Bulletin of the *Transilvania* University of Braşov • Vol 4(53), No. 2 - 2011
Series III: Mathematics, Informatics, Physics, 43-56

THEORY OF FINSLER SPACES WITH (γ, β) METRICS

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Communicated to:
Finsler Extensions of Relativity Theory, August 29 - September 4, 2011, Braşov, Romania

Abstract

In the present paper, we introduce the concept of (γ, β)-metric and a number of propositions and theorems have worked for a (γ, β)-metric, where $\gamma^3 = a_{ijk}(x)y^i y^j y^k$ is a cubic metric and $\beta = b_i(x)y^i$ is a one form metric.

2000 *Mathematics Subject Classification:* 53B40, 53C60.
Key words: Finsler space with (γ, β)-metric, cubic metric, one form metric, curvature tensor, torsion tensor.

1 Preliminaries

Let M^n be an n-dimensional differentiable manifold and TM^n be its tangent bundle. The manifold M^n is covered by neighbourhoods (U), in each U of which we have a local coordinate systen (x^i). A tangent vector at a point $x = (x^i)$ of U is written as $y^i(\frac{\partial}{\partial x^i})_x$, and we have a local coordinate system (x^i, y^i) of TM^n over the U.

A (Finslerian) tensor field, for instance, of type (1, 1) in M^n is by definition a collection of n^2 functions $T^i_j(x, y)$ of variables (x^i, y^i) which obey the usual transformation law of components of a tensor field,

$$\bar{T}^a_b(\bar{x}, \bar{y}) = T^i_j(x, y)\bar{X}^a_i \underline{X}^j_b$$

when a local coordinate system (x^i) is changed for $(\bar{x}^a)$, where $\bar{X}^a_i = \frac{\partial \bar{x}^a}{\partial x^i}$ and $\underline{X}^j_b = \frac{\partial x^j}{\partial \bar{x}^b}$.

throughout the following the symbols ∂_i and $\dot{\partial}_i$ stand for partial differentiations $\frac{\partial}{\partial x^i}$ and

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[1]Author is very thankful to NBHM-DAE of Goverment of INDIA for their financial assistance as a Postdoctoral Fellowship

$\frac{\partial}{\partial y^i}$, respectively.

A Finsler connection $F\Gamma$ is defined as tried $(F^i_{jk}(x,y), N^i_j(x,y), V^i_{jk}(x,y))$ as follows:

(1) $F^i_{jk}(x,y)$ are called the connection coefficients of h-connection whch obey the usual transformation law of connection coefficients of a connection

$$\bar{F}^a_{bc}(\bar{x},\bar{y}) = F^i_{jk}(x,y)\bar{X}^a_i \underline{X}^j_b \underline{X}^k_c + \bar{X}^a_i \partial_c \underline{X}^i_b$$

('h' is the abbreviation of 'horizontal'.)

(2) $N^i_j(x,y)$ are called the connection coefficients of non-linear connection which obey the transformation law

$$\bar{N}^a_b(\bar{x},\bar{y}) = N^i_j(x,y)\bar{X}^a_i \underline{X}^j_b + \bar{X}^a_i \partial_b \underline{X}^i_c \bar{y}^c$$

(3) $V^i_{jk}(x,y)$ are called the connection coefficients of v-connection which are components of a tensor field of (1,2)-type. ('v' is the abbreviation of 'vertical'.)

A Finsler connection $F\Gamma$ serves the purpose of constructing covariant derivatives a Finslerian tensor field. For a tensor field $T^i_j(x,y)$, for instance, of (1,1)-type we have first the h-covariant derivative $\bigtriangledown^h T$ whose components are given by

$$T^i_{j|k} = \delta_k T^i_j + T^r_j F^i_{rk} - T^i_r F^r_{jk} \tag{1}$$

where, δ_k is a differential operator $\delta_k = \partial_k - N^r_k \dot{\partial}_r$. Secondly, we have the v-covariant derivative $\bigtriangledown^v T$ whose components are given by

$$T^i_j|_k = \dot{\partial}_k T^i_j + T^r_j V^i_{rk} - T^i_r V^r_{jk} \tag{2}$$

It is noteworthy that y^i of local coordinate system (x^i, y^i) of TM^n constitute a Finslerian contravariant vector field y. The components of $\bigtriangledown^h y$ and $\bigtriangledown^v y$ are, respectively written as

$$y^i_{|j}(= D^i_j) = -N^i_j + y^r F^i_{rj}, \qquad y^i|_j = \delta^i_j + y^r V^i_{rj} \tag{3}$$

The tensor D of components D^i_j is called the deflection tensor of $F\Gamma$. Therefore, $D^i_j = N^i_j$ and $y^r V^i_{rj} = 0$ may be desirable conditions for a Finsler connection.

Throughout the following the index 0 denotes the transvection by y^i. Thus $y^r V^i_{rj}$ is written as V^i_{0j}.

For a tensor field T of components $T^i_j(x,y)$ we get a tensor field $\bigtriangledown^0 T$ of components $\dot{\partial}_k T^i_j$. Therefore, we can get a Finsler connection $(F^i_{jk}, N^i_j, 0)$ from a given Finsler connection $(F^i_{jk}, N^i_j, V^i_{jk})$. $\bigtriangledown^0 T$ is called 0-covariant derivative of T.

We shall write the important commutation formulae of covariant derivatives; for contravariant vector field (X^i) we have

$$\begin{cases} X^i_{|j|k} - X^i_{|k|j} = X^r R^i_{rjk} - X^i_{|r} T^r_{jk} - X^i|_r R^r_{jk} \\ X^i_{|j}|_k - X^i|_{k|j} = X^r P^i_{rjk} - X^i_{|r} V^r_{jk} - X^i|_r P^r_{jk} \\ X^i|_j|_k - X^i|_k|_j = X^r S^i_{rjk} - X^i|_r S^r_{jk} \end{cases} \tag{4}$$

Thus we get three kinds of curvature tensors:

$R^2 = (R^i_{hjk})$....... h-curvature, $P^2 = (P^i_{hjk})$....... hv-curvature
$S^2 = (S^i_{hjk})$....... v-curvature

and five kinds of torsion tensors:

$T = (T^i_{jk})$.......(h)h-torsion, $R^1 = (R^i_{jk})$.......(v)h-torsion,
$V = (V^i_{jk})$.......(h)hv-torsion, $P^1 = (P^i_{jk})$.......(v)hv-torsion
$S^1 = (S^i_{jk})$.......(v)v-torsion

It is noted that the v-connection (V^i_{jk}) also plays a role of torsion tensor and

$$T^i_{jk} = F^i_{jk} - F^i_{kj}, \qquad P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}, \qquad S^i_{jk} = V^i_{jk} - V^i_{kj}$$

It should be also noteworthy that if the desirable conditions, $y^i_{|j} = 0$ and $y^i|_j = \delta^i_j$ as above mentioned, are satisfied, then we have

$$R^i_{jk} = R^i_{0jk}, \qquad P^i_{jk} = P^i_{0jk}, \qquad S^i_{jk} = S^i_{0jk}$$

A fundamental function or a Finsler metric is a scalar field L(x,y)which satisfies the following three conditions:

(1) It is defined and differentiable for any point of $TM^n - (0)$.

(2) It is positively homogeneous of first degree in y^i, that is, L(x,py)=pL(x,y) for any positive number p.

(3) It is regular, that is,

$$g_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j \left(\frac{L^2}{2}\right)$$

constitute the regular matrix (g_{ij}).

The inverse matrix of (g_{ij}) is indicated by (g^{ij}). The homogeneity condition (2) enables us to consider the integral

$$s = \int_a^b L(x, \frac{dx}{dt})dt$$

along an arc $x^i = x^i(t)$, independently of the choice of parameter t except the orientation. The manifold M^n equipped with a fundamental function L(x,y) is called a Finsler space $F^n = (M^n, L)$ and the s is called the length of the arc. Thus the following two conditions are desirable for L(x,y) from the geometrical point of view:

(4) It is positive-valued for any point $TM^n - (0)$.

(5) $g_{ij}(x, y)$ define a positive-definite quadratic form.

Here we have to remark that there are some cases where the conditions (1), (4) and (5) should be restricted to some domain of $TM^n - (0)$.

The value L(x,y) is called the length of the tangent vector y at a point x. We get $L^2(x, y) = g_{ij}(x, y)y^i y^j$. The set $(\frac{y}{L(x,y)} = 1)$ in the tangent space at x or geometrically the set of all the end points of such y is called the indicatrix at x. If we have an equation f(x,y)=0 of the indicatrix at x, then the fundamental function L is defined by $(\frac{f(x,y)}{L}) = 0$.

The tensor g_{ij} is called the fundamental tensor. From L we get two other important tensors:

$$l_i = \dot{\partial}_i L, \qquad h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$$

The former is called the normalized supporting element, because $l^i = g^{ir}l_r$ is written as $\frac{y^i}{L(x,y)}$ and satisfies $L(x, y) = 1$. The latter is called the angular metric tensor. It satisfies $h_{ij}y^j = 0$ and the rank of (h_{ij}) is equal to $n - 1$.

In the theory of Finsler spaces two Finsler connections $B\Gamma$ and $C\Gamma$ have been important from the beginning.

The Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ is uniquely determined from function L(x,y) of F^n by the following five axioms:-

(B1) $L_{|i} = 0$, i.e. L-metrical

(B2) (h)h-torsion: $T^i_{jk} = 0$

(B3) Deflection: $D^i_j = 0$

(B4) (v)hv-torsion: $P^i_{jk} = 0$

(B5) (h)hv-torsion: $C^i_{jk} = 0$

The Cartan connection $C\Gamma = (\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$ is uniquely determined from function L(x,y) of F^n by the following five axioms:-

(C1) $g_{ij|k} = 0$ i.e. h-metrical

(C2) (h)h-torsion: $T^i_{jk} = 0$

(C3) Deflection tensor field $D^i_j = 0$

(C4) $g_{ij}|_k = 0$, i.e. v-metrical

(C5) (h)h-torsion: $S^i_{jk} = 0$

2 Introduction

M. Matsumoto in the year 1979 [5], introduce the concept of cubic metric on a differentiable manifold with the local co-ordinate x^i, defined by

$$L(x, y) = (a_{ijk}(x)y^i y^j y^k)^{\frac{1}{3}} \tag{5}$$

where, $a_{ijk}(x)$ are components of a symmetric tensor field of (0,3)-type depending on the position x alone, and a Finsler space with a cubic metric is called the *cubic Finsler space.*

We have had few paper studing cubic Finsler spaces [7, 2, 9, 10, 11, 12] although there are various papers on geometry of spaces with a cubic metric as a generalization of Euclidean or Riemannian geometry.

In paper [8] concerned with the unified field theory of gravitation and electromagnetism Randers wrote:

"Perhaps the most characteristic property of the physical world is the unidirection of time-like intervals. Since there is no obvious reason why this asymmetry should disappear in the mathematical description it is of interest to consider the possibility of a metric with asymmetrical property.

It is known that many reasons speak for the necessity of a quadratic indicatrix. The only way of introducing an asymmetry while retaining the quadratic indicatrix is to displace the center of the indicatrix. In other words, we adopt as indicatrix an eccentric quadratic hypersurface. This involves the definition of a vector at each point of the space, determining the displacement of the centre of the indicatrix. The formula for the lenght ds of a line-element dx^i must necessarily be homogeneous of first degree in dx^i. The simplest "eccentric" line-element possessing this property, and of course being invariant, is

$$ds = (a_{ij}(x)dx^i dx^j)^{\frac{1}{2}} + b_i(x)dx^i \tag{6}$$

where, $a_{ij}(x)$ is the fundamental tensor of the Riemannian affine connection, and b_i is a covariant vector determining the displacement of the centre of the indicatrix."

In present paper, we shall study Finsler space with the fundamental function $L(\gamma, \beta)$ on the lines of the Finsler space with the (α, β)-metric as studied by M. Matsumoto in paper [12]. In Section 3 of the paper, we have work out some basic tensors namely h_{ij}, g_{ij}, C_{ijk} and g^{ij} and work out four propositions and one theorem. Later on in the Section 4, we have workout again three propositions and one theorem regarding the Finsler space with (γ, β)-metric.

3 Basic tensors of (γ, β)-metric

Definition: A Finsler metric L(x,y) is called a (γ, β)-metric, when L is positively homogeneous function $L(\gamma, \beta)$ of first degree in two variables, γ and β, where $\gamma^3 = a_{ijk}(x)y^i y^j y^k$ is cubic metric and $\beta = b_i(x)y^i$ is a one form metric [4].

Throughout the present paper, the following notations are adopted

$$a_{ijk}(x)y^j y^k = a_i, \;\; 2a_{ijk}y^k = a_{ij}, \;\; a^{ij}b_j = B^i, \;\; a^{ij}a_j = a^i$$

where, (a^{ij}) is the inverse matrix of (a_{ij}). Further, subscripts γ, β denote partial differentiations with respect to γ, β respectively.

As for a (γ, β)-metric,

$$L = L(\gamma, \beta) \tag{7}$$

Differentiating (7), with respect to y^i, we get

$$l_i = \dot{\partial}_i L = \frac{L_\gamma}{\gamma^2} a_i + L_\beta b_i \tag{8}$$

Equation (8) can also be written as

$$y_i = L l_i = L(\dot{\partial}_i L) = \frac{L L_\gamma}{\gamma^2} a_i + L L_\beta b_i \tag{9}$$

Again differentiating (8), with respect to, y^j, we have the angular metric tensor $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$, as

$$h_{ij} = p_{-1}a_{ij} + q_0 b_i b_j + q_{-2}(a_i b_j + a_j b_i) + q_{-4} a_i a_j \tag{10}$$

where,

$$(10)' \qquad p_{-1} = \tfrac{L L_\gamma}{\gamma^2}, \qquad q_0 = L L_{\beta\beta}, \qquad q_{-2} = \tfrac{L L_{\gamma\beta}}{\gamma^2}, \qquad q_{-4} = \tfrac{L}{\gamma^4}(L_{\gamma\gamma} - \tfrac{2L_\gamma}{\gamma})$$

Owing to the homogenity or $h_{ij}y^j = 0$, we have two identities,

$$(p_{-1} + q_{-2}\beta + q_{-4}\gamma^3)a_i + (q_0\beta + q_{-2}\gamma^3)b_i = 0$$

Since a_i and b_i are two independent vector fields, hence, we must have

$$\begin{cases} p_{-1} + q_{-2}\beta + q_{-4}\gamma^3 = 0 \\ q_0\beta + q_{-2}\gamma^3 = 0 \end{cases} \tag{11}$$

Remark: In (10) the subscripts of coefficients $p_{-1}, q_0, q_{-2}, q_{-4}$ are used to indicate respective degrees of homogeneity.

Again,

$$g_{ij} = h_{ij} + l_il_j = p_{-1}a_{ij} + p_0b_ib_j + p_{-2}(a_ib_j + a_jb_i) + p_{-4}a_ia_j \tag{12}$$

where,

$$(12)' \qquad p_0 = q_0 + L_\beta^2, \qquad p_{-2} = q_{-2} + \frac{L_\gamma L_\beta}{\gamma^2}, \qquad p_{-4} = q_{-4} + \frac{L_\gamma^2}{\gamma^4}$$

Using (11) and (12)', we get

$$\begin{cases} p_0\beta + p_{-2}\gamma^3 = LL_\beta \\ p_{-2}\beta + p_{-4}\gamma^3 = 0 \end{cases} \tag{13}$$

It is well known that

Proposition 1:[3] Let a non-singular symmetric n-matrix (A_{ij}) and n quantities c_i be given, and put $B_{ij} = A_{ij} + c_ic_j$. The inverse matrix (B^{ij}) of (B_{ij}) and the $\det(B_{ij})$ are given by,

$$B^{ij} = A^{ij} - \frac{1}{(1+c^2)}c^ic^j, \qquad det(B_{ij}) = A(1 + c^2)$$

where, (A^{ij}) is the inverse matrix of (A_{ij}), $A = det(A_{ij})$, $c^i = A^{ij}c_j$, and $c^2 = c^ic_i$.

From (12), the components g_{ij} may be written as,

$$g_{ij} = p_{-1}a_{ij} + c_ic_j + d_id_j$$

where, we put,

$$c_i = \pi b_i, \qquad d_i = \pi_0b_i + \pi_{-2}a_i$$
$$\pi^2 + \pi_0^2 = p_0, \qquad \pi_0\pi_{-2} = p_{-2}, \qquad \pi_{-2}^2 = p_{-4}$$

Then putting, $B_{ij} = p_{-1}a_{ij} + c_ic_j$, then we have, $g_{ij} = B_{ij} + d_id_j$

From, defintion of B^{ij}, we have $B_{ij}B^{jk} = \delta_i^k$

Then,

$$B^{ij} = \frac{1}{p_{-1}}(a^{ij} - \frac{c^i c^j}{p_{-1}+c^2})$$

where, a^{ij} is reciprocal of a_{ij}, $c^i = a^{ij}c_j$, and $c^2 = c^i c_i$ Now, by using Proposition 1, we have

$$g^{ij} = B^{ij} - \frac{d^i d^j}{1+d^2}$$

where, $d^i = B^{ij}d_j$, $d^i d_i = d^2$

$$|g_{ij}| = |B_{ij}|(1+d^2) = |p_{-1}a_{ij}|\frac{(p_{-1}+c^2)}{p_1}(1+d^2) = p_{-1}^{n-1}a(p_{-1}+c^2)(1+d^2)$$

where a is the determinant of a_{ij}.

$$g^{ij} = \frac{1}{p_{-1}}a^{ij} - \frac{c^i c^j}{p_{-1}(p_{-1}+c^2)} - \frac{d^i d^j}{1+d^2}$$

Now,

$$d^i = B^{ij}d_j = \frac{1}{p_{-1}}[\frac{(\pi_0 p_{-1} - \pi^2\pi_{-2}\bar{a})}{(p_{-1}+c^2)}B^i + \pi_{-2}a^i]$$

where, $B^i b_i = b^2 = a^{im}b_m b_i$, $a_i B^i = a^{im}a_i b_m = a^i b_i = \bar{a}$, $\pi^2 b^2 = c^2$

Again,

$$d^i d^j = \frac{1}{p_{-1}^2}[\frac{(\pi_0 p_{-1} - \pi^2\pi_{-2}\bar{a})^2}{(p_{-1}+c^2)^2}B^i B^j + \frac{(\pi_0 p_{-1} - \pi^2\pi_{-2}\bar{a})}{(p_{-1}+c^2)}\pi_{-2}(B^i a^j + a^i B^j) + \pi_{-2}^2 a^i a^j]$$

$$\text{or} \quad d^i d^j = \frac{1}{p_{-1}^2}[\frac{(\pi_0 p_{-1} - \pi^2\pi_{-2}\bar{a})^2}{(p_{-1}+c^2)^2}B^i B^j + \frac{(p_{-2}p_{-1} - \pi^2 p_{-4}\bar{a})}{(p_{-1}+c^2)}(B^i a^j + a^i B^j) + p_{-4}a^i a^j]$$

Now,

$$d^2 = d_i d^i = \frac{1}{p_{-1}(p_{-1}+c^2)}[\pi_0^2 b^2 p_{-1} + 2p_{-1}p_{-2}\bar{a} - p_{-2}^2\bar{a}^2 + p_{-4}p_{-1}a^2 + p_{-4}c^2 a^2]$$

Again,

$$|g_{ij}| = p_{-1}^{n-1}a(p_{-1}+c^2)(1+d^2) = p_{-1}^{n-1}a\tau_{-2}$$

where, $\tau_{-2} = p_{-1}(p_{-1} + p_0 b^2 + p_{-2}\bar{a}) + (p_{-2}p_{-1}\bar{a} - p_{-2}^2\bar{a}^2) + p_{-4}p_{-1}a^2 + p_{-4}c^2 a^2$. Thus, the reciprocal of (g_{ij}) is given by

$$g^{ij} = \frac{1}{p_{-1}}a^{ij} - s_2 B^i B^j - s_0(B^i a^j + B^j a^i) - s_{-2}a^i a^j \tag{14}$$

where,

$$(14)' \qquad s_2 = \frac{\pi_0 p_{-1}^2 + \pi^2(\tau_{-2} + \pi^2 p_{-4}\bar{a}^2 - 2p_{-2}\bar{a})}{\tau_{-2}p_{-1}(p_{-1}+c^2)}, \qquad s_0 = \frac{p_{-2}p_{-1} - \pi^2 p_{-4}\bar{a}}{\tau_{-2}p_{-1}}$$

$$s_{-2} = \frac{p_{-1}p_{-4} + c^2 p_{-4}}{\tau_{-2}p_{-1}}$$

Differentiating (12) by y^k, we get,

$$2C_{ijk} = 2p_{-1}a_{ijk} + p_{0\beta}b_ib_jb_k + \Pi_{(ijk)}(P_ia_{jk} + p_{-2\beta}a_ib_jb_k + \frac{p_{-2\gamma}}{\gamma^2}a_ia_jb_k) + \frac{p_{-4\gamma}}{\gamma^2}a_ia_ja_k \tag{15}$$

where, $\Pi_{(ijk)}$ represents the sum of cyclic permutation of i, j, k.

$$P_i = p_{-4}a_i + p_{-2}b_i$$

or,

$$(15a) \qquad 2p_{-1}C_{ijk} = 2p_{-1}^2a_{ijk} + p_{0\beta}b_ib_jb_k + \Pi_{(ijk)}(P_ih_{jk} + r_{-4}a_ib_jb_k + r_{-6}a_ia_jb_k) + r_{-8}a_ia_ja_k$$

where,

$$(15a)' \qquad r_{-2} = p_{-1}p_{0\beta} - 3p_{-2}q_0, \qquad r_{-4} = p_{-1}p_{-2\beta} - q_0p_{-4} - 2p_{-2}q_{-2}$$
$$r_{-6} = p_{-1}p_{-4\beta} - 2p_{-4}q_{-2} - p_{-2}q_{-4}, \qquad r_{-8} = p_{-1}\frac{p_{-4\gamma}}{\gamma^2} - 3p_{-4}q_{-4}$$

Proposition 2: The normalized supporting element l_i, angular metric tensor h_{ij}, metric tensor g_{ij} and (h)hv-torsion tensor C_{ijk} of Finsler space with (γ, β)-metric are given by (8), (10), (12) and (15a) respectively.

Proposition 3: The reciprocal of the metric tensor g_{ij} of (γ, β)-metric is given by (14).

Proposition 4: The coefficients $r_{-8}, r_{-6}, r_4, r_{-2}$ defined in $(15a)'$ satisfy the following relation

$$r_{-\mu}\beta + r_{-\mu-2}\gamma^3 = 0, \qquad \mu = 2, 4, 6. \tag{16}$$

Proof: Using (11), (13) and $(15a)'$, we easily get the relation (16).

Now, from (13) and (16), we have

$$p_{-4} = \phi p_{-2}, \qquad r_{-\mu-2} = \phi^{\frac{\mu}{2}} r_{-2}, \qquad \mu = 2, 4, 6. \tag{17}$$

where, $\phi = -\frac{\beta}{\gamma^3}$.

Using relation (17) in (15a), we easily get

$$(15b) \qquad 2p_{-1}C_{ijk} = 2p_{-1}^2a_{ijk} + \Pi_{(ijk)}(H_{jk}P_i)$$

where,

$$H_{ij} = h_{ij} + \frac{r_{-2}}{3p_{-2}^3}P_iP_j$$

Further, by direct computation from C_{ijk} and g^{ij}, we have,

$$C_i = p_{-1}a_{ijk}g^{jk} + Aa_i + Bb_i \tag{18}$$

where A and B are certain scalar.

If the Finsler space (F^n) is C-reducible, then

$$C_{ijk} = \frac{1}{(n+1)}\Pi_{(ijk)}(h_{ij}C_k) \tag{19}$$

from (15b) and (19), it follows that,

$$a_{ijk} + \frac{r_{-2}}{2p_{-1}^2 p_{-2}^3}P_iP_jP_k = \Pi_{(ijk)}(h_{ij}N_k) \tag{20}$$

where, $N_k = \frac{2p_{-1}}{(n+1)}C_k - P_k$ Conversely, if (20) is satisfied for certain covariant vector N_k, then from (15b) we have

$$2p_{-1}C_{ijk} = \Pi_{(ijk)}(h_{ij}(P_k + N_k)) \tag{21}$$

which gives (19). Thus, we have,

Theorem 1: A Finsler space with (γ, β)-metric is C-reducile iff equation (20) holds.

4 Important tensors of (γ, β)-metric

It follow from (15b) and (14), that the components C^i_{jk} of the (h)hv-torsion tensor $C\Gamma$ are given by,

$$\begin{aligned} 2p_{-1}C^i_{jk} = 2p_{-1}^2 a^i_{jk} + (\delta^i_j P_k - l^i l_j P_k) + (\delta^i_k P_j - l^i l_k P_j) \\ + \frac{r_{-2}}{p_{-2}^3}P^iP_jP_k + h_{jk}P^i \end{aligned} \tag{22}$$

where, $P_i g^{ik} = P^j, \qquad g^{ij}l_j = l^i, \qquad a_{rjk}g^{ri} = a^i_{jk}$

Again from (15b) and (22), we have,

$$4p_{-1}^2 C_{hk}^r C_{rij} = 4p_{-1}^4 a_{hk}^r a_{rij} + 2p_{-1}^2 \Pi_{(ijk)}(a_{ijk}P_h) - \frac{r_{-2}\bar{P}}{p_{-2}^3 L^2}\Pi_{(ijk)}(P_iP_jP_kl_h) - \frac{2p_{-1}^2}{L}[a_{ij}(l_hP_k + l_kP_h)] + \frac{2p_{-1}^2 r_{-2}}{p_{-2}^3} a_{rij}P^rP_hP_k + \frac{2p_{-1}^2 r_{-2}}{p_{-2}^3} a_{hk}^r P_rP_iP_j + 2p_{-1}^2 a_{rij}P^r h_{hk} + 2p_{-1}^2 a_{hk}^r P_r h_{ij} - 2p_{-1}^2[a_{hk}^r l_r(l_iP_j + l_jP_i)] + [h_{ih}P_jP_k + h_{ik}P_hP_j + h_{hk}P_iP_j + h_{hj}P_kP_i + h_{jk}P_iP_h + h_{hk}P_iP_j + h_{ij}P_kP_h + h_{ij}P_kP_h] + \frac{r_{-2}}{p_{-2}^3}\frac{P^2}{L^4}[h_{ij} + \frac{r_{-2}}{p_{-2}^3}P_iP_j]P_kP_h - \frac{\bar{P}}{L^2}h_{ij}(l_hP_k + l_kP_h) - \frac{\bar{P}}{L^2}h_{hk}(l_iP_j + l_jP_i) + \frac{r_{-2}}{p_{-2}^3}\frac{P^2}{L^4}h_{hk}P_iP_j + \frac{4r_{-2}}{p_{-2}^3}P_iP_jP_kP_h. \tag{23}$$

where, $a_{irk}l^r = \frac{a_{ik}}{L}$, $P^rl_r = \frac{\bar{P}}{L^2}$, $P^rP_r = \frac{P^2}{L^4}$, $P^rg_{ir} = P_i$, $\delta_i^r a_{rhk} = a_{ihk}$.

From (23), the v-curvature tensor S_{hijk} of $C\Gamma$ is written as,

$$4p_{-1}^2 S_{hijk} = 4p_{-1}^2 \Theta_{(jk)}(C_{hk}^r C_{rij})$$

where, $\Theta_{(jk)}$ anti-symmetric with respect to indecies j and k.

Thus,

$$4p_{-1}^2 S_{hijk} = \Theta_{(jk)}[4p_{-1}^4 a_{hk}^r a_{rij} + 2p_{-1}^2(a_{rij}P_rH_{hk} + a_{hk}^r P_r H_{ij}) - (l_hP_k + l_kP_h)A_{ij} - (l_iP_j + l_jP_i)A_{hk} + H'_{ij}P_hP_k + H'_{hk}P_iP_j] \tag{24}$$

where,

$$A_{ij} = 2p_{-1}^2 a_{ij} - \tfrac{\bar{P}}{L^2}h_{ij}$$
$$H'_{ij} = 2p_{-1}^2 \tfrac{2r_{-2}}{3p_{-2}^3} a_{rij}P_r + (1 + \tfrac{P^2}{L^4})h_{ij}$$

Proposition 5: The v-curvature tensor of a Finsler space with (γ, β)-metric is given by (24).

Next, h- and v-covariant derivatives $X_{i|j}, X_i|_j$ of a covariant vector field X_i with respect to the Cartan connection $C\Gamma$ are defined by,

$$X_{i|j} = \partial_j X_i - (\dot{\partial}_r X_i)N_j^r - X_r F_{ij}^r$$
$$X_i|_j = \dot{\partial}_j X_i - X_r C_{ij}^r$$

where, $(F^i_{jk}, N^i_j(=F^i_{0j}), C^i_{jk})$ are connection coefficients of $C\Gamma$ and suffix '0' means the contraction by supporting element y^i [3, 1].

If $b_{i|h} = 0$, then for $L(\gamma, \beta)$-metric, we have,

$$a_{i|j} = 0 \qquad a_{ij|k} = 0 \tag{25}$$

because, $l_{i|j} = 0$ and $h_{ij|k} = 0$. Then, the h-covariant differentiation of (15b), we have,

$$C_{ijk|h} = p_{-1} a_{ijk|h} \tag{26}$$

Therefore, the v(hv)-torsion tensor P_{ijk} is written as,

$$P_{ijk} = C_{ijk|h} y^h = C_{jk|0} = p_{-1} a_{ijk|0} \tag{27}$$

Definition : [3] A Finsler space is called a Berwald space, if tensor $C_{ijk|h}$ vanishes identically and called a Landsberg space if $C_{ijk|0}$ vanishes identically.

Theorem 2: If b_i is h-covarintly constant (resp. $b_{i|0}$), then a Finsler space with (γ, β)-metric is a Berwald space (resp. Landsberg space) iff the tensor $a_{ijk|h}$ (resp. $a_{ijk|0}$) vanishes identically.

Now, the hv-curvature tensor P_{hijk} [3, 1] is given by,

$$P_{hijk} = \Theta_{(hi)}(C_{ijk|h} + C^r_{hj} C_{rik|0})$$

Now,

$$C^r_{hj} C_{rik|0} = p^2_{-1} a^r_{hj} a_{rik|0} + \tfrac{1}{2} a_{hik|0} P_j - \tfrac{1}{2L} a_{ik|0}(l_h P_j + l_j P_h) + \tfrac{1}{2} a_{jik|0} P_h + \tfrac{r_{-2}}{2p^3_{-2}} a_{rik|0} P^r P_j P_h + \tfrac{1}{2} h_{jh} P^r a_{rik|0}$$

Thus,

$$\begin{aligned} P_{hijk} = \Theta_{(hi)}[a_{ijk|h} + \frac{1}{2} a_{ijk|0} P_h - \frac{1}{2L} a_{ik|0}(l_h P_j + l_j P_h) \\ + a_{rik|0} P^r H_{jh} + a_{rik|0} A^r_{hj}] \end{aligned} \tag{28}$$

where, $A^r_{hj} = p^2_{-1} a^r_{hj} + \frac{1}{b} \frac{r_{-2}}{p^3_{-2}} P^r P_j P_h$.

Proposition 6: The (v)hv-torsion tensor P_{ijk} and hv-curvature tensor P_{hijk} for (γ, β)-metric is given by (27) and (28) respectively.

Now, the T-tensor is given by [3, 1]

$$T_{hijk} = LC_{hij}|_k + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij} + l_h C_{ijk}$$

Now, the v-derivative of C_{hij} is given by

$$
\begin{aligned}
2p_{-1}C_{hij}|_k = 2p^2_{-1}|_k a_{hij} - 2p_{-1}|_k C_{hij} - \frac{1}{L}[h_{jk}(l_iP_h + l_hP_i) + \\
h_{ik}(l_jP_h + l_hP_j) + h_{hk}(l_iP_j + l_jP_i)] + \Pi_{(hij)}(H_{hi}P_j|_k) + \\
\frac{2r_{-2}}{3p^3_{-2}}\Pi_{(hij)}(P_hP_iP_j|_k)
\end{aligned}
\tag{29}
$$

Using (8), (15b) and (29), the T-tensor for (γ, β)-metric is given by

$$
\begin{aligned}
T_{hijk} = \frac{1}{2P_{-1}}[2p^2_{-1}|_k La_{hij} - 2p_{-1}|_k LC_{hij} - [h_{jk}(l_iP_h + l_hP_i) + \\
h_{ik}(l_jP_h + l_hP_j) + h_{hk}(l_iP_j + l_jP_i)] + L\Pi_{(hij)}(H_{hi}P_j|_k) + \\
\frac{2r_{-2}}{3p^3_{-2}}L\Pi_{(hij)}(P_hP_iP_j|_k) + p^2_{-1}\Pi_{(hijk)}(l_ha_{ijk}) \\
+\Pi_{(hijk)}(H_{hi}(l_jP_k + l_kP_j))]
\end{aligned}
\tag{30}
$$

Thus,

Proposition 7: The T-tensor T_{hijk} for (γ, β)-metric is given by (30).

Acknowledgements: *The authors would like to express their sincere gratitude to Prof. B. N. Prasad for their invaluable suggestions and criticism also.*

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