of $N = \overset{\alpha}{N}$ and $D = \overset{\alpha}{D}$ are expressed in terms of $\gamma^i{}_{jk}$ and B as:

$$G^i{}_j = \gamma^i{}_{jk} y^k + B^i{}_j, \; G^i{}_{jk} = \gamma^i{}_{jk} + B^i{}_{jk}. \tag{20}$$

From the homogeneity of degree 2 of B in the fiber coordinates, it follows: $B^i{}_j y^j = 2B^i$, $\quad B^i{}_{jk} y^k = B^i{}_j, \quad B^i{}_{jkl} y^l = 0.$

Extremal curves for $\overset{\alpha}{L}$ obey the geodesic equations (11)), while deviations of these extremal curves are given by (12).

4 Basic equations of gravitational and electromagnetic fields

In the following, we will apply the above construction to the case when g_{ij} describes the gravitational field and $A = A_i dx^i$ in (14), is the 4-potential of the electromagnetic field. The differential forms A and $F := dA = \frac{1}{2}F_{ij}dx^i \wedge dx^j$ can be lifted to horizontal forms on TM, which we will denote in the same way (the notations in (16) remain unchanged). Unless elsewhere specified, the parameter $\alpha \neq 0$ is arbitrary.

The electromagnetic 2-form F can be expressed in terms of $l = l_i dx^i$. More precisely, we have, [14],

$$D_{\delta_j} l_i = \frac{\alpha}{2} F_{ij}; \tag{21}$$

the above can be also written in the form (which reminds [12], [9], [11])

$$\alpha F_{ij} = D_{\delta_j} l_i - D_{\delta_i} l_j, \tag{22}$$

or, in coordinate-free notation: $\alpha F = h(dl)$.

Einstein and Maxwell equations can be expressed in terms of tidal tensors attached to $D = \overset{\alpha}{D}$, $\alpha \neq 0$. Consider the *angular metric, [1]:* $h = h_{ij}(x,y)dx^i \otimes dx^j$, given by:

$$h_{ij} = g_{ij} - l_i l_j.$$

Then:

- *homogeneous Maxwell equations* $\nabla_{\partial_i} F_{jk} + \nabla_{\partial_k} F_{ij} + \nabla_{\partial_j} F_{ki} = 0$ are written, [14], as:

$$\tilde{E}_{[ij]} = 0, \tag{23}$$

where $\tilde{E}_{ij} = h_{ik}E^k{}_j$ and square brackets denote antisymmetrization;

- *inhomogeneous Maxwell equations* $\nabla_{\partial_i}F^{ij} = 4\pi J^i$, [8], are expressed in terms of tidal tensors as:

$$E^i{}_i = e^i{}_i - 4\pi\alpha\rho_c \|y\|^2 + B^l{}_i B^i{}_l, \tag{24}$$

where $\rho_c = -J^i l_i$ and $e^i{}_i = \overset{0}{E}{}^i{}_i$;

- *Einstein field equations* can also be expressed in terms of tidal tensors. The Ricci tensor of $D = \overset{\alpha}{D}$ is $R_{ij} = -\frac{1}{2}(E^l{}_l)_{\cdot ij}$. We have proved, [15], that for $\frac{3\alpha^2}{2} = 1$, the Ricci scalar $R = \overset{\alpha}{R}$ is dynamically equivalent to the Lagrangian

$$\tilde{R} = r + F_{ij}F^{ij}$$

on M – i.e., to the Lagrangian which leads to the usual Einstein-Maxwell equations, [8]. Here, $\tilde{R} = R_{ij} - \overset{0}{D}_{\delta_k}(B^k{}_{ij})$ and $\tilde{R} = g^{ij}\tilde{R}_{ij}$. Then, Einstein field equations

$$r_{ij} - \frac{1}{2}rg_{ij} = 8\pi(\overset{em}{T}_{ij} + \overset{m}{T}_{ij})$$

(where $\overset{em}{T}_{ij}$ is the stress-energy tensor of the electromagnetic field and $\overset{m}{T}_{ij}$, the stress-energy tensor of matter and/or other fields) are equivalent to:

$$\tilde{R}_{ij} - \frac{1}{2}\tilde{R}g_{ij} + \mathcal{B}_{\cdot ij} = 8\pi\overset{m}{T}_{ij}, \tag{25}$$

where $\mathcal{B} := \frac{3}{2}\frac{B^l B_l}{\|y\|^2} + \frac{1}{2}B^i{}_h B^h{}_i$; thus, the electromagnetic stress-energy tensor $\overset{em}{T}_{ij}$ is included in the Einstein tensor $\mathcal{G}_{ij} = \tilde{R}_{ij} - \frac{1}{2}\tilde{R}g_{ij} + \mathcal{B}_{\cdot ij}$.

- *Equations of motion of a charged particle*, [8], are (15):

$$\overset{\alpha}{D}y^i = 0, \quad y = \dot{x}, \tag{26}$$

in which, this time, we set $\alpha = \frac{q}{m}$.

For particles having the same ratio $\frac{q}{m}$, worldline deviation equations are:

$$\overset{\alpha}{D}{}^2 w^i = E^i{}_j w^j, \quad \alpha = \frac{q}{m}. \tag{27}$$

In the particular cases when only one of the two physical fields is present, we find similar equations to those underlying the gravito-electromagnetic analogy in [6], [7] – just, without resorting to any restriction upon w in defining tidal tensors.

a) *Gravity only*: In this case, we have $B^i = 0$, $G^i{}_{jk} = \gamma^i{}_{jk}$ and $R_j{}^i{}_{kl} = r_j{}^i{}_{kl}$ and, from (25), $r = -8\pi \overset{m}{T}{}^l{}_l$. The tidal tensor is $e^i{}_j := r_l{}^i{}_{jk} y^l y^k$; thus, contracting (25) by $y^i y^j$, we can write:

$$\begin{array}{ccc} r_{ij} = 8\pi(\overset{m}{T}_{ij} - \frac{1}{2} g_{ij} \overset{m}{T}{}^l{}_l) & \Leftrightarrow & \frac{1}{\|y\|^2} e^i{}_i = -4\pi(2\rho_m - T^i{}_i) \\ r_{jk} = r_{kj} & \Leftrightarrow & \tilde{e}_{[ij]} = 0, \end{array} \tag{28}$$

where $\rho_m = T_{ij} l^i l^j$.

b) Electromagnetism in flat Minkowski space: In this case, $\gamma^i{}_{jk} = 0$, $G^i{}_{jk} = B^i{}_{jk}$ and Maxwell equations are written as:

$$\begin{array}{ccc} \nabla_{\partial_i} F^{ij} = 4\pi J^i & \Leftrightarrow & \frac{1}{\|y\|^2} E^i{}_i = -4\pi\alpha\rho_c + \frac{1}{\|y\|^2} B^i{}_h B^h{}_i \\ \nabla_{\partial_i} F_{jk} + \nabla_{\partial_k} F_{ij} + \nabla_{\partial_j} F_{ki} = 0 & \Leftrightarrow & \tilde{E}_{[ij]} = 0. \end{array} \tag{29}$$

5 KCC invariants and their meaning

The solutions of a system of ordinary differential equations on a manifold is determined, up to a change of coordinates, by five invariants, [2], called the Kosambi–Cartan-Chern invariants (or the *KCC-invariants*) of the system.

Consider a globally defined ODE system on M, and G^i – the coefficients of the corresponding semispray:

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0.$$

The KCC invariants of the system are:

1) The *deviation tensor* $\mathcal{E}^i = 2G^i - N^i{}_j y^j$;

2) The *Jacobi endomorphism* $\Phi = \Phi^i{}_j dx^j \otimes \dot{\delta}_i$, where

$$\Phi^i{}_j = 2G^i_{,j} - S(G^i{}_j) - G^i{}_r G^r{}_j.$$

3) The curvature $R^i{}_{jk}$ of the semispray connection N.

4) The horizontal curvature component $R_j{}^i{}_{kl}$ of the Berwald connection.

5) The mixed curvature component of the Berwald connection (the *Douglas tensor*): $G_j{}^i{}_{kl} = G^i_{\cdot jkl}$.

In our case, $N = \overset{\alpha}{N}$ are spray connections, which entails $\mathcal{E}^i = 0$.

A direct computation shows that $\Phi^i{}_j = -E^i{}_j$, i.e., the Jacobi endomorphism and the tidal tensor E are related by the the tangent structure $J = dx^i \otimes \dot{\delta}_i$:

$$\Phi = -J \circ E,$$

The third and the fourth invariants are related to the second one (i.e., to $E^i{}_j$) by:

$$\begin{aligned} E^i{}_j &= R^i{}_{jk} y^k = R_l{}^i{}_{jk} y^l y^k, \\ R^i{}_{jk} &= \frac{1}{3}(E^i{}_{j\cdot k} - E^i{}_{k\cdot j}), \quad R_j{}^i{}_{kl} = R^i{}_{kl\cdot j} \ ; \end{aligned}$$

any of these invariants characterizes worldline deviation. The vanishing of $e^i{}_j = \overset{0}{E}{}^i{}_j$ (equivalently, of $\overset{0}{R}{}^i{}_{jk}$ or of $r_j{}^i{}_{kl} = \overset{0}{R}{}^i{}_{jkl}$) indicates the absence of the gravitational field (flat space-time).

The fifth invariant is:

$$G_j{}^i{}_{kl} = B^i{}_{\cdot jkl}$$

and it contains information regarding electromagnetism. If $B^i{}_{\cdot jkl} = 0$, then the electromagnetic 2-form F vanishes.

The justification of the latter statement is the following. If $B^i{}_{\cdot jkl} = 0$, then in (19), we have $B^i{}_{jk} = B^i{}_{jk}(x)$, i.e., $B^i = -\alpha \|y\| F^i{}_j(x) y^j$ is bilinear in y. Supposing $F^i{}_j \neq 0$, it follows that $\|y\|$ should be linear, $\|y\| = a_i(x) y^i$ and thus, $g_{ij} = a_i a_j$ (which is a degenerate metric tensor); since this is impossible, it follows that $F^i{}_j = 0$.

Consequently, a nonzero Douglas tensor $G_j{}^i{}_{kl}$ (equivalently: a nonzero contortion $B^i{}_{jk}$) indicates the presence of the electromagnetic field.

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