Bulletin of the *Transilvania* University of Braşov • Vol 4(53), No. 2 - 2011
Series III: Mathematics, Informatics, Physics, 63-72

ON AN INTEGRAL OPERATOR

Virgil PESCAR[1]

Abstract

In this paper we derive some criteria for univalence of a new integral operator for analytic functions in the open unit disk.

2000 *Mathematics Subject Classification:* 30C45.
Key words: analytic, integral operator, univalence.

1 Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$. We consider $\mathcal{P}$ the class of functions $p(z) = 1 + b_1 z + b_2 z^2 + \dots$ which are analytic in $\mathcal{U}$, with positive real part in $\mathcal{U}$.

In this work, we introduce a new integral operator, which is defined by

$$E_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{\alpha_j} (g_j(u))^{\beta_j}\, du, \tag{1.1}$$

for functions $f_j \in \mathcal{A}$, $g_j \in \mathcal{P}$ and $\alpha_j,\ \beta_j$ be complex numbers, $j = \overline{1,n}$.

From (1.1), for $\beta_j = 0$, α_j be complex numbers, $f_j \in \mathcal{A}$, $j = \overline{1,n}$, we obtain the integral operator defined in [5].

For $\alpha_j = 0$, β_j be complex numbers, $g_j \in \mathcal{P}$, $j = \overline{1,n}$, from (1.1) we get the integral operator which is defined in [6].

If we take $\beta_j = 0$, $j = \overline{1,n}$, $\alpha_1 = \alpha$, $\alpha_j = 0$, $j = \overline{2,n}$, $f_1 = f$, from (1.1) we have the integral operator Kim-Merkes [2].

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: virgilpescar@unitbv.ro

2 Preliminary results

To discuss our problems for univalance of the integral operator $E_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}$ we need the following lemmas.

Lemma 1. *[1]. If the function f is analytic in $\mathcal{U}$ and*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 2. *[4]. Let γ be a complex number, $Re\,\gamma > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2.2}$$

for all $z \in \mathcal{U}$, then for any complex number δ, $Re\,\delta \geq Re\,\gamma$, the function

$$F_\delta(z) = \left[\delta \int_0^z u^{\delta-1} f'(u)du\right]^{\frac{1}{\delta}} \tag{2.3}$$

is regular and univalent in $\mathcal{U}$.

Lemma 3. *(Schwarz [3]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ \left(z \in \mathcal{U}_R\right), \tag{2.4}$$

the equality (in the inequality (2.2) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3 Main results

Theorem 1. *Let α_j, β_j be complex numbers, K_j, L_j positive real numbers, $j = \overline{1,n}$, and the functions, $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \dots$ and $g_j \in \mathcal{P}$, $g_j(z) = 1 + b_{1j}z + \dots$, $j = \overline{1,n}$. If*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq K_j, \quad \left(j = \overline{1,n}; z \in \mathcal{U}\right), \tag{3.1}$$

$$\left|\frac{zg_j'(z)}{g_j(z)}\right| \leq L_j, \quad \left(j = \overline{1,n}; z \in \mathcal{U}\right), \tag{3.2}$$

and

$$\sum_{j=1}^{n} \left[|\alpha_j| K_j + |\beta_j| L_j \right] \leq \frac{3\sqrt{3}}{2}, \tag{3.3}$$

then the integral operator $E_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}$*, defined by (1.1), is in the class* $\mathcal{S}$*.*

Proof. The function $E_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)$ is regular in $\mathcal{U}$ and

$$E_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(0) = E'_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(0) - 1 = 0.$$

We have

$$\frac{zE''_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)}{E'_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)} = \sum_{j=1}^{n} \left[\alpha_j \left(\frac{zf'_j(z)}{f_j(z)} - 1 \right) \right] + \sum_{j=1}^{n} \left[\beta_j \frac{zg'_j(z)}{g_j(z)} \right], \tag{3.4}$$

and hence, we obtain

$$\left(1 - |z|^2\right) \left| \frac{zE''_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)}{E'_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)} \right| \leq \left(1 - |z|^2\right) \sum_{j=1}^{n} \left[|\alpha_j| \left| \frac{zf'_j(z)}{f_j(z)} - 1 \right| + |\beta_j| \left| \frac{zg'_j(z)}{g_j(z)} \right| \right] \tag{3.5}$$

for all $z \in \mathcal{U}$.

By (3.1), (3.2) and Lemma 3, we obtain

$$\left| \frac{zf'_j(z)}{f_j(z)} - 1 \right| \leq K_j |z|, \quad \left(j = \overline{1,n}; z \in \mathcal{U}\right), \tag{3.6}$$

$$\left| \frac{zg'_j(z)}{g_j(z)} \right| \leq L_j |z|, \quad \left(j = \overline{1,n}; z \in \mathcal{U}\right) \tag{3.7}$$

and by (3.5), we have

$$\left(1 - |z|^2\right) \left| \frac{zE''_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)}{E'_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)} \right| \leq \left(1 - |z|^2\right) |z| \left\{ \sum_{j=1}^{n} \left[|\alpha_j| K_j + |\beta_j| L_j \right] \right\} \tag{3.8}$$

for all $z \in \mathcal{U}$.

Because

$$\max_{|z| \leq 1} \left[\left(1 - |z|^2\right) |z| \right] = \frac{2}{3\sqrt{3}},$$

from (3.3) and (3.8) we get

$$\left(1 - |z|^2\right) \left| \frac{zE''_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)}{E'_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(z)} \right| \leq 1, \quad (z \in \mathcal{U}), \tag{3.9}$$

for all $z \in \mathcal{U}$ and by Lemma 1, it results that the integral operator $E_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}$ is in the class $\mathcal{S}$. □

Theorem 2. *Let* $\alpha_j,\ \beta_j,\ \gamma$ *be complex numbers,* $j=\overline{1,n}$, $0 < Re\ \gamma \le 1$ *and the functions* $f_j \in \mathcal{A},\ f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots,\ g_j \in \mathcal{P},\ g_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots,\ j = \overline{1,n}$.

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \le \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}}}{2}, \quad \left(j = \overline{1,n}; z \in \mathcal{U}\right), \tag{3.10}$$

$$\left|\frac{zg_j'(z)}{g_j(z)}\right| \le \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}}}{2}, \quad \left(j = \overline{1,n}; z \in \mathcal{U}\right), \tag{3.11}$$

and

$$\sum_{j=1}^{n} \left[|\alpha_j| + |\beta_j|\right] \le 1, \tag{3.12}$$

then the integral operator $E_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}$, *defined by (1.1), belongs to the class* $\mathcal{S}$.

Proof. From (3.4) we obtain:

$$\frac{1-|z|^{2Re\ \gamma}}{Re\ \gamma}\left|\frac{zE''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}{E'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}\right| \le$$

$$\le \frac{1-|z|^{2Re\ \gamma}}{Re\ \gamma}\sum_{j=1}^{n}\left[|\alpha_j|\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| + |\beta_j|\left|\frac{zg_j'(z)}{g_j(z)}\right|\right], \tag{3.13}$$

for all $z \in \mathcal{U}$.

By (3.10), (3.11) and Lemma 3, we get

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \le \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}}}{2}|z|, \quad \left(j = \overline{1,n}; z \in \mathcal{U}\right), \tag{3.14}$$

$$\left|\frac{zg_j'(z)}{g_j(z)}\right| \le \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}}}{2}|z|, \quad \left(j = \overline{1,n}; z \in \mathcal{U}\right), \tag{3.15}$$

and hence, by (3.13), we get

$$\frac{1-|z|^{2Re\ \gamma}}{Re\ \gamma}\left|\frac{zE''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}{E'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}\right| \le$$

$$\le \frac{1-|z|^{2Re\ \gamma}}{Re\ \gamma}\cdot|z|\cdot\frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}}}{2}\cdot\sum_{j=1}^{n}\left[|\alpha_j| + |\beta_j|\right], \tag{3.16}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\le 1}\left[\frac{(1-|z|^{2Re\,\gamma})|z|}{Re\,\gamma}\right]=\frac{2}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}},$$

from (3.12) and (3.16) we obtain that

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zE''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}{E'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}\right|\le 1, \tag{3.17}$$

for all $z \in \mathcal{U}$ and by Lemma 2, for $\delta = 1$ and $f = E_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}$, it results that the integral operator $E_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}$, defined by (1.1), belongs to the class $\mathcal{S}$. □

Theorem 3. *Let $\alpha_j,\ \beta_j,\ \gamma$ be complex numbers, $f_j \in \mathcal{S},\ f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots,$ $g_j \in \mathcal{P},\ g_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots,\ j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n}\left[4|\alpha_j|+2|\beta_j|\right]\le 1,\quad (j=\overline{1,n}), \tag{3.18}$$

then the integral operator $E_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}$, defined by (1.1), belongs to the class $\mathcal{S}$.

Proof. From (3.4) we have

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}{E'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}\right|\le$$

$$\le\left(1-|z|^2\right)\sum_{j=1}^{n}\left[|\alpha_j|\left|\frac{zf'_j(z)}{f_j(z)}-1\right|+|\beta_j|\left|\frac{zg'_j(z)}{g_j(z)}\right|\right], \tag{3.19}$$

for all $z \in \mathcal{U}$.

Since $f_j \in \mathcal{S}$ and $g_j \in \mathcal{P}$, $j = \overline{1,n}$, we have

$$\left|\frac{zf'_j(z)}{f_j(z)}\right|\le\frac{1+|z|}{1-|z|},\quad \left(j=\overline{1,n}; z\in\mathcal{U}\right),$$

$$\left|\frac{zg'_j(z)}{g_j(z)}\right|\le\frac{2|z|}{1-|z|^2},\quad \left(j=\overline{1,n}; z\in\mathcal{U}\right),$$

and hence, by (3.19) we obtain

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}{E'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}\right|\le\sum_{j=1}^{n}\left[4|\alpha_j|+2|\beta_j|\right], \tag{3.20}$$

for all $z \in \mathcal{U}$. From (3.18) and (3.20) we get

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}{E'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}(z)}\right| \leq 1, \tag{3.21}$$

for all $z \in \mathcal{U}$ and by Lemma 1 we obtain that the integral operator $E_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n}$ is in the class $\mathcal{S}$.

□

4 Corollaries

Corollary 1. *Let α_j be complex numbers, K_j positive real numbers, $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $j = \overline{1,n}$. If*

$$\left|\frac{zf'_j(z)}{f_j(z)} - 1\right| \leq K_j, \quad (j = \overline{1,n}; z \in \mathcal{U}) \tag{4.1}$$

and

$$\sum_{j=1}^{n} \left[|\alpha_j| K_j\right] \leq \frac{3\sqrt{3}}{2}, \tag{4.2}$$

then the function

$$H_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{\alpha_j} du \tag{4.3}$$

is in the class $\mathcal{S}$.

Proof. We take $\beta_j = 0,\ j = \overline{1,n}$ in Theorem 1. □

Corollary 2. *Let α be a complex number $(\alpha \neq 0)$, $f \in \mathcal{A}$, $f(z) = z + a_{21}z^2 + \ldots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2|\alpha|}, \quad (z \in \mathcal{U}), \tag{4.4}$$

then the function

$$H_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\alpha du \tag{4.5}$$

belongs to the class $\mathcal{S}$.

Proof. For $\beta_j = 0,\ j = \overline{1,n},\ \alpha_1 = \alpha,\ \alpha_j = 0,\ j = \overline{2,n},\ f_1 = f$ in Theorem 1, we obtain the Corollary 2. □

Corollary 3. *Let β_j be complex numbers, L_j positive real numbers and $g_j \in \mathcal{P}$, $g_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$. If*

$$\left|\frac{zg_j'(z)}{g_j(z)}\right| \leq L_j, \quad (j = \overline{1,n}; z \in \mathcal{U}), \tag{4.6}$$

and

$$\sum_{j=1}^{n} [|\beta_j| L_j] \leq \frac{3\sqrt{3}}{2}, \tag{4.7}$$

then the function

$$G_{\beta_1,\ldots,\beta_n}(z) = \int_0^z \prod_{j=1}^{n} (g_j(u))^{\beta_j} \, du \tag{4.8}$$

is in the class $\mathcal{S}$.

Proof. We take $\alpha_j = 0$, $j = \overline{1,n}$, in Theorem 1. □

Corollary 4. *Let α_j, γ be complex numbers, $j = \overline{1,n}$, $0 < Re\ \gamma \leq 1$ and $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma + 1}{2Re\ \gamma}}}{2}, \quad (j = \overline{1,n}; z \in \mathcal{U}) \tag{4.9}$$

and

$$|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \leq 1, \tag{4.10}$$

then the function $H_{\alpha_1,\ldots,\alpha_n}(z)$, defined by (4.3), belongs to the class $\mathcal{S}$.

Proof. For $\beta_j = 0$, $j = \overline{1,n}$, from Theorem 2 we obtain the Corollary 4. □

Corollary 5. *Let α, γ be complex numbers, $|\alpha| \leq 1$, $0 < Re\ \gamma \leq 1$ and $f \in \mathcal{A}$, $f(z) = z + a_{21}z^2 + a_{31}z^3 + \ldots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma + 1}{2Re\ \gamma}}}{2}, \quad (z \in \mathcal{U}), \tag{4.11}$$

then the function $H_\alpha(z)$, defined by (4.5), is in the class $\mathcal{S}$.

Proof. We take $\beta_j = 0$, $j = \overline{1,n}$, $\alpha_1 = \alpha$, $\alpha_j = 0$, $j = \overline{2,n}$, $f_1 = f$ in Theorem 2. □

Corollary 6. *Let* β_j, γ *be complex numbers,* $j = \overline{1,n}$, $0 < Re\ \gamma \leq 1$ *and* $g_j \in \mathcal{P}$, $g_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.

If

$$\left| \frac{zg_j'(z)}{g_j(z)} \right| \leq \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma + 1}{2Re\ \gamma}}}{2}, \quad (j = \overline{1,n}; z \in \mathcal{U}) \tag{4.12}$$

and

$$|\beta_1| + |\beta_2| + \ldots + |\beta_n| \leq 1, \tag{4.13}$$

then the function $G_{\beta_1,\ldots,\beta_n}(z)$, *defined by (4.8), belongs to the class* $\mathcal{S}$.

Proof. We consider $\alpha_j = 0,\ j = \overline{1,n}$ in Theorem 2. □

Corollary 7. *Let* α_j *be complex numbers,* $f_j \in \mathcal{S}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $j = \overline{1,n}$.

If

$$|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \leq \frac{1}{4}, \tag{4.14}$$

then the function $H_{\alpha_1,\ldots,\alpha_n}(z)$, *defined by (4.3), is in the class* $\mathcal{S}$.

Proof. From Theorem 3, if we take $\beta_j = 0,\ j = \overline{1,n}$, we obtain the Corollary 7. □

Corollary 8. *Let* α *be a complex number,* $f \in \mathcal{S}$, $f(z) = z + a_{21}z^2 + a_{31}z^3 + \ldots$.

If

$$|\alpha| \leq \frac{1}{4} \tag{4.15}$$

then the function $H_\alpha(z)$, *defined by (4.5), is in the class* $\mathcal{S}$.

Proof. For $\beta_j = 0,\ j = \overline{1,n},\ \alpha_1 = \alpha,\ \alpha_j = 0,\ j = \overline{2,n},\ f_1 = f$ in Theorem 3, we get the Corollary 8. □

Corollary 9. *Let* β_j *be complex numbers,* $g_j \in \mathcal{P}$, $g_j(z) = 1 + b_{1j}z + \ldots$, $j = \overline{1,n}$.

If

$$|\beta_1| + |\beta_2| + \ldots + |\beta_n| \leq \frac{1}{2}, \tag{4.16}$$

then the function $G_{\beta_1,\ldots,\beta_n}(z)$, *defined by (4.8), belongs to the class* $\mathcal{S}$.

Proof. We take $\alpha_j = 0,\ j = \overline{1,n}$ in Theorem 3. □

5 References

[1] Becker, J., *Löwnersche Differentialgleichung Und Quasikonform Fortsetzbare Schlichte Functionen*, J. Reine Angew. Math. **255** (1972), 23-43.

[2] Kim, Y. J. , E. P. Merkes, *On an Integral of Powers of a Spirallike Function*, Kyungpook Math. J. **12** (1972), 249-253.

[3] Mayer, O., *The Functions Theory of One Variable Complex*, Bucureşti, 1981.

[4] Pascu, N. N., *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilow (Braşov), 1987, University of Braşov, 43-48, 1987.

[5] Pescar, V., Breaz, V.D., *The Univalence of Integral Operators*, Monograph, Academic Publishing House, Sofia, 2008.

[6] Pescar, V., *On the univalence of an integral operator* (to appear).