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ON $\mathbb{R}-$COMPLEX FINSLER SPACES WITH KROPINA METRIC

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Abstract

In the present paper the notion of $\mathbb{R}-$complex Finsler space with Kropina metric is defined . The fundamental tensor fields g_{ij} and $g_{i\bar{j}}$ are determined and the determinant and the inverse of these tensor fields are given. Also some properties of these spaces are studied. A special aproach is dedicated to the non-Hermitian $\mathbb{R}-$complex Finsler space with Kropina metric. Some examples of Hermitian and non-Hermitian $\mathbb{R}-$complex Finsler space with Kropina metric are given.

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1 $\mathbb{R}$ - complex Finsler spaces

In a previous paper [14], we extended the known definition of a complex Finsler space ([1, 2, 13, 16]), reducing the scalars to $\lambda \in \mathbb{R}$. The outcome was a new class of Finsler space called by us the $\mathbb{R}$- complex Finsler spaces [14]. Our interest in this class of Finsler spaces issues from the fact that the Finsler geometry means, first of all, distance and this refers to curves depending on the real parameter.

In the present paper, following the ideas from real Finsler spaces with Kropina metrics ([6, 16, 10, 11, 12]), we introduce the similar notions on $\mathbb{R}-$ complex Finsler spaces.

In this section we keep the general setting from [13, 14] and subsequently we recall only some needed notions.

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n, \left(z^k\right)$ be local complex coordinates in a chart (U,φ) and $T'M$ its holomorphic tangent bundle. It has a natural structure of complex manifold, $\dim_{\mathbb{C}} T'M = 2n$ and the induced coordinates in a local chart on $u \in T'M$ are denoted by $u = \left(z^k, \eta^k\right)$. The changes of local coordinates in u are given by the rules

$$z'^k = z'^k\left(z\right)\ ;\ \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{1.1}$$

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The natural frame $\left\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\right\}$ of $T'_u(T'M)$ transforms with the Jacobi matrix of (1.1) changes, $\frac{\partial}{\partial z^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial z'^j} + \frac{\partial^2 z'^j}{\partial z^k \partial z^h}\eta^h \frac{\partial}{\partial \eta'^j}$; $\frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial \eta'^j}$.

A complex nonlinear connection, briefly (c.n.c.), is a supplementary distribution $H(T'M)$ to the vertical distribution $V(T'M)$ in $T'(T'M)$. The vertical distribution is spanned by $\left\{\frac{\partial}{\partial \eta^k}\right\}$ and an adapted frame in $H(T'M)$ is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial n^j}$, where N^j_k are the coefficients of the (c.n.c.) and they have a certain rule of change at (1.1) so that $\dfrac{\delta}{\delta z^k}$ transform like vectors on the base manifold M(d-tensor in [14] terminology). Next we use the abbreviations: $\partial_k = \dfrac{\partial}{\partial z^k}, \delta_k = \dfrac{\delta}{\delta z^k}, \dot{\partial}_k = \dfrac{\partial}{\partial \eta^k}$ and $\partial_{\bar{k}}, \dot{\partial}k, \delta_{\bar{k}}$ for their conjugates. The dual adapted basis of $\left\{\delta_k, \dot{\partial}_k\right\}$ are $\left\{dz^k, \delta\eta^k = d\eta^k + N^k_j dz^j\right\}$ and $\left\{d\bar{z}^k, \delta\bar{\eta}^k\right\}$ theirs conjugates.

We recall that the homogeneity of the metric function of a complex Finsler space ([1, 2, 13, 16]) is with respect to all complex scalars and the metric tensor of the space, is a Hermititian one. In [14] we slightly changed the definition of a complex Finsler space.

An $\mathbb{R}-$ *complex Finsler metric* on M is a continuous function $F : T'M \longrightarrow \mathbb{R}_+$ satisfying:

i) $L := F^2$ is smooth on $\widetilde{T'M}$ (except the 0 sections);

ii) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = |\lambda| F(z, \eta, \bar{z}, \bar{\eta}), \forall \lambda \in \mathbb{R}$;

It follows that L is $(2,0)$ homogeneous with respect to the real scalars λ, and in [14] we proved that the following identities are fulfilled:

$$\frac{\partial L}{\partial \eta^i}\eta^i + \frac{\partial L}{\partial \bar{\eta}^i}\bar{\eta}^i = 2L; \quad g_{ij}\eta^i + g_{\bar{j}i}\bar{\eta}^i = \frac{\partial L}{\partial \eta^j}; \tag{1.2}$$
$$\frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0; \quad \frac{\partial g_{i\bar{k}}}{\partial \eta^j}\eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0.$$
$$2L = g_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j;$$

where:

$$g_{ij} := \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}\ ; \quad g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}; \quad g_{\bar{i}\bar{j}} := \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j} \tag{1.3}$$

are the metric tensors of the space.

2 $\mathbb{R}-$complex Finsler spaces with Kropina metric

As noticed in paper [14] an $\mathbb{R}-$ complex Finsler space produce two tensor fields g_{ij} and $g_{i\bar{j}}$. For a properly Hermitian geometry $g_{i\bar{j}}$ be invertible is a compulsory requirement, but from some physicist point of view, for which Hermitian condition is an impediment, it seems more appropriate that g_{ij} be an invertible metric tensor. These problems led us to in [14] to speak about Hermitian $\mathbb{R}-$ complex Finsler spaces (i.e. $\det\left(g_{i\bar{j}}\right) \neq 0$) and non - Hermitian $\mathbb{R}-$ complex Finsler spaces (i.e. $\det(g_{ij}) \neq 0$). The present section applies our results to ℝ− complex Finsler spaces with Kropina metric, better illustrating the interest for this work. As in [12] we have:

Definition 2.1. *An ℝ−complex Finsler space (M, F) is called with (α, β) −metric if the fundamental function $F(z, \eta, \bar{z}, \bar{\eta})$ is ℝ− homogeneous by means of functions $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$− depend by $z^i, \eta^i, \bar{z}^i$ and $\bar{\eta}^i, (i = 1, ..., n)$ by means of $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$, i.e. :*

$$F(z, \eta, \bar{z}, \bar{\eta}) = F(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})) \tag{2.1}$$

where

$$\begin{aligned} \alpha^2(z, \eta, \bar{z}, \bar{\eta}) &= \frac{1}{2}\left(a_{ij}\eta^i\eta^j + a_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2a_{i\bar{j}}\eta^i\bar{\eta}^j\right) = Re\left\{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\right\}, \\ \beta(z, \eta, \bar{z}, \bar{\eta}) &= \frac{1}{2}\left(b_i\eta^i + b_{\bar{i}}\bar{\eta}^i\right) = Re\left\{b_i\eta^i\right\}, \end{aligned} \tag{2.2}$$

with:

$$a_{ij} = a_{ij}(z), a_{i\bar{j}} = a_{i\bar{j}}(z), b_i = b_i(z), \tag{2.3}$$

$b_i(z)dz^i$ is a 1− form on the complex manifold M.

We denote:

$$L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})) = F^2(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})). \tag{2.4}$$

Remark 2.1 F^2 is a $\mathbb{R} - (\alpha, \beta)$ complex Finsler metric.

Definition 2.2. *An ℝ−complex Finsler space with (α, β)-metric is called an ℝ−complex Kropina space or a ℝ−complex Finsler space with Kropina metric if:*

$$L(\alpha, \beta) = \left(\frac{\alpha^2}{\beta}\right)^2, \beta \neq 0. \tag{2.4$'$}$$

It follows that $F(\alpha, \beta) = \dfrac{\alpha^2}{\beta}, \beta \neq 0$.

Taking into account the 2−homogeneity condition of L :

$$L(\alpha(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}), \beta(z, \lambda\eta, \bar{z}, \lambda\bar{\eta})) = \lambda^2 L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \lambda \in R_+, \tag{2.5}$$

we have:

Proposition 2.1. *([5]) In an ℝ−complex Finsler space with (α, β)-metric the following equalities hold:*

$$\begin{aligned} &\alpha L_\alpha + \beta L_\beta = 2L,\ \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \\ &\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta,\ \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \end{aligned} \tag{2.6}$$

where:

$$L_\alpha := \frac{\partial L}{\partial \alpha},\ L_\beta = \frac{\partial L}{\partial \beta},\ L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta},\ L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2},\ L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}. \tag{2.6$'$}$$

Particular case: For a $\mathbb{R}-$complex Finsler space with Kropina metric, we have:

$$\begin{aligned} L_\alpha &= \frac{4\alpha^3}{\beta^2},\ L_\beta = -\frac{2\alpha^4}{\beta^3},\ \alpha L_\alpha + \beta L_\beta = 2L \\ L_{\alpha\alpha} &= \frac{12\alpha^2}{\beta^2},\ L_{\alpha\beta} = -\frac{8\alpha^3}{\beta^3},\ L_{\beta\beta} = \frac{6\alpha^4}{\beta^4}. \end{aligned} \tag{2.6$''$}$$

In the following, we propose to determine the metric tensors of an $\mathbb{R}-$ complex Finsler space with Kropina metric, i.e. $g_{ij} := \partial^2 L(z,\eta,\bar{z},\lambda\bar{\eta})/\ \partial\eta^i\partial\eta^j$;
$g_{i\bar{j}} := \partial^2 L(z,\eta,\bar{z},\lambda\bar{\eta})/\ \partial\eta^i\partial\bar{\eta}^j$, each of these being of interest in the following.

We consider:

$$\begin{aligned} \frac{\partial\alpha}{\partial\eta^i} &= \frac{1}{2\alpha}\left(a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j\right) = \frac{1}{2\alpha}l_i,\ \frac{\partial\beta}{\partial\eta^i} = \frac{1}{2}b_i, \\ \frac{\partial\alpha}{\partial\bar{\eta}^i} &= \frac{1}{2\alpha}\left(a_{\bar{i}\bar{j}}\bar{\eta}^j + a_{i\bar{j}}\eta^j\right) = \frac{1}{2\alpha}l_{\bar{i}},\ \frac{\partial\beta}{\partial\bar{\eta}^i} = \frac{1}{2}b_{\bar{i}}, \end{aligned} \tag{2.7}$$

where:

$$l_i := a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j,\ l_{\bar{j}} := a_{\bar{i}\bar{j}}\bar{\eta}^i + a_{i\bar{j}}\eta^i. \tag{2.7$'$}$$

We find immediately:

$$l_i\eta^i + l_{\bar{j}}\bar{\eta}^j = 2\alpha^2. \tag{2.8}$$

We denote:

$$\eta_i = \frac{\partial L}{\partial\eta^i}. \tag{2.9}$$

Consequently, we obtain:

$$\eta_i = \rho_0 l_i + \rho_1 b_i, \tag{2.10}$$

where:

$$\rho_0 = \frac{1}{2}\alpha^{-1}L_\alpha\ \left(0-\text{homogeneity}\right),\ \rho_1 = \frac{1}{2}L_\beta\ \left(1-\text{homogeneity}\right), \tag{2.10$'$}$$

Differentiating $(2.10)'$ by η^j and $\bar{\eta}^j$ respectively we obtain:

$$\begin{aligned} \frac{\partial\rho_0}{\partial\eta^j} &= \rho_{-2}l_j + \rho_{-1}b_j,\ \frac{\partial\rho_0}{\partial\bar{\eta}^j} = \rho_{-2}l_{\bar{j}} + \rho_{-1}b_{\bar{j}}, \\ \frac{\partial\rho_1}{\partial\eta^i} &= \rho_{-1}l_j + \mu_0 b_i,\ \frac{\partial\rho_1}{\partial\bar{\eta}^i} = \rho_{-1}l_{\bar{i}} + \mu_0 b_{\bar{i}}, \end{aligned} \tag{2.11}$$

where:

$$\rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3},\ \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha},\ \ \mu_0 = \frac{L_{\beta\beta}}{4}. \tag{2.11$'$}$$

Proposition 2.2. *The invarians of an $\mathbb{R}-$complex Finsler space with Kropina metric: $\rho_0, \rho_1, \rho_{-2}, \rho_{-1}$ are given by:*

$$\begin{aligned} &\rho_0 := \tfrac{2F}{\beta};\ \ \rho_1 := -\tfrac{F^2}{\beta},\quad \beta \neq o; \\ &\rho_{-2} := \tfrac{2}{\beta^2};\ \ \rho_{-1} := \tfrac{-2F}{\beta^2};\ \ \mu_0 := \tfrac{3F^2}{2\beta^2};\ \ \beta \neq o. \end{aligned} \tag{2.11$''$}$$

Subscripts $-2,-1,0,1$ give us the degree of homogeneity of these invariants.
We have immediately:

Proposition 2.3. *The fundamental tensor fields of an* $\mathbb{R}-$*complex Finsler space with Kropina metric are given by:*

$$g_{ij}=\frac{2F}{\beta}a_{ij}+\frac{2}{\beta^2}l_il_j+\frac{3F^2}{2\beta^2}b_ib_j+\frac{-2F}{\beta^2}\left(b_jl_i+b_il_j\right). \tag{2.12}$$

$$g_{i\bar{j}}=\frac{2F}{\beta}a_{i\bar{j}}+\frac{2}{\beta^2}l_il_{\bar{j}}+\frac{3F^2}{2\beta^2}b_ib_{\bar{j}}+\frac{-2F}{\beta^2}\left(b_{\bar{j}}l_i+b_il_{\bar{j}}\right). \tag{2.13}$$

or, in the equivalent form:

$$g_{ij}=\frac{2F}{\beta}a_{ij}-\frac{2}{\beta^2}l_il_j+\frac{F^2}{2\beta^2}b_ib_j+\frac{1}{F^2}\eta_i\eta_j, \tag{2.12'}$$

$$g_{i\bar{j}}=\frac{2F}{\beta}a_{i\bar{j}}-\frac{2}{\beta^2}l_il_{\bar{j}}+\frac{F^2}{2\beta^2}b_ib_{\bar{j}}+\frac{1}{F^2}\eta_i\eta_{\bar{j}}, \tag{2.13'}$$

Proof. Using the relations (2.11") in Theorem 2.1 [5] by direct calculus we have the results □

The next objective is to obtain the determinant and the inverse of the tensor field g_{ij}. The solution is obtained by adapting Proposition 11.2.1, p. 287 from [6] and Proposition 2.2 from [4] for an arbitrary non-singular non-Hermitian matrix (Q_{ij}). The result is:

Proposition 2.4. *Suppose:*

- (Q_{ij}) *is a non-singular* $n\times n$ *complex matrix with inverse* (Q^{ji});
- C_i *and* $C_{\bar{\imath}}:=\overline{C_i}$, $i=1,..,n$, *are complex numbers;*
- $C^i:=Q^{ji}C_j$ *and its conjugates;* $C^2:=C^iC_i=\bar{C}^iC_{\bar{\imath}}$; $H_{ij}:=Q_{ij}\pm C_iC_j$

Then

i) $\det(H_{ij})=(1\pm C^2)\det(Q_{ij})$

ii) Whenever $1\pm C^2\neq 0$, *the matrix* (H_{ij}) *is invertible and in this case its inverse is* $H^{ji}=Q^{ji}\mp\frac{1}{1\pm C^2}C^iC^j$.

Theorem 2.1. *For a non-Hermitian* $\mathbb{R}-$*complex Finsler space with Kropina metric,* $\left(L(\alpha,\beta)=\left(\frac{\alpha^2}{\beta}\right)^2,\ \beta\neq 0,\right.$ *with* $\left.a_{i\bar{j}}=0\right)$, *we have:*

i) the contravariant tensor g^{ij} *of the fundamental tensor field* g_{ij} *is:*

$$\begin{aligned}g^{jl}=&\frac{\beta}{2F}a^{jl}+\frac{2\beta(F^3\omega-\beta^3)}{M}\eta^j\eta^l+\frac{F^2\beta(4F\gamma-3\beta^3)}{2M}b^jb^l+\\&+\frac{2\alpha^2(\beta^3+F^2\varepsilon)}{M}b^j\eta^l+\frac{2F\beta(\beta^3-F^2\delta)}{M}\eta^jb^l,\end{aligned} \tag{2.14}$$

where:

$$M = 4F^4(\varepsilon\delta - \gamma\omega) + 4\alpha^2\beta(\beta\gamma - \alpha^2\varepsilon - \alpha^2\delta) + 3\alpha^6\omega + \beta^6,$$

$$l_i = a_{ij}\eta^j,\ \eta_i = \frac{2F}{\beta}a_{ij}\eta^j - \frac{F}{\beta^2}b_i, \tag{2.15}$$

$$\gamma = a_{jk}\eta^j\eta^k = l_k\eta^k,\ \varepsilon = b_j\eta^j,\ \omega = b_jb^j,\ b^k = a^{jk}b_j,\ b_l = b^k a_{kl},$$
$$\delta = a_{jk}n^jb^k = l_kb^k,\ l^j = a^{ji}l_i = n^j.$$

ii) $det(g_{ij}) = (2q^2)^n B\ \dfrac{4+q^2A}{4}\ \dfrac{\alpha^2-\gamma}{\alpha^2}\ det(a_{ij})$, *where:*

$$A = \frac{\omega\alpha^2 - \omega\gamma + \varepsilon^2}{\alpha^2-\gamma}$$

$$B = 2 - \frac{2\beta\varepsilon - \omega}{2\alpha^2\beta^2} + \frac{\alpha^2}{2(\alpha^2-\gamma)} - \frac{\alpha^2q^2\varepsilon^2}{2(\alpha^2-\gamma)^2(4+q^2A)} - \frac{q^2(2\beta\varepsilon-\omega)^2}{2\alpha^2\beta^2(4+q^2A)} - \frac{\varepsilon q^2(2\beta\varepsilon-\omega)}{\beta(\alpha^2-\gamma)(4+q^2A)}$$

Proof. To prove the claims we apply the above Proposition in a recursive algorithm in three steps. We write g_{ij} from $2.12'$ in the form:

$$g_{ij} = 2q^2(a_{ij} - \frac{1}{\alpha^2}l_il_j + \frac{q^2}{4}b_ib_j + \frac{1}{2q^4\alpha^2}\eta_i\eta_j)$$

I. In the first step, we set $Q_{ij} = a_{ij}$ and $c_i = \frac{1}{2}l_i$. By applying the Proposition 2.4 we obtain $Q^{ji} = a^{ji}, c^2 = \frac{\gamma}{\alpha}^2$, $1 - c^2 = \frac{\alpha^2-\gamma}{\alpha^2} \neq 0$ and $c^i = \frac{\eta^i}{\alpha}$. So, the matrix $H_{ij} = a_{ij} - \frac{1}{\alpha^2}l_il_j$ is invertible with $H^{ji} = a^{ji} + \frac{1}{\alpha^2-\gamma}\eta^i\eta^j$ and $det(a_{ij} - \frac{1}{\alpha^2}l_il_j) = \frac{\alpha^2-\gamma}{\alpha^2}det(a_{ij})$. II. Now, we consider $Q_{ij} = a_{ij} - \frac{1}{\alpha^2}l_il_j$ and $c_i = \frac{q}{2}b_i$. By applying the Proposition 2.4 we obtain this time: $Q^{ji} = a^{ji} + \frac{1}{\alpha^2-\gamma}\eta^i\eta^j$, $c^2 = \frac{q^2}{4}(\omega + \frac{1}{\alpha^2-\gamma}\varepsilon^2)$, $1 + c^2 = 1 + \frac{q^2(\alpha^2\omega - \gamma\omega + \varepsilon^2)}{4(\alpha^2-\gamma)} = \frac{4+q^2A}{4} \neq 0$, where $A = \frac{\alpha^2\omega - \gamma\omega + \varepsilon^2}{\alpha^2-\gamma}$ and $c^i = \frac{q}{2}(b^i + \frac{\varepsilon}{\alpha^2-\gamma}\eta^i)$.

It results that the inverse of $H_{ij} = a_{ij} - \frac{1}{\alpha^2}l_il_j + \frac{q^2}{4}b_ib_j$ exists and it is $H^{ji} = a^{ji} + \frac{1}{\alpha^2-\gamma}(1 - \frac{q^2\varepsilon^2}{(4+q^2A)(\alpha^2-\gamma)})\eta^i\eta^j - \frac{q^2}{4+q^2A}b^ib^j - \frac{q^2\varepsilon}{(4+q^2A)(\alpha^2-\gamma)}(\eta^ib^j + b^i\eta^j)$ and $det(a_{ij} - \frac{1}{\alpha^2}l_il_j + \frac{q^2}{4}b_ib_j) = \frac{4+q^2A}{4}\ \frac{\alpha^2-\gamma}{\alpha^2}det(a_{ij})$.

III. Finally, we put $Q_{ij} = a_{ij} - \frac{1}{\alpha^2}l_il_j + \frac{q^2}{4}b_ib_j$ and $c_i = \frac{1}{\sqrt{2}\alpha q^2}\eta_i$.

From here, we obtain:

$$Q^{ji} = a^{ji} + \frac{1}{\alpha^2-\gamma}(1 - \frac{q^2\varepsilon^2}{(4+q^2A)(\alpha^2-\gamma)})\eta^i\eta^j - \frac{q^2}{4+q^2A}b^ib^j - \\ - \frac{q^2\varepsilon}{(4+q^2A)(\alpha^2-\gamma)}(\eta^ib^j + b^i\eta^j),$$

$$\begin{aligned} c^2 &= \Big[a^{ji} + \frac{1}{\alpha^2-\gamma}(1 - \frac{q^2\varepsilon^2}{(4+q^2A)(\alpha^2-\gamma)})\eta^i\eta^j - \frac{q^2}{4+q^2A}b^ib^j - \\ &\quad - \frac{q^2\varepsilon}{(4+q^2A)(\alpha^2-\gamma)}(\eta^ib^j + b^i\eta^j)\Big]\frac{1}{2\alpha^2q^4}\eta_i\eta_j \\ &= 1 - \frac{2\beta\varepsilon-\omega}{2\alpha^2\beta^2} + \frac{\alpha^2}{2(\alpha^2-\gamma)} - \frac{\alpha^2q^2\varepsilon^2}{2(\alpha^2-\gamma)(4+q^2A)} \\ &\quad - \frac{q^2}{(4+q^2A)2\alpha^2}(2\varepsilon - \frac{\omega}{\beta})^2 - \frac{\varepsilon q^2}{(4+q^2A)(\alpha^2-\gamma)}(2\varepsilon - \frac{\omega}{\beta}), \\ 1 + c^2 &= 2 - \frac{2\beta\varepsilon-\omega}{2\alpha^2\beta^2} + \frac{\alpha^2}{2(\alpha^2-\gamma)} - \frac{\alpha^2q^2\varepsilon^2}{2(\alpha^2-\gamma)(4+q^2A)} - \frac{q^2(2\beta\varepsilon-\omega)^2}{2\alpha^2\beta^2(4+q^2A)} \\ &\quad - \frac{\varepsilon q^2(2\beta\varepsilon-\omega)}{\beta(\alpha^2-\gamma)(4+q^2A)} \neq 0 \end{aligned}$$

and

$$c^i = \frac{1}{\sqrt{2}\alpha q^2}(M\eta^i + Nb^i),$$

where $M = \frac{\sqrt{2}}{\alpha} + + \frac{\alpha}{\sqrt{2}(\alpha^2-\gamma)}\ (1 - \frac{q^2\varepsilon^2}{(4+q^2A)(\alpha^2-\gamma)}) - \frac{q^4\varepsilon(2\beta\varepsilon-\omega)}{\sqrt{2}\beta(4+q^2A)(\alpha^2-\gamma)}$ and $N = -\frac{1}{\sqrt{2}\alpha\beta} - \frac{q^2(2\varepsilon\beta-\omega)}{\sqrt{2}(4+q^2A)\beta\alpha} - \frac{\alpha\varepsilon q^2}{\sqrt{2}(4+q^2A)(\alpha^2-\gamma)}$.

By applying the Proposition 2.4 we obtain that the inverse of

$$H_{ij} = a_{ij} - \frac{1}{\alpha^2}l_il_j + \frac{q^2}{4}b_ib_j + \frac{1}{2\alpha^2q^4}\eta_i\eta_j$$

is

$$\begin{aligned} H^{ji} &= a^{ji} + \frac{1}{\alpha^2-\gamma}(1 - \frac{q^2\varepsilon^2}{(4+q^2A)(\alpha^2-\gamma)})\eta^i\eta^j - \frac{q^2}{4+q^2A}b^ib^j \\ &\quad - \frac{q^2\varepsilon}{(4+q^2A)(\alpha^2-\gamma)}(\eta^ib^j + b^i\eta^j) - \frac{1}{B}\frac{1}{2\alpha^2q^4}(M\eta^i + Nb^i)(M\eta^j + Nb^j) \end{aligned}$$

and $det(a_{ij} - \frac{1}{\alpha^2}l_il_j + \frac{q^2}{4}b_ib_j + \frac{1}{2\alpha^2q^4}\eta_i\eta_j) = B\frac{4+q^2A}{4}\frac{\alpha^2-\gamma}{}\alpha^2 det(a_{ij})$, where $B = 1 + c^2$. But, $g_{ij} = 2q^2H_{ij}$, with H_{ij} from III. Thus, $g^{ij} = \frac{1}{2q^2}H^{ij}$ and $det(g_{ij}) = (2q^2)^nB\frac{4+q^2A}{4}\frac{\alpha^2-\gamma}{\alpha^2}det(a_{ij})$. From here, immediately results i) and ii). □

Proposition 2.5. *In a non-Hermitian $\mathbb{R}-$ complex Finsler space with Kropina metric we have the following properties:*

$$\gamma + \overline{\gamma} = l_i\eta^i + l_{\bar{j}}\eta^{\bar{j}} = a_{ij}\eta^j\eta^i + a_{\bar{j}\bar{k}}\eta^{\bar{k}}\eta^{\bar{j}} = 2\alpha^2 \tag{2.16}$$

$$\varepsilon + \overline{\varepsilon} = b_j\eta^j + b_{\bar{j}}\eta^{\bar{j}} = 2\beta, \delta = \varepsilon, \tag{2.17}$$

where:

$l_i = a_{ij}\eta^j, \eta_i = \frac{2F}{\beta}a_{ij}\eta^j - \frac{F}{\beta^2}b_i,\ \gamma = a_{jk}\eta^j\eta^k = l_k\eta^k, \varepsilon = b_j\eta^j, \omega = b_jb^j,$

$b^k = a^{jk}b_j, b_l = b^ka_{kl}, \delta = a_{jk}\eta^jb^k = l_kb^k, l^j = a^{ji}l_i = \eta^j.$

Example 1. We set α as

$$\alpha^2(z,\eta) := \frac{(1+\varepsilon|z|^2)\sum_{k=1}^n Re(\eta^k)^2 - \varepsilon Re < z,\eta >^2}{(1+\varepsilon|z|^2)^2}, \tag{2.18}$$

where $|z|^2 := \sum_{k=1}^n z^k\overline{z}^k$, $< z,\eta >:= \sum_{k=1}^n z^k\overline{\eta}^k$, defined over the disk $\Delta_r^n = \left\{z \in \mathbf{C}^n,\ |z| < r,\ r := \sqrt{\frac{1}{|\varepsilon|}}\right\}$ if $\varepsilon < 0$, on $\mathbf{C}^n$ if $\varepsilon = 0$ and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$. By computation, we obtain $a_{ij} = \frac{1}{1+\varepsilon|z|^2}\left(\delta_{ij} - \varepsilon\frac{\overline{z}^i\overline{z}^j}{1+\varepsilon|z|^2}\right)$ and $a_{i\bar{j}} = 0$ and so, $\alpha^2(z,\eta) = \frac{1}{2}\left(a_{ij}\eta^i\eta^j + a_{\bar{i}\bar{j}}\overline{\eta}^i\overline{\eta}^j\right)$. Now, taking $\beta(z,\eta) := Re\frac{<z,\eta>}{1+\varepsilon|z|^2}$, where $b_i := \frac{\overline{z}^i}{1+\varepsilon|z|^2}$, we obtain some examples of non - Hermitian $\mathbb{R}$ - complex Kropina metrics:

$$F_\varepsilon := \frac{\frac{(1+\varepsilon|z|^2)\sum_{k=1}^n Re(\eta^k)^2 - \varepsilon Re<z,\eta>^2}{(1+\varepsilon|z|^2)^2}}{Re\frac{<z,\eta>}{1+\varepsilon|z|^2}}. \tag{2.19}$$

Theorem 2.2. *For a Hermitian $\mathbb{R}-$complex Finsler space with Kropina metric ($F = \frac{\alpha^2}{\beta},\ \beta \neq 0,\ a_{ij} = 0$) we have:*

$$\begin{aligned} g^{\bar{j}k} &= \frac{\beta}{2F}a^{\bar{j}k} + \frac{\beta(-4\beta+F\omega)}{2F^2N}\eta^{\bar{j}}\eta^k + \frac{\beta(-3F\beta+\alpha^2)}{2FN}b^{\bar{j}}b^k + \\ &+ \frac{\beta(4\beta-\overline{\varepsilon})}{2FN}b^{\bar{j}}\eta^k + \frac{\beta(4\beta-\varepsilon)}{2FN}\eta^{\bar{j}}b^k, \end{aligned} \tag{2.20}$$

where:

$$\begin{aligned} &N = |\varepsilon|^2 - \alpha^2\omega + 3F\beta\omega + 8\beta^2 - 8\beta Re(\varepsilon) \\ &\alpha^2 = a_{i\bar{j}}\eta^i\eta^{\bar{j}} = l_{\bar{i}}\eta^{\bar{i}}, l^j = a^{\bar{i}j}l_{\bar{i}} = \eta^j \\ &b^k = a^{\bar{j}k}b_{\bar{j}}, \varepsilon = l_{\bar{i}}b^{\bar{i}}, \overline{\varepsilon} = b_{\bar{i}}\eta^{\bar{i}}, \omega = b_{\bar{i}}b^{\bar{i}}. \end{aligned} \tag{2.21}$$

Proof. Assuming $a_{ij} = 0$, from (2.7)', (2.10) and Proposition 2.2 it follows:

$$l_i = a_{i\bar{j}}\eta^{\bar{j}}, \eta_i = \frac{2F}{\beta}a_{i\bar{j}}\eta^{\bar{j}} - \frac{F^2}{\beta}b_i$$

Considering:

$$g^{\bar{j}k} = \frac{\beta}{2F} a^{\bar{j}k} + C^* \eta^{\bar{j}} \eta^k + D^* b^{\bar{j}} b^k + E^* b^{\bar{j}} \eta^k + F^* \eta^{\bar{j}} b^k$$

and on the condition:

$$g_{i\bar{j}} g^{\bar{j}k} = \delta_i^k,$$

by direct calculus we obtain:

$$C^* = \frac{\beta(-4\beta + F\omega)}{2F^2 N}, D^* = \frac{\beta(-3F\beta + \alpha^2)}{2FN}, E^* = \frac{\beta(4\beta - \bar{\varepsilon})}{2FN}, F^* = \frac{\beta(4\beta - \varepsilon)}{2FN}$$

where N is given in (2.21) □

Example 2. We consider α given by

$$\alpha^2(z, \eta) := \frac{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |< z, \eta >|^2\right)}{(1 + \varepsilon|z|^2)^2}, \tag{2.22}$$

defined over the disk $\Delta_r^n = \left\{z \in \mathbf{C}^n,\ |z| < r,\ \ r := \sqrt{\frac{1}{|\varepsilon|}}\right\}$ if $\varepsilon < 0$, on $\mathbf{C}^n$ if $\varepsilon = 0$ and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$, where $|< z, \eta >|^2 :=< z, \eta > \overline{< z, \eta >}$. By computation, we obtain $a_{ij} = 0$ and $a_{i\bar{j}} = \frac{1}{1+\varepsilon|z|^2}\left(\delta_{i\bar{j}} - \varepsilon\frac{\bar{z}^i z^j}{1+\varepsilon|z|^2}\right)$ and so, $\alpha^2(z, \eta) = a_{i\bar{j}}(z)\eta^i\bar{\eta}^j$. Thus it determines purely Hermitian metrics which have special properties. They are Kähler with constant holomorphic curvature $\mathcal{K}_\alpha = 4\varepsilon$. Particularly, for $\varepsilon = -1$ we obtain the *Bergman metric* on the unit disk $\Delta^n := \Delta_1^n$; for $\varepsilon = 0$ the *Euclidean metric* on $\mathbf{C}^n$, and for $\varepsilon = 1$ the *Fubini-Study metric* on $P^n(\mathbf{C})$. Setting $\beta(z, \eta)$ as in Example1, we obtain some examples of Hermitian ℝ - complex Kropina metrics:

$$F_\varepsilon := \frac{\frac{|\eta|^2+\varepsilon\left(|z|^2|\eta|^2-|<z,\eta>|^2\right)}{(1+\varepsilon|z|^2)^2}}{Re\frac{<z,\eta>}{1+\varepsilon|z|^2}}. \tag{2.23}$$

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