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A NOTE ON THE STICKY BROWNIAN MOTION ON $\mathbb{R}$

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Abstract

We consider a degenerate stochastic differential equation which describes an arbitrary sticky Brownian motion on $\mathbb{R}$ with sticky point 0. We obtain a representation formula different from the one in [3], which describes the solutions in terms of time delays.

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1 Introduction

Consider the (degenerate) stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s)\, dB_s, \tag{1}$$

where

$$\sigma(y) = \begin{cases} 1, & y \neq 0 \\ 0, & y = 0 \end{cases} \tag{2}$$

and $(B_t)_{t\geq 0}$ is a 1-dimensional Brownian motion starting at 0 on a probability space $(\Omega, \mathcal{F}, P)$.

If $x = 0$, it is easy to see that $X_t \equiv 0$ and $X_t = B_t$ are two solutions to (1), thus pathwise uniqueness fails for this equation. Moreover, by a result of Engelbert and Schmidt ([3]), since the null set of σ is $\mathcal{N} = \{0\}$ and σ^{-2} is locally integrable on $\mathbb{R}$, the solution of (1) is not even weakly unique. In [3], the authors also showed that the general solution can be obtained from B_t by delaying it when it is at 0, and thus it can be viewed as a *sticky Brownian motion* on $\mathbb{R}$ with sticky point 0 (a process which behaves like the ordinary Brownian motion away from 0, and it spends a positive amount of Lebesgue time at 0).

Sticky (reflecting) Brownian motion on $[0, \infty)$ has been considered by other authors, see for example [6], [7] and the references cited therein. For a connection between the

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sticky Brownian motion on $\mathbb{R}$ and on $[0,\infty)$, one can consider $|X_t|$, where X_t is a solution of (1).

The main result of the paper is Theorem 1, in which we obtain a new representation of the solutions of (1) different from the one of Engelbert and Schmidt, which uses the notion of time delays (see Definition 4.1 and Theorem 5.5 in [3]). As a corollary, we obtain the pathwise uniqueness of solutions of (1) which spend zero Lebesgue time at 0.

We were led to considering the equation (1) by the recent extension of the *mirror coupling* of reflecting Brownian motions introduced by K. Burdzy and the co-authors (see [1] and the references cited therein). Trying to extend the construction to the case when the two reflecting Brownian motion live in different domains, we were led to the problem of constructing two 1-dimensional Brownian motions which have the same increments when they coincide and opposite increments when they are different, that is, given a 1-dimensional Brownian motion B_t to construct another 1-dimensional Brownian motion W_t adapted to the filtration generated by B_t which solves

$$W_t = w + \int_0^t G\left(W_t - B_t\right) dB_t, \qquad t \geq 0,$$

where $G(y) = 1 - 2\sigma(y)$, $y \in \mathbb{R}$.

With the substitution $X_t = -\frac{1}{2}\left(W_t - B_t\right)$ the above equation reduces to the stochastic differential equation (1) for $x = -\frac{w}{2}$. Theorem 1 gives an explicit form of the solutions of this latter problem, and Corollary 1 shows that the solution is pathwise unique when restricting the class of solutions to those that spend zero time at 0. For more details on the extension of the mirror coupling of reflecting Brownian motions, see [4].

2 Main results

We will use the terminology of [2] referring to the weak/strong solutions and the corresponding weak and strong (pathwise) uniqueness of stochastic differential equation. We recall the following:

Definition 1. *Given a one-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ starting at $B_0 = 0$ on a probability space $(\Omega, \mathcal{F}, P)$, a strong solution to*

$$X_t = x + \int_0^t \sigma\left(X_s\right) dB_s \tag{3}$$

which spends zero time at 0 is a continuous process $X = (X_t)_{t\geq 0}$ adapted to the filtration generated by B, solves (3), and satisfies $\int_0^\infty 1_{\{0\}}\left(X_s\right) ds = 0$ a.s.

We say that pathwise uniqueness holds for (3) among functions which spend zero time at 0 if whenever X and $\widetilde{X}$ are two solutions on the same probability space corresponding to the same Brownian motion B (relative to possibly different filtrations), we have

$$P(X_t = \widetilde{X}_t, \quad t \geq 0) = 1.$$

Before giving the main result, we will give a description of the solutions of (1) in terms of time changes. This result is in the spirit of time delays (see [3]), and we present it here in order to show the differences from the representation obtained in Theorem 1 below.

We observe that any solution of (1) is a continuous local martingale with quadratic variation process given by

$$A_t = \int_0^t \sigma(X_s)\, ds = t - \int_0^t 1_{\{0\}}(X_s)\, ds = t - \Lambda_t,$$

where

$$\Lambda_t = \int_0^t 1_{\{0\}}(X_s)\, ds$$

represents the Lebesgue measure of the time spent by X at 0.

If $\Lambda_t \not\equiv 0$, then the right-continuous inverse $\alpha_t = \inf\{s \geq 0 : A_s > t\}$ of A_t might have jumps. However, noticing that

$$t \wedge A_\infty - s \wedge A_\infty = A_{\alpha_t} - A_{\alpha_s} = \int_{\alpha_s}^{\alpha_t} \sigma(X_u)\, du, \qquad 0 \leq s \leq t, \tag{4}$$

we obtain

$$\lim_{s \nearrow t} \int_{\alpha_s}^{\alpha_t} \sigma(X_u)\, du = 0, \qquad t > 0, \tag{5}$$

and using the continuity of X_t it follows that the time-changed process $(X_{\alpha_t})_{t\geq 0}$ is continuous in $t \geq 0$.

By Lévy's characterization of Brownian motion, it follows that the time-changed solution X_{α_t} is a one-dimensional (possibly stopped) $\widetilde{\mathcal{F}}_{\alpha_t}$-Brownian motion $(\widetilde{B}_t)_{t\geq 0}$ starting at x, and therefore

$$X_t = \widetilde{B}_{A_t}, \qquad t \geq 0. \tag{6}$$

An arbitrary solution of (1) is therefore a one-dimensional Brownian motion run with the clock A_t. In the particular case $x = 0$, the solutions $X_t \equiv 0$ and $X_t \equiv B_t$ correspond to the choices $A_t \equiv 0$, respectively $A_t \equiv t$.

In general, the Brownian motion $\widetilde{B}_t$ is defined on a standard extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ of the underlying probability space $(\Omega, \mathcal{F}, P)$. It is remarkable that in the present case we can give an explicit description of the solution $X_t = \widetilde{B}_{A_t}$ in terms of B_t, as follows:

Theorem 1. *If B_t is a one dimensional Brownian motion starting at $B_0 = 0$ and X_t is a continuous process which solves*

$$X_t = x + \int_0^t \sigma(X_s)\, dB_s, \qquad t \geq 0, \tag{7}$$

then X_t has the representation

$$X_t = B_t - B_{t_\Lambda}, \qquad t \geq 0, \tag{8}$$

where t_Λ is the first point of increase of $\Lambda_s = \int_0^s 1_{\{0\}}(X_u)\, du$ after the time t, that is

$$t_\Lambda = \inf\{s \geq t : \Lambda_s > \Lambda_t\} \in [0, \infty], \tag{9}$$

and in the case $t_\Lambda = \infty$ we set

$$B_\infty = \begin{cases} -x, & t_\infty = 0 \\ B_{t_\infty}, & t_\infty > 0 \end{cases}, \tag{10}$$

with $t_\infty = \inf\{s \geq 0 : \Lambda_s = \Lambda_\infty\}$ denoting the last point of increase of Λ.

Proof. Let X be a solution to (7) and let $(\mathcal{F}_s)_{s\geq 0}$ denote the common filtration with respect to which X and B are adapted.

First note that for $t \geq 0$ arbitrarily fixed, $t_\Lambda = \inf\{s \geq t : \Lambda_s > \Lambda_t\} \in [t, \infty]$ is a stopping time with respect to the filtration $\mathcal{F}_s$, since

$$\{t_\Lambda < u\} = \begin{cases} \varnothing, & u \leq t \\ \{\Lambda_t < \Lambda_u\}, & t < u \end{cases} \in \mathcal{F}_u, \qquad u \geq 0.$$

For any $s \geq t$, $t_\Lambda \wedge s$ is a bounded stopping time, and from (7) we obtain

$$X_{t_\Lambda \wedge s} - X_t = \int_t^{t_\Lambda \wedge s} \sigma(X_u)\, dB_u = B_{t_\Lambda \wedge s} - B_t + \int_t^{t_\Lambda \wedge s} 1_{\{0\}}(X_u)\, dB_u. \tag{11}$$

The stochastic integral term on the right is zero. To see this, note that

$$\int_t^{t_\Lambda \wedge s} 1_{\{0\}}(X_u)\, dB_u = \int_0^\infty 1_{[t, t_\Lambda \wedge s]}(u) 1_{\{0\}}(X_u)\, dB_u, \tag{12}$$

and that the integrand on the right is a measurable, $\mathcal{F}_s$-adapted process, with

$$E \int_0^\infty \left(1_{[t, t_\Lambda \wedge s]}(u) 1_{\{0\}}(X_u)\right)^2 du = E \int_t^{t_\Lambda \wedge s} 1_{\{0\}}(X_u) du \leq s - t < \infty.$$

Considering the sequence $f^{(n)} \equiv 0$ $(n \geq 1)$ of identically zero processes, since $\Lambda_{t_\Lambda} = \Lambda_t$ by the definition of the stopping time t_Λ, we obtain

$$0 \leq \lim_{n\to\infty} E \int_0^\infty \left(f_u^{(n)} - 1_{[t, t_\Lambda \wedge s]}(u) 1_{\{0\}}(X_u)\right)^2 du = E\left(\Lambda_{t_\Lambda \wedge s} - \Lambda_t\right) \leq 0,$$

and therefore $f^{(n)}$ is an approximating sequence which can be used to define the stochastic integral in (12). Since $\int_0^\infty f_u^{(n)} dB_u = 0$ for any $n \geq 1$, it follows that $\int_t^{t_\Lambda \wedge s} 1_{\{0\}}(X_u)\, dB_u = 0$, concluding the proof of the claim.

From (11) it follows that

$$X_{t_\Lambda \wedge s} - X_t = B_{t_\Lambda \wedge s} - B_t, \qquad \text{a.s. for any } s \geq t, \tag{13}$$

and we distinguish the following cases.

i) On the set $\{t_\Lambda < \infty\}$, passing to the limit with $s \to \infty$ in the above equality and using the continuity of the processes X and B, we obtain $X_{t_\Lambda} - X_t = B_{t_\Lambda} - B_t$. Since by definition t_Λ is a point of increase of Λ, it follows that $X_{t_\Lambda} = 0$, and we obtain $X_t = B_t - B_{t_\Lambda}$, which concludes the proof in this case.

ii) On the set $\{t_\Lambda = \infty\}$, the equality (13) becomes $X_s - X_t = B_s - B_t$ for any $s \geq t$, which shows that $X_\cdot - X_t - (B_\cdot - B_t)$ is identically zero on the interval $[t, \infty)$.

Since $\{s_\Lambda = \infty\} \subset \{t_\Lambda = \infty\}$ for any $s \leq t$, a similar argument (with s instead of t) shows that $X_\cdot - X_t - (B_\cdot - B_t)$ is also identically zero on any interval $[s, \infty)$ with $s_\Lambda = \infty$. It is not difficult to see that $\inf\{s \in \mathbb{Q}^+ : s_\Lambda = \infty\} = t_\infty$, and using the continuity of the processes X and B it follows that $X_{t_\infty} - X_t - (B_{t_\infty} - B_t) = 0$ on the set $\{t_\infty < \infty\} \subset \{t_\Lambda = \infty\}$.

If $t_\infty = 0$, then $X_{t_\infty} = X_0 = x$ and $B_{t_\infty} = B_0 = 0$, and the above becomes $X_t = B_t + x$, which is the same as (8) with the convention $B_{t_\Lambda} = B_\infty := -x$.

If $t_\infty > 0$, from the definition of t_∞ it follows that $X_{t_\infty} = 0$, so the above becomes in this case $X_t = B_t - B_{t_\infty}$, which is the same as (8) with the convention $B_{t_\Lambda} = B_\infty := B_{t_\infty}$, concluding the proof. □

As an example describing a generic sticky Brownian motion on $\mathbb{R}$ with sticky point 0 we have the following:

Example 1. *Given a Brownian motion B_t starting at 0, the process X_t defined by*

$$X_t = \begin{cases} x + B_t, & t \leq \tau \\ 0, & \tau < t \leq \tau + t_0 \\ B_t - B_{\tau + t_0}, & t > \tau + t_0 \end{cases}, \tag{14}$$

where $\tau = \inf\{s > 0 : B_s = -x\}$ and $t_0 > 0$, behaves like an ordinary Brownian motion except for the time interval $[\tau, \tau + t_0]$ when it "sticks" to 0.

It is not difficult to see that X_t is a solution of (7), and in the notation of the theorem above we have $t_\infty = \tau + t_0$, $B_\infty = B_{t_\infty} = B_{\tau + t_0}$,

$$\Lambda_t = \begin{cases} 0, & t \leq \tau \\ t - \tau, & \tau < t < \tau + t_0 \\ t_0 - \tau, & t \geq \tau + t_0 \end{cases} \quad \text{and} \quad t_\Lambda = \begin{cases} \tau, & t \leq \tau \\ t, & \tau < t < \tau + t_0 \\ \infty, & t \geq \tau + t_0 \end{cases}.$$

From Theorem 1 it follows that X_t has the representation $X_t = B_t - B_{t_\Lambda}$, $t \geq 0$, which can be verified immediately.

Remark 1. *Note that if in particular the solution X_t of (7) spends zero Lebesgue time at the origin, then $\Lambda_t \equiv 0$ and $t_\Lambda \equiv \infty$, and therefore (8) becomes in this case*

$$X_t = B_t - B_\infty = x + B_t, \qquad t \geq 0,$$

which shows that pathwise uniqueness holds for the solutions of (7) which spend zero time at the origin.

We obtained therefore the following:

Corollary 1. *Pathwise uniqueness holds for the solutions of the stochastic differential equation*

$$X_t = x + \int_0^t \sigma(X_s)\, dB_s \tag{15}$$

which spend zero time at 0. Moreover, a strong solution is explicitly given by $X_t = x + B_t$, $t \geq 0$.

Proof. We only need to prove that $X_t = x + B_t$ is a strong solution to (15). This follows easily by the same arguments as in the proof of the previous theorem. □

Remark 2. *Consider the (degenerate) stochastic differential equation*

$$X_t = \int_0^t |X_s|^\alpha \, dB_s, \qquad t \geq 0, \tag{16}$$

where B_t is a 1-dimensional Brownian motion.

By a classical result of Yamada and Watanabe ([5]), pathwise uniqueness holds for (16) if $\alpha \in [1/2, \infty)$, and this result is sharp. Extending this result, in [2] the authors showed that for $\alpha \in (0, 1/2)$ pathwise uniqueness still holds for (16) when restricting the class of solutions to those that spend zero time at 0, and they proved the existence of a strong solution of (16).

Observing that for $\alpha \searrow 0$ we have $\sigma_\alpha(x) = |x|^\alpha \to 1_{\mathbb{R}-\{0\}}(x) = \sigma(x)$, Corollary 1 above shows that the same conclusion holds for the limiting case $\alpha = 0$.

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