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A SUFFICIENT CONDITION FOR UNIVALENCE

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Abstract

In this paper we obtain sufficient conditions for the analyticity and the univalence of the functions defined by an integral operator.

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1 Introduction

We denote by $U_r = \{ z \in \mathbb{C} : \ |z| < r\}$ the disk of z-plane, where $r \in (0,1]$, $U_1 = U$ and $I = [0, \infty)$.
Let A be the class of analytic functions f in the unit disk U, such that $f(0) = 0, \ f'(0) = 1$. Let S denote the class of functions $f \in A$, f univalent in U. The subclasses of S consisting of starlike functions and α-convex functions will be denoted by S^*, respectively M_α.

Definition 1. ([2]). *Let $f \in A$, $f(z)f'(z) \neq 0$ for $z \in U$ and $\alpha > 0$. We denote by*

$$M(\alpha, f) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(\frac{zf''(z)}{f'(z)} + 1)$$

If $\Re\, M(\alpha, f) > 0$ in U, then f is said to be an α-convex function ($f \in M_\alpha$).

Theorem 1. ([2]). *The function $f \in M_\alpha$ if and only if there exists a function $g \in S^*$ such that*

$$f(z) = \left(\frac{1}{\alpha} \int_0^z \frac{g^{1/\alpha}(u)}{u} du \right)^\alpha \tag{1}$$

In order to prove our main result we need the theory of Löewner chains.

Theorem 2. ([3]). *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots, \ \ a_1(t) \neq 0$ be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and locally uniformly with respect to U_r. For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

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where $p(z,t)$ is analytic in U and satisfies the condition $Re\ p(z,t) > 0$, for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

2 Main results

In this section, making use of Theorem 2, we obtain a sufficient condition for the analyticity and the univalence of the functions defined by the operator introduced by P. T. Mocanu ([2]) in the integral representation of α-convex functions.

Theorem 3. *Let $f \in A$ and α be a complex number, $\Re\alpha > \frac{1}{2}$. If the inequality*

$$\left| \frac{1-\alpha}{\alpha}|z|^{2\alpha} + (1-|z|^{2\alpha})\left(\frac{zf'(z)}{f(z)} - 1\right) \right| \le 1 \tag{2}$$

is true for all $z \in U$, then the function

$$F(z) = \left(\alpha \int_0^z \frac{f^\alpha(u)}{u} du\right)^{1/\alpha} \tag{3}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. Let us prove that there exists $r \in (0,1]$ such that the function $L : U_r \times I \longrightarrow \mathbb{C}$ defined as

$$L(z,t) = \left[\int_0^{e^{-t}z} \frac{f^\alpha(u)}{u} du + (e^{2\alpha t} - 1) f^\alpha(e^{-t}z)\right]^{1/\alpha} \tag{4}$$

is analytic in U_r for all $t \in I$.

From the analyticity of the function f it follows that the function $h(z) = \frac{f(z)}{z}$ is analytic in U and since $h(0) = 1$ there is a disk U_{r_1} in which $h(z) \neq 0$. Therefore we can choose the uniform branch of $(h(z))^\alpha$ equal to 1 at the origin, denoted by h_1. It is easy to see that the function

$$h_2(z,t) = \int_0^{e^{-t}z} u^{\alpha-1} h_1(u) du$$

can be written as $h_2(z,t) = z^\alpha h_3(z,t)$, where h_3 is analytic in U_{r_1}, $h_3(0,t) = e^{-\alpha t}/\alpha$. The function $h_4(z,t) = h_3(z,t) + (e^{2\alpha t} - 1)e^{-\alpha t} h_1(e^{-t}z)$ is also analytic in U_{r_1} and $h_4(0,t) = e^{\alpha t}\left[1 + \frac{1-\alpha}{\alpha} e^{-2\alpha t}\right]$. We shall prove that $h_4(0,t) \neq 0$ for any $t \in I$. We have $h_4(0,0) = 1/\alpha$. Assume that there exists $t_0 > 0$ such that $h_4(0,t_0) = 0$. Then $e^{2\alpha t_0} = (\alpha - 1)/\alpha$. Since $\Re\alpha > 1/2$ is equivalent with $|(\alpha-1)/\alpha| < 1$ it follows $|e^{2\alpha t_0}| < 1$ and we conclude that $h_4(0,t) \neq 0$ for all $t \in I$. Therefore, there is a disk U_{r_2}, $0 < r_2 \le r_1$, in which $h_4(z,t) \neq 0$ for all $t \in I$. We choose the uniform branch of $[h_4(z,t)]^{1/\alpha}$ analytic in U_{r_2}, denoted by $h_5(z,t)$, that is equal to

$$a_1(t) = e^t \left[1 + \frac{1-\alpha}{\alpha} e^{-2\alpha t}\right]^{1/\alpha}$$

at the origin. From these considerations, it results that the relation (4) may be written as $L(z,t) = zh_5(z,t) = a_1(t)z + a_2(t)z^2 + \dots$.

Under the assumption of the theorem, we have $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. Since $L(z,t)$ is an analytic function in U_{r_2}, it results that there exist a number $0 < r_3 < r_2$ and a constant $k = k(r_3)$ such that $\mid L(z,t)/a_1(t) \mid < k$, $z \in U_{r_3}$, and hence $\{L(z,t)/a_1(t)\}$ forms a normal family in U_{r_3}.

It can be easy see that $\frac{\partial L(z,t)}{\partial t}$ is an analytic function in U_{r_3} and therefore $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_3}. We define

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}$$

and we will prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$. Let $w(z,t)$ be the function defined by

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}.$$

After computation, we obtain

$$w(z,t) = \frac{1-\alpha}{\alpha}e^{-2\alpha t} + (1-e^{-2\alpha t})\left(\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} - 1\right) \tag{5}$$

We have

$$w(z,0) = \frac{1-\alpha}{\alpha} \quad \text{and} \quad w(0,t) = \frac{1-\alpha}{\alpha}e^{-2\alpha t}$$

Since $\Re\alpha > \frac{1}{2}$ we obtain that

$$\mid w(z,0) \mid < 1 \quad \text{and also} \quad \mid w(0,t) \mid < 1 \tag{6}$$

Let t be a fixed positive number, $z \in U,\ z \neq 0$. Since $|e^{-t}z| \le e^{-t} < 1$ for all $z \in \overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ we conclude that the function $w(z,t)$ is analytic in $\overline{U}$. Using the maximum modulus principle it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|w(z,t)| < \max_{|\xi|=1} |w(\xi,t)| = |w(e^{i\theta},t)|, \tag{7}$$

We denote $u = e^{-t} \cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (5) we get

$$w(e^{i\theta},t) = \frac{1-\alpha}{\alpha}|u|^{2\alpha} + (1-|u|^{2\alpha})\left(\frac{uf'(u)}{f(u)} - 1\right)$$

Since $u \in U$, the inequality (2) implies $|w(e^{i\theta},t)| \le 1$ and from (6) and (7) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \ge 0$.

From Theorem 2 it results that the function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t \in I$. For $t = 0$ it follows that the function

$$L(z,0) = \left(\int_0^z \frac{f^\alpha(u)}{u}du\right)^{1/\alpha}$$

is analytic and univalent in U and then the function F defined by (3) is also analytic and univalent in U. □

Corollary 1. *Let $f \in A$ and α be a complex number, $\Re\alpha > \frac{1}{2}$. If for all $z \in U$*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\Re\alpha}{|\alpha|}, \tag{8}$$

then the function F defined by (3) is analytic and univalent in U.

Proof. It is known that for all $z \in U$, $z \neq 0$ and $\Re\alpha > 0$ we have

$$\left| \frac{1-|z|^{2\alpha}}{\alpha} \right| \leq \frac{1-|z|^{2\Re\alpha}}{\Re\alpha} \tag{9}$$

In view of (8) and (9) and since $|\alpha - 1| < |\alpha|$ we obtain

$$\left| \frac{1-\alpha}{\alpha}|z|^{2\alpha} + (1-|z|^{2\alpha}) \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| \leq$$

$$\left|\frac{1-\alpha}{\alpha}|z|^{2\alpha}\right| + \left|\frac{1-|z|^{2\alpha}}{\alpha}\right| \left|\alpha \left(\frac{zf'(z)}{f(z)} - 1 \right)\right| \leq |z|^{2\Re\alpha} + \frac{1-|z|^{2\Re\alpha}}{\Re\alpha}|\alpha| \left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1$$

From Theorem 3 it follows that the function F is analytic and univalent in U.

□

Remark 1. The condition (8) implies $f \in S^*$. For α real number, $\alpha > 0$, from Theorem 1 we get that F is an α-convex function.

Example 1. *Let α, b be complex numbers, $\Re\alpha > \frac{1}{2}$, $|b| \geq 1 + \frac{|\alpha|}{\Re\alpha}$. The function*

$$F(z) = \left(\alpha \int_0^z \frac{u^{\alpha-1}}{(u+b)^\alpha} du \right)^{1/\alpha}$$

is analytic and univalent in U.

Proof. We consider the function $f(z) = \frac{bz}{z+b} = z + \ldots$ which satisfies the condition (8) of Corollary 1. □

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