Bulletin of the *Transilvania* University of Braşov • Vol 4(53), No. 2 - 2011
Series III: Mathematics, Informatics, Physics, 89-96

AFFINE DEFORMATIONS OF MINKOWSKI SPACES

J. SZILASI and L. TAMÁSSY[1]

Communicated to:
Finsler Extensions of Relativity Theory, August 29 - September 4, 2011, Braşov, Romania

Abstract

We investigate generalized Berwald spaces $\mathcal{B}^n$ over $\mathbb{R}^n$ (local theory). These are Finsler spaces admitting metric linear connections Γ^* over $T\mathbb{R}^n$. (If Γ^* is torsion free, then $\mathcal{B}^n$ is a Berwald space.)

An affine deformation is a regular linear transformation of each $T_p\mathbb{R}^n$. This takes the indicatrices of a Minkowski space $\mathcal{M}^n$ into other indicatrices, and thus it leads to a new Finsler space.

We prove that any $\mathcal{B}^n$ is the affine deformation of an $\mathcal{M}^n$, and conversely. We show that any $\mathcal{B}^n$ can be represented by a pair $(V^n, \mathcal{M}^n)$ of a Riemannian and a Minkowski space. Several properties of $\mathcal{B}^n$ will be expressed by properties of V^n or $\mathcal{M}^n$. Also the linear automorphisms of the indicatrices will be investigated.

2000 *Mathematics Subject Classification:* 53B40, 51B20.
Key words: Minkowski space, generalized Berwald space, affine deformation.

1 Introduction

In metric differential geometry, like Finsler geometry, one can study arc length, angle, area, geodesics, motion, etc. using the metric only, but a number of important questions need relation between the tangent vectors. This is provided by a connection. But a connection is really effective if it is linear and metric. Unfortunately such a connection does not exist (in general) in Finsler geometry, at least not among the tangent vectors of the base manifold $\mathbb{R}^n$. This deficiency was surmounted by introducing the line-elements. However this made the apparatus more combined and difficult. Nevertheless there are important special Finsler spaces in which there exists metric linear connection in the tangent bundle $T\mathbb{R}^n$. Such are, among others, the Riemannian and the Minkowski spaces.

In this paper we investigate affine deformations of Minkowski spaces. Affine deformation means a regular linear transformation in each tangent space $T_x\mathbb{R}^n$. We show that also the spaces arising from a Minkowski space by affine deformation admit linear metric connections for the vectors of $T\mathbb{R}^n$. We detect and describe several properties of the

[1]University of Debrecen, Institute of Mathematics, H–4010 Debrecen, P.O. Box 12, Hungary, E-mail: szilasi@science.unideb.hu and tamassy@science.unideb.hu

affine deformation of Minkowski spaces. Several results are near to those of Y. Ichijyō [I]. We often apply direct geometric considerations rather than analytic calculations. These sometimes lead to quick results.

2 Relations between generalized Berwald spaces and Minkowski spaces

Our investigations will be local, nevertheless more results remain valid also over certain global manifolds. So our base manifolds will be $\mathbb{R}^n$. The indicatrices $I(x)$ of a Finsler space $F^n = (\mathbb{R}^n(x), \mathcal{F}(x,y))$ are defined by

$$I(x) := \{y \in T_x\mathbb{R}^n \mid \mathcal{F}(x,y) = 1\}.$$

The indicatrix bundle $\{I(x)\}$ and the Finsler metric function $\mathcal{F}(x,y)$ uniquely determine each other

$$\{I(x)\} \stackrel{1:1}{\longleftrightarrow} \mathcal{F}(x,y).$$

Thus we can write $F^n = (\mathbb{R}^n(x), I(x))$ in place of $F^n = (\mathbb{R}^n(x), \mathcal{F}(x,y))$. Indicatrices $I(x)$ seem to be more adequate for geometric considerations, than the Finsler function $\mathcal{F}(x,y)$.

Definition. An affine deformation in $\mathbb{R}^n$ is a family of invertible (or, equivalently, regular) linear transformations

$$\mathfrak{a}(x) : T_x\mathbb{R}^n \to T_x\mathbb{R}^n, \tag{1}$$

depending smoothly on x. If the components of $\mathfrak{a}(x)$ are denoted by $a^i_k(x)$, $\det(a^i_k(x)) \neq 0$, then (1) means $a^i_k(x)\xi^k = \bar{\xi}^i$, $\xi \in T_x\mathbb{R}^n$. This family (which is, in fact, a $(1,1)$ tensor on $\mathbb{R}^n$) is denoted by $\mathfrak{A}$.

$$\mathfrak{a}(x)I(x) = \bar{I}(x) \tag{2}$$

is an affine deformation of the indicatrix $I(x)$, and it is another indicatrix $\bar{I}(x)$. Thus the affine deformation $\mathfrak{A}F^n$ of a Finsler space F^n is the Finsler space

$$\mathfrak{A}F^n := (\mathbb{R}^n(x), \mathfrak{a}(x)I(x)) = (\mathbb{R}^n(x), \bar{I}(x)).$$

The affine deformation $\mathfrak{A}E^n$ of the Euclidean space $E^n = (\mathbb{R}^n(x), S)$, where S is the Euclidean unit sphere, is a Riemannian space V^n:

$$\mathfrak{A}E^n = (\mathbb{R}^n(x), \mathfrak{a}(x)S) = (\mathbb{R}^n(x), Q(x)) =: V^n,$$

where

$$\mathfrak{a}(x)S = Q(x) \tag{3}$$

are ellipsoids. Given a bundle $\{Q(x)\}$, (3) is solvable for $\mathfrak{a}(x)$. So every V^n is an affine deformation of E^n.

Let $\mathcal{M}^n = (\mathbb{R}^n(x), I_0)$ be a Minkowski space in an adapted coordinate system (x), where I_0 is independent of x. Then the indicatrices $I_0(x)$ are parallel translates of each other, and

$$\mathfrak{A}\mathcal{M}^n = (\mathbb{R}^n(x), \mathfrak{a}(x)I_0) = (\mathbb{R}^n(x), \bar{I}(x)) = F^n$$

is a Finsler space, which is not a Minkowski space any longer (except if $\mathfrak{a}(x)$ does not depend on x).

Finally any two Riemannian spaces V_1^n and V_2^n are affine deformations of each other. The affine deformation of a Finsler space is another Finsler space, however two Finsler spaces are not affine deformation of each other in general.

A Berwald space is a Finsler space in which the coefficients of the Berwald connection are independent of the point x. They admit metric linear connections in the tangent bundle, but this property is not characteristic for them. (For details we refer to [SzLK].) The Finsler spaces admitting metric linear connections in the tangent bundle $T\mathbb{R}^n$ are the *generalized Berwald spaces* denoted in this paper by $\mathcal{B}^n$.

Our basic result is

Theorem 1. *Every generalized Berwald space $\mathcal{B}^n$ is an affine deformation of a Minkowski space, and conversely.*

This result is closely related, and partially coincides with that of Y. Ichijyō [I] (see also L. Tamássy [Ta]).

Proof. I/ *Any $\mathfrak{A}\mathcal{M}^n$ admits a metric linear connection Γ^*, and thus it is a $\mathcal{B}^n$.*

Let (x) be an adapted coordinate system for the Minkowski space $\mathcal{M}^n = (\mathbb{R}^n(x), I_0)$. Its affine deformation is $\mathfrak{A}\mathcal{M}^n = (\mathbb{R}^n(x), \mathfrak{a}(x)I_0) = F^n = (\mathbb{R}^n(x), I(x))$. Let x_0 and $\bar{x}$ be two arbitrary points of $\mathbb{R}^n$, and $x(t)$ a curve, such that $x(0) = x_0$ and $\bar{x} = x(\bar{t})$. Let $\mathfrak{b}(x)$ be the inverse of $\mathfrak{a}(x)$, and $\mathfrak{i} : T_{x_0}\mathbb{R}^n \to T_{\bar{x}}\mathbb{R}^n$ the canonical linear isomorphism. We define

$$\mathfrak{p}(x_0, \bar{x}) := \mathfrak{a}(\bar{x}) \circ \mathfrak{i} \circ \mathfrak{b}(x_0).$$

Displaying by a sketchy diagram:

$$\begin{array}{cccc}
T\mathbb{R}^n & \xi(0) & & \xi(\bar{t}) \\
\mathfrak{A}\mathcal{M}^n & I(x_0) & \xrightarrow{\ \mathfrak{p}\ } & I(\bar{x}) \\
 & \Big\uparrow{\scriptstyle \mathfrak{a}(x_0)} & & \Big\uparrow{\scriptstyle \mathfrak{a}(\bar{x})} \\
\mathcal{M}^n & I_0 & \xrightarrow{\ \mathfrak{i}\ } & I_0 \\
\mathbb{R}^n & x(0) & \xrightarrow{\ x(t)\ } & \bar{x} = x(\bar{t})
\end{array}$$

Obviously, $\mathfrak{p}$ is a linear transformation. It takes $T_{x_0}\mathbb{R}^n$ into $T_{\bar{x}}\mathbb{R}^n$, and a vector $\xi_0 \in T_{x_0}\mathbb{R}^n$ into $\xi(x(\bar{t})) \in T_{\bar{x}}\mathbb{R}^n$. Then, in local coordinates,

$$\xi^i(x(t)) = a_k^i(x(t)) b_s^k(x_0) \xi_0^s.$$

From this

$$\frac{d\xi^i}{dt}\Big|_{t=0} = \frac{\partial a_k^i}{\partial x^r}(x_0) b_s^k(x_0) \dot{x}^r \xi_0^s,$$

hence

$$\Gamma^{*\,i}_{r\ s}(x) := -\frac{\partial a_k^i}{\partial x^r}(x) b_s^k(x) \tag{4}$$

are the coefficients of a linear connection Γ^* induced by the parallel translation $\mathfrak{p}(x_0, x)$. However $\mathfrak{p}(x_0, x)I_0 = I(x)$. This means that Γ^* is metric. Thus $\mathfrak{A}\mathcal{M}^n$ is a $\mathcal{B}^n$.

II/ *Any $\mathcal{B}^n = (\mathbb{R}^n(x), \bar{I}(x))$ is an affine deformation of an $\mathcal{M}^n$.*

$\mathcal{B}^n$ determines a metric, linear connection Γ^*, and a parallel translation $\mathfrak{p}$. First we construct a Minkowski space $\mathcal{M}^n$, for which (x) is an adapted coordinate system. Let x_0 be an arbitrarily chosen fixed point of $\mathbb{R}^n$, and let $I_0 := \bar{I}(x)$. Then $I_0 = \mathfrak{i}_1 I_0(x)$ and $I(x_0) = \mathfrak{i}_2 I_0$. (Replace $\mathfrak{i}$ by $\mathfrak{i}_1^{-1}$ and $\mathfrak{a}(x_0)$ by $\mathfrak{i}_2$ in the diagram above.) We define

$$\mathfrak{a}(x) := \mathfrak{p}(x_0, x) \circ \mathfrak{i}_2 \circ \mathfrak{i}_1.$$

This $\mathfrak{a}(x)$ takes the indicatrix $I_0(x) = I_0$ of $\mathcal{M}^n$ into the indicatrix $\bar{I}(x)$ of $\mathcal{B}^n$, therefore $\mathcal{B}^n = \mathfrak{A}\mathcal{M}^n$. □

Theorem 2. *The curvature R^* of Γ^* vanishes.*

This is so, for the parallel translation $\mathfrak{p}$ arisen from the metric linear connection Γ^* is independent of the curve joining x_0 and $\bar{x}$.

3 Properties of the generalized Berwald spaces

M. Matsumoto and H. Shimada introduced and investigated Finsler spaces with 1-form metric [MS]. These are Finsler spaces $F^n = (\mathbb{R}^n(x), \mathcal{F}(x, y))$, such that there exists a scalar function $G : \mathbb{R}^n \to \mathbb{R}$, such that

$$\mathcal{F}(x^i, y^i) = G(a^i_k(x) y^k). \tag{5}$$

We show

Theorem 3. *Finsler spaces F^n with 1-form metric are generalized Berwald spaces $\mathcal{B}^n$, and conversely.*

Proof. I/ *Any Finsler space $F^n = (\mathbb{R}^n(x), \mathcal{F}(x, y))$ with 1-form metric is a generalized Berwald space $\mathcal{B}^n$.*

Let $F^n = (\mathbb{R}^n(x), \mathcal{F}(x, y))$ be a Finsler space with 1-form metric. Then we have a function $G : \mathbb{R}^n \to \mathbb{R}$ and a bundle $\mathfrak{A} = \{\mathfrak{a}(x)\}$ with coefficients $a^i_k(x)$, which satisfy (5). Let us deform F^n by $\mathfrak{B} = \{\mathfrak{b}(x)\}$, where $\mathfrak{b} = \mathfrak{a}^{-1}$. Then

$$\mathfrak{B}F^n = (\mathbb{R}^n(x), G(b^j_i(x) a^i_k(x) y^k)) = (\mathbb{R}^n(x), G(y)).$$

The metric function of $\mathfrak{B}F^n$ is $G(y)$. This does not depend of x. Therefore $\mathfrak{B}F^n$ is a Minkowski space $\mathcal{M}^n$. From this $F^n = \mathfrak{A}\mathcal{M}^n$, which, by Theorem 1 is a $\mathcal{B}^n$.

II./ Conversely, *any $\mathcal{B}^n = \mathfrak{A}\mathcal{M}^n$ is a Finsler space $F^n = (\mathbb{R}^n(x), \mathcal{F}(x, y))$ with 1-form metric.*

Let $\mathcal{M}^n = (\mathbb{R}^n(x), G(y)) = (\mathbb{R}^n(x), I_0)$. Then

$$\mathcal{B}^n = \mathfrak{A}\mathcal{M}^n = (\mathbb{R}^n(x), G(\mathfrak{a}\, y)) = (\mathbb{R}^n(x), G(a^i_k(x) y^k)).$$

But $\mathcal{B}^n$ is a Finsler space $F^n = (\mathbb{R}^n(x), \mathcal{F}(x, y))$. Then the metric function of $F^n = \mathcal{B}^n$ has the form $\mathcal{F}(x^i, y^i) = G(a^i_k(x) y^k)$, that is $\mathcal{B}^n$ is a Finsler space with 1-form metric. □

Let us consider a Riemannian space V^n, whose indicatrices are the ellipsoids $Q(x)$. Any ellipsoid $Q(x)$ is the affine deformation of the Euclidean sphere: $Q(x) = \mathfrak{a}(x)S$, as we have seen it in (3). Given a $Q(x)$ and an S, $\mathfrak{a}(x)$ is unique in (3) up to a rotation of S, which can be fixed by requiring that the coordinate axes of E^n are mapped by $\mathfrak{a}(x)$ into the axes of the ellipsoids. Conversely, given an $\mathfrak{a}(x)$, the image $Q(x)$ is unique. In this sense the relation between $\mathfrak{a}(x)$ and $Q(x)$, and between $\mathfrak{A}$ and $V^n = (\mathbb{R}^n(x), Q(x))$ is $1:1$. Then $(V^n, \mathcal{M}^n)$ determines $\mathcal{B}^n = \mathfrak{A}\mathcal{M}^n$, and conversely. This is a representation of $\mathcal{B}^n$. These yield

Theorem 4. *Any generalized Berwald space $\mathcal{B}^n$ is determined by a pair $(V^n, \mathcal{M}^n)$, and conversely.*

Proposition 1. *The generalized Berwald spaces $(V_0^n, \mathcal{M}^n)$ with fix V_0^n and arbitrary $\mathcal{M}^n$ have the same metric linear connection Γ^*.*

This is so, for by (4) Γ^* is determined by $\mathfrak{a}(x)$, that is by V_0^n, and it is independent of $\mathcal{M}^n$.

We present several further properties of generalized Berwald spaces. Proofs sometimes will only be sketchy.

Theorem 5. A/ $\mathcal{B}^n = (V^n, \mathcal{M}^n) = \mathfrak{A}\mathcal{M}^n$ *is the* $\mathcal{M}^n$ *appearing in the representation if and only if for every* $i_0 \in \{1, \dots, n\}$ *the 1-form* $(a_k^{i_0}(x))$ *is closed. In this case* $V^n = E^n$.

B/ $\mathcal{B}^n = (V^n, \mathcal{M}^n)$ *is the Riemannian space* V^n *appearing in the representation if and only if* $\mathcal{M}^n = E^n$.

C/ $\mathcal{B}^n = E^n$ *if and only if* $\mathcal{M}^n = E^n$*, and, if the* $(a_k^{i_0}(x))$*'s are closed 1-forms.*

Proof. A/ If the $(a_k^{i_0}(x))$'s are closed, then $\mathfrak{a}(x)$ is a coordinate transformation: (a) $(x^i) \to (\bar{x}^i)$. Then $I_0 \to \mathfrak{a}(x)I_0 = \bar{I}(x)$ is the result of (a), that is $\mathcal{M}^n = (\mathbb{R}^n(x), I_0)$ and $(\mathbb{R}^n(x), \bar{I}(x)) = \mathcal{B}^n$ have the same structures, but in different coordinate systems. Furthermore if the $(a_k^{i_0}(x))$'s are closed 1-forms, then, because of $Q(x) = \mathfrak{a}(x)S$, V^n has vanishing curvature, that is $V^n = E^n$.

The converse can easily be seen.

B/ We know that

$$\mathcal{B}^n = (V^n, \mathcal{M}^n) = \mathfrak{A}\mathcal{M}^n = (\mathbb{R}^n(x), \bar{I}(x)), \quad V^n = (\mathbb{R}^n(x), Q(x)),$$

$$Q(x) \overset{(b)}{=} \mathfrak{a}(x)S \quad \text{and} \quad \mathfrak{a}(x)I_0 \overset{(c)}{=} \bar{I}(x).$$

In our case $Q(x) = \bar{I}(x)$, and thus we have $I_0 = S$ by (b) and (c). This means that $\mathcal{M}^n = \mathcal{B}^n$. The converse follows similarly.

One can see that $(V^n, \mathcal{M}^n) = V_1^n$ with a $V_1^n \neq V^n$ is not possible. These mean that if a $\mathcal{B}^n$ is a Riemannian space, then it must be the Riemannian space appearing in the representation $(V^n, \mathcal{M}^n)$, and in this case $\mathcal{M}^n = E^n$.

C/ We have seen that $\mathcal{M}^n = E^n$ if and only if $\mathcal{B}^n = V^n$ with $Q(x) = \mathfrak{a}(x)S$. Then $V^n = (\mathbb{R}^n(x), Q(x)) = \mathfrak{A}E^n$. If $a_k^{i_0}(x)$ are closed 1-forms, then $\mathfrak{A}E^n = E^n$, and $V^n = E^n$. $\square$

Theorem 6. A/ *Between Minkowski spaces there exists no proper conformal relation.*

B/ $\mathcal{B}_1^n$ *is conformal to* $\mathcal{B}_2^n$ *if and only if* $\mathcal{M}_1^n = \mathcal{M}_2^n$ *and* V_1^n *is conformal to* V_2^n.

C/ $\mathcal{B}^n$ *is conformally flat if and only if* $\mathcal{M}^n = E^n$ *and* V^n *is conformally flat.*

D/ *Any conformally flat Finsler space* F^n *is a Riemannian space* V^n.

Proof. A/ Let (x) be an adapted coordinate system for $\mathcal{M}_1^n = (\mathbb{R}^n(x), I_1)$ and $\mathcal{M}_2^n = (\mathbb{R}^n(x), I_2)$, where I_1 and I_2 are independent of x. If they are in conformal relation, then $I_2 = \sigma(x)I_1$. However I_1 and I_2 are independent of x. Consequently σ must also be a constant c. If we denote $\bar{x}^i = cx^i$, then $I_1(x) = I_2(\bar{x})$, and $\mathcal{M}_1^n = \mathcal{M}_2^n$.

B/ If $\mathcal{B}_1^n = (\mathbb{R}^n(x), \bar{I}_1(x)) = \mathfrak{A}_1\mathcal{M}_1^n = \mathfrak{A}_1(\mathbb{R}^n(x), I_1)$ is conformal to $\mathcal{B}_2^n = (\mathbb{R}^n(x), \bar{I}_2(x)) = \mathfrak{A}_2\mathcal{M}_2^n = \mathfrak{A}_2(\mathbb{R}^n(x), I_2)$, then $\bar{I}_1(x) \overset{(d)}{=} \sigma(x)\bar{I}_2(x)$, and $I_1(x) \overset{(e)}{=} \tilde{\mathfrak{a}}(x)I_2$, where I_1 and I_2 are the indicatrices of $\mathcal{M}_1^n$ and $\mathcal{M}_2^n$. From (d) we obtain $\mathcal{M}_1^n = \mathcal{M}_2^n (= \mathcal{M}_0^n)$, and $\mathfrak{a}_1 = \mathfrak{a}_2 (= \mathfrak{a}_0)$. These, with (e) yield a conformal relation between V_1^n and V_2^n.

C/ This is a special case of B/, where $\mathcal{B}_2^n = E^n$.

D/ If $F^n = (\mathbb{R}^n, I(x))$ is conformally flat, then $I(x) = \sigma(x)S$. But $\sigma(x)S$ is an ellipsoid $Q(x)$. Thus $F^n = (\mathbb{R}^n(x), Q(x)) = V^n$. □

A characterization of the generalized Berwald spaces is given by

Theorem 7. *A Finsler space* $F^n = (\mathbb{R}^n(x), I(x))$ *is a* $\mathcal{B}^n$ *if and only if there exists an indicatrix* $I(x_0)$*, which is in affine relation with any other* $I(x)$ *(i.e. if any* $I(x)$ *is the affine deformation of* $I(x_0)$*).*

Namely in this case we have a linear transformation $\mathfrak{p}(x_1, x_2)$, which takes $I(x_1)$ into $I(x_2)$, and this induces a metric linear connection Γ^*.

From these it is easy to see that a $\mathcal{B}^n$ is a V^n if and only if one of its indicatrices is an ellipsoid.

Theorem 8. *A* $\mathcal{B}^n = (V^n, \mathcal{M}^n)$ *is projectively flat if and only if* V^n *is projectively flat.*

The proof relies on the observation that a curve is a geodesic iff the osculation points of the geodesic spheres centered at the curve lie on the curve.

4 Decomposition of M and motions

Definition. Given a Finsler space $F^n = (\mathbb{R}^n, I)$, a regular linear transformation $\mathfrak{k}(x)$ of the tangent space $T_x\mathbb{R}^n$ is an *affine automorphism* of $I(x)$ if it takes the indicatrix $I(x)$ as a whole into itself:

$$\mathfrak{k}(x)I(x) = I(x).$$

The set $\{\mathfrak{k}(x)\}$ of all $\mathfrak{k}(x)$ at a point x form an Abelian group $\mathfrak{K}(x)$. In case of a $\mathcal{B}^n$ $\mathfrak{K}$ is independent of x. $I(x)$ is called *rigid*, if $\mathfrak{K}(x)$ consists of the identity only: $|\mathfrak{K}(x)| = 1$, and it is called *mobile* if $|\mathfrak{K}(x)| > 1$.

P. Gruber [G] proved (see also A. C. Thompson [Th] p. 83) that in case of an F^2, $I(x_0)$ is an ellipse iff $|\mathfrak{K}(x_0)| = \infty$. If $|\mathfrak{K}(x)| = \infty$ for every x, then $F^2 = V^2$.

In case of a $\mathcal{B}^2 = (\mathbb{R}^n, I)$ from $|\mathfrak{K}(x_0)| = \infty$ it follows that $\mathcal{B}^2 = V^2$. Also if $\mathbb{R}^n$ is simply connected, and the holonomy group of $\mathcal{B}^2$ is not the identity, then $\mathcal{B}^2 = V^2$.

In a Finsler space $F^n = (\mathbb{R}^n, I)$ two points x_1 and x_2 are in *affine relation* if the indicatrices $I(x_1)$ and $I(x_2)$ are affine deformations of each other:

$$x_1 \sim x_2 \Leftrightarrow I(x_2) = \mathfrak{a}(x_1, x_2) I(x_1),$$

where $\mathfrak{a}(x_1, x_2)$ is a regular linear map taking $T_{x_1}\mathbb{R}^n$ into $T_{x_2}\mathbb{R}^n$. This equivalence relation decomposes $\mathbb{R}^n$ into equivalence classes M_α, $\alpha \in \mathcal{A}$. One can prove that each M_α is a closed set with respect to the metric arising from the Finsler function. F^n restricted to an M_α is a $\mathcal{B}^n \restriction \mathcal{M}_\alpha$. Conformal or isometric mappings $F^n = (\mathbb{R}^n, I) \to \bar{F}^n = (\mathbb{R}^n, \bar{I})$ keep the equivalence structure.

In case of a motion the orbits are within an M_α.

Theorem 9. *If $F^n = (\mathbb{R}^n, I)$ admits a transitive continuous group of motions, then F^n is a $\mathcal{B}^n$.*

Proof. In this case $\mathbb{R}^n$ constitutes a single equivalence class, therefore $F^n = (\mathbb{R}^n, I)$ is the same as $\mathcal{B}^n = (\mathbb{R}^n, I)$. Our theorem follows also from the fact that the tangent linear map of a transitive motion takes $I(x_0)$ into any other $I(x)$. Then the indicatrices of F^n are in affine relation with each other, and then, by Theorem 7, $F^n = \mathcal{B}^n$. □

We state that any equivalence class consisting of a single point must be a fixed point of any motion of F^n. Finally, we can prove rather easily that if a 2-dimensional Finsler space F^2 with reversible metric (i.e., with symmetric indicatrices) admits a continuous group of motions, and in the decomposition of $\mathbb{R}^n$ there are two equivalence classes M_1 and M_2 consisting of a single point x_1 and x_2, resp., and the injectivity radii ι satisfy $\iota(x_1) + \iota(x_2) > \varrho(x_1, x_2)$ (ϱ denotes the Finslerian distance), then F^2 is diffeomorphic to S^2.

References

[G] Gruber, P. M., *Minimal ellipsoids and their duals*, Rend. Circ. Math. Palermo **37** (1988), 35–64.

[I] Ichijyō, Y., *Finsler manifolds modeled on Minkowski spaces*, J. Math. Kyoto Univ. **16** (1976), 639–652.

[MS] Matsumoto, M. and Shimada, H., *On Finsler spaces with 1-form metric*, Tensor N. S. **32** (1978), 161–179.

[SzLK] Szilasi, J., Lovas, R. L. and Kertész, D. Cs., *Several ways to a Berwald manifold and some steps beyond*, to appear in Extracta Mathematicae.

[Ta] Tamássy, L., *Point Finsler spaces with metric linear connections*, Publ. Math. Debrecen **56** (2000), 643–655.

[Th] Thompson, A. C., *Minkowski geometry*, Cambridge Univ. Press, Cambridge 1996.