JET FINSLER-LIKE GEOMETRY FOR THE x-DEPENDENT CONFORMAL DEFORMATION OF AN ONE-PARAMETER FAMILY OF BERWALD-MOÓR METRICS OF ORDER FOUR

Mircea NEAGU[1]

Abstract

The aim of this paper is to develop on the 1-jet space $J^1(\mathbb{R}, M^4)$ the Finsler-like geometry (in the sense of distinguished (d-) connection, d-torsions, d-curvatures and some gravitational-like and electromagnetic-like geometrical models) for the x-conformal deformed rheonomic Berwald-Moór metric of order four.

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1 Introduction

The geometric-physical Berwald-Moór structure ([6], [13], [12]) was intensively investigated by P.K. Rashevski [18] and further fundamented and developed by D.G. Pavlov, G.I. Garas'ko and S.S. Kokarev ([15], [16], [9], [17]). At the same time, the physical studies of Asanov [1] or Garas'ko and Pavlov [9] emphasize the importance of the Finsler geometry characterized by the total equality in rights of all non-isotropic directions, in the theory of space-time structure, gravitation and electromagnetism. For such a reason, one underlines the important role played by the Berwald-Moór metric

$$F : TM \to \mathbb{R}, \qquad F(y) = \sqrt[n]{y^1 y^2 ... y^n},$$

whose tangent Finslerian geometry is studied by the geometers Matsumoto and Shimada [10] or Balan [3]. In such a perspective, according to the recent geometric-physical ideas proposed by Garas'ko in [8] and [7], we consider that a Finsler-like geometric-physical study for x-dependent conformal deformations of the jet Berwald-Moór structure is required. Note that, based on the works of Matsumoto and Shimada, Balan and Nicola [5] already presented the equation of motion in the x-conformally deformed 4-dimensional Berwald-Moór framework on tangent spaces.

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[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: mircea.neagu@unitbv.ro

In such a geometrical and physical context, this paper investigates on the 1-jet space $J^1(\mathbb{R}, M^4)$ the Finsler-like geometry (together with a theoretical-geometric gravitational field-like theory) of the *x-conformal deformed rheonomic Berwald-Moór metric of order four*[2]

$$\overset{*}{F}(t, x, y) = e^{\sigma(x)}\sqrt{h^{11}(t)}\left[y_1^1 y_1^2 y_1^3 y_1^4\right]^{1/4}, \tag{1}$$

where $\sigma(x)$ is a smooth non-constant function on M^4, $h^{11}(t)$ is the dual of a Riemannian metric $h_{11}(t)$ on $\mathbb{R}$, and

$$(t, x, y) = (t, x^1, x^2, x^3, x^4, y_1^1, y_1^2, y_1^3, y_1^4)$$

are the coordinates of the 1-jet space $J^1(\mathbb{R}, M^4)$, which transform by the rules (the Einstein convention of summation is assumed everywhere):

$$\widetilde{t} = \widetilde{t}(t), \quad \widetilde{x}^i = \widetilde{x}^i(x^j), \quad \widetilde{y}_1^i = \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{dt}{d\widetilde{t}} \cdot y_1^j, \tag{2}$$

where $i, j = \overline{1, 4}$, rank $(\partial \widetilde{x}^i / \partial x^j) = 4$ and $d\widetilde{t}/dt \neq 0$. It is important to note that, based on the geometrical ideas promoted by Miron and Anastasiei in the classical Lagrangian geometry of tangent bundles [11], together with those used by Asanov in the geometry of 1-jet spaces [2], the differential geometry (in the sense of d-connections, d-torsions, d-curvatures, gravitational and electromagnetic geometrical theories) produced by a jet rheonomic Lagrangian function $L : J^1(\mathbb{R}, M^n) \to \mathbb{R}$ is now completely done in the monograph [4]. In what follows, we apply the general geometrical results from [4] to the *x*-conformal deformed rheonomic Berwald-Moór metric of order four (1).

2   The canonical nonlinear connection

Let us rewrite the *x*-conformal deformed rheonomic Berwald-Moór metric of order four (1) in the form
$$\overset{*}{F}(t, x, y) = e^{\sigma(x)}F(t, y),$$

where
$$F(t, y) = \sqrt{h^{11}(t)} \cdot \left[y_1^1 y_1^2 y_1^3 y_1^4\right]^{1/4}$$

can be regarded as a *t*-parameter family of Berwald-Moór metrics of order four. Hereinafter, using the notation $G_{1111} := y_1^1 y_1^2 y_1^3 y_1^4$, the *fundamental metrical d-tensor* produced by the metric (1) is given by the formula

$$\overset{*}{g}_{ij}(t, x, y) = \frac{h_{11}(t)}{2}\frac{\partial^2 \overset{*}{F}{}^2}{\partial y_1^i \partial y_1^j} = e^{2\sigma(x)}g_{ij}(y), \tag{3}$$

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[2]We assume that $y_1^1 y_1^2 y_1^3 y_1^4 > 0$. This is the domain where we can *y*-differentiate the function $\overset{*}{F}(t, x, y)$.

where[3] (see [4])

$$g_{ij} = \frac{h_{11}(t)}{2} \frac{\partial^2 F^2}{\partial y_1^i \partial y_1^j} = \frac{(1 - 2\delta_{ij})}{8} \frac{G_{1111}^{1/2}}{y_1^i y_1^j} \text{ (no sum by } i \text{ or } j\text{).} \tag{4}$$

Moreover, the matrix $\overset{*}{g} = (\overset{*}{g}_{ij})$ admits the inverse $\overset{*}{g}^{-1} = (\overset{*}{g}^{jk})$, whose entries are

$$\overset{*}{g}^{jk} = 2e^{-2\sigma(x)}(1 - 2\delta^{jk})G_{1111}^{-1/2}y_1^j y_1^k \text{ (no sum by } j \text{ or } k\text{).} \tag{5}$$

Let us consider that the Christoffel symbol of the Riemannian metric $h_{11}(t)$ on $\mathbb{R}$ is

$$\kappa_{11}^1 = \frac{h^{11}}{2} \frac{dh_{11}}{dt},$$

where $h^{11} = 1/h_{11} > 0$. Then, using a general formula from [4] and taking into account that we have

$$\frac{\partial G_{1111}}{\partial y_1^i} = \frac{G_{1111}}{y_1^i},$$

we find the following geometrical result:

Proposition 1. *For the x-conformal deformed rheonomic Berwald-Moór metric of order four (1), the energy action functional*

$$\overset{*}{\mathbf{E}}(t, x(t)) = \int_a^b \overset{*}{F^2}(t, x, y)\sqrt{h_{11}}dt = \int_a^b e^{2\sigma(x)} \left[y_1^1 y_1^2 y_1^3 y_1^4 \right]^{1/2} \cdot h^{11}\sqrt{h_{11}}dt,$$

*where $y = dx/dt$, produces on the 1-jet space $J^1(\mathbb{R}, M^4)$ the **canonical nonlinear connection***

$$\overset{*}{\Gamma} = \left(M_{(1)1}^{(i)} = -\kappa_{11}^1 y_1^i, \; N_{(1)j}^{(i)} = 4\sigma_i y_1^i \delta_j^i \right), \tag{6}$$

where $\sigma_i = \partial\sigma/\partial x^i$.

Proof. For the energy action functional $\overset{*}{\mathbf{E}}$, the associated Euler-Lagrange equations can be written in the equivalent form (see [4])

$$\frac{d^2 x^i}{dt^2} + 2H_{(1)1}^{(i)}\left(t, x^k, y_1^k\right) + 2G_{(1)1}^{(i)}\left(t, x^k, y_1^k\right) = 0, \tag{7}$$

where the local components

$$H_{(1)1}^{(i)} \overset{def}{=} -\frac{1}{2}\kappa_{11}^1(t)y_1^i$$

and

$$G_{(1)1}^{(i)} \overset{def}{=} \frac{h_{11}\overset{*}{g}^{ip}}{4} \left[\frac{\partial^2 \overset{*}{F^2}}{\partial x^r \partial y_1^p}y_1^r - \frac{\partial \overset{*}{F^2}}{\partial x^p} + \frac{\partial^2 \overset{*}{F^2}}{\partial t \partial y_1^p} + \right.$$

$$\left. + \frac{\partial \overset{*}{F^2}}{\partial y_1^p}\kappa_{11}^1(t) + 2h^{11}\kappa_{11}^1\overset{*}{g}_{pr}y_1^r \right] = 2\sigma_i\left(y_1^i\right)^2$$

---

[3]Throughout this paper the Latin letters i, j, k, m, r, ... take values in the set $\{1, 2, 3, 4\}$.

represent, from a geometrical point of view, a *spray* on the 1-jet space $J^1(\mathbb{R}, M^4)$. The *canonical nonlinear connection* associated to this spray has the components (see [4])

$$M_{(1)1}^{(i)} \overset{def}{=} 2H_{(1)1}^{(i)} = -\kappa_{11}^1 y_1^i, \quad N_{(1)j}^{(i)} \overset{def}{=} \frac{\partial G_{(1)1}^{(i)}}{\partial y_1^j} = 4\sigma_i y_1^i \delta_j^i.$$

$\square$

3 The Cartan canonical $\overset{*}{\Gamma}$-linear connection. Its d-torsions and d-curvatures

The nonlinear connection (6) produces the dual *adapted bases* of d-vector fields (no sum by i)

$$\left\{ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \kappa_{11}^1 y_1^p \frac{\partial}{\partial y_1^p} \; ; \; \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - 4\sigma_i y_1^i \frac{\partial}{\partial y_1^i} \; ; \; \frac{\partial}{\partial y_1^i} \right\} \subset \mathcal{X}(E) \tag{8}$$

and d-covector fields (no sum by i)

$$\left\{ dt \; ; \; dx^i \; ; \; \delta y_1^i = dy_1^i - \kappa_{11}^1 y_1^i dt + 4\sigma_i y_1^i dx^i \right\} \subset \mathcal{X}^*(E), \tag{9}$$

where $E = J^1(\mathbb{R}, M^4)$. The naturalness of the geometrical adapted bases (8) and (9) is coming from the fact that, via a transformation of coordinates (2), their elements transform as the classical tensors. Therefore, the description of all subsequent geometrical objects on the 1-jet space $J^1(\mathbb{R}, M^4)$ (e.g., the Cartan canonical linear connection, its torsion and curvature etc.) will be done in local adapted components. Consequently, by direct computations, we obtain the following geometrical result:

Proposition 2. *The Cartan canonical $\overset{*}{\Gamma}$-linear connection, produced by the x-conformal deformed rheonomic Berwald-Moór metric of order four (1), has the following adapted local components (no sum by i, j or k):*

$$C\overset{*}{\Gamma} = \left(\kappa_{11}^1, \; G_{j1}^k = 0, \; L_{jk}^i = 4\delta_j^i \delta_k^i \sigma_i, \; C_{j(k)}^{i(1)} = \mathbf{C}_{jk}^i \cdot \frac{y_1^i}{y_1^j y_1^k} \right), \tag{10}$$

where

$$\mathbf{C}_{jk}^i = \frac{2\delta_j^i + 2\delta_k^i + 2\delta_{jk} - 8\delta_j^i \delta_{jk} - 1}{8} =$$

$$= \begin{cases} -\dfrac{1}{8}, & i \neq j \neq k \neq i \\[2mm] \dfrac{1}{8}, & i = j \neq k \; or \; i = k \neq j \; or \; j = k \neq i \\[2mm] -\dfrac{3}{8}, & i = j = k. \end{cases}$$

Proof. The adapted components of the Cartan canonical connection are given by the formulas (see [4])

$$G_{j1}^k \stackrel{def}{=} \frac{\overset{*}{g}{}^{km}}{2} \frac{\delta \overset{*}{g}_{mj}}{\delta t} = 0, \quad L_{jk}^i \stackrel{def}{=} \frac{\overset{*}{g}{}^{im}}{2} \left(\frac{\delta \overset{*}{g}_{jm}}{\delta x^k} + \frac{\delta \overset{*}{g}_{km}}{\delta x^j} - \frac{\delta \overset{*}{g}_{jk}}{\delta x^m} \right) = 4\delta_j^i \delta_k^i \sigma_i,$$

$$C_{j(k)}^{i(1)} \stackrel{def}{=} \frac{\overset{*}{g}{}^{im}}{2} \left(\frac{\partial \overset{*}{g}_{jm}}{\partial y_1^k} + \frac{\partial \overset{*}{g}_{km}}{\partial y_1^j} - \frac{\partial \overset{*}{g}_{jk}}{\partial y_1^m} \right) = \frac{\overset{*}{g}{}^{im}}{2} \frac{\partial \overset{*}{g}_{jk}}{\partial y_1^m}.$$

Using the derivative operators (8), the direct calculations lead us to the required results. Moreover, it is important to note that the vertical d-tensor $C_{j(k)}^{i(1)}$ also has the properties (see also [10] and [14]):

$$C_{j(k)}^{i(1)} = C_{k(j)}^{i(1)}, \quad C_{j(m)}^{i(1)} y_1^m = 0, \quad C_{j(m)}^{m(1)} = 0, \quad C_{i(k)|m}^{m(1)} = 0, \tag{11}$$

with sum by m, where

$$C_{i(k)|j}^{l(1)} \stackrel{def}{=} \frac{\delta C_{i(k)}^{l(1)}}{\delta x^j} + C_{i(k)}^{r(1)} L_{rj}^l - C_{r(k)}^{l(1)} L_{ij}^r - C_{i(r)}^{l(1)} L_{kj}^r.$$

□

Proposition 3. *The Cartan canonical connection of the x-conformal deformed rheonomic Berwald-Moór metric of order four (1) has **two** effective local torsion d-tensors:*

$$R_{(1)ij}^{(r)} = 4\left(\delta_i^r \sigma_{rj} - \delta_j^r \sigma_{ri} \right) y_1^r, \quad P_{i(j)}^{r(1)} = \frac{2\delta_i^r + 2\delta_j^r + 2\delta_{ij} - 8\delta_i^r \delta_{ij} - 1}{8} \cdot \frac{y_1^r}{y_1^i y_1^j},$$

where $\sigma_{pq} := \dfrac{\partial^2 \sigma}{\partial x^p \partial x^q}$.

Proof. Generally, an *h-normal* Γ-*linear connection* on the 1-jet space $J^1(\mathbb{R}, M^4)$ has *eight* effective local d-tensors of torsion (for more details, see [4]). For the Cartan canonical connection (10) these reduce only to *two* (the other six are zero):

$$R_{(1)ij}^{(r)} \stackrel{def}{=} \frac{\delta N_{(1)i}^{(r)}}{\delta x^j} - \frac{\delta N_{(1)j}^{(r)}}{\delta x^i}, \quad P_{i(j)}^{r(1)} \stackrel{def}{=} C_{i(j)}^{r(1)}.$$

□

Proposition 4. *The Cartan canonical connection of the x-conformal deformed rheonomic Berwald-Moór metric of order four (1) has **three** effective local curvature d-tensors:*

$$R_{ijk}^l = \frac{\partial L_{ij}^l}{\partial x^k} - \frac{\partial L_{ik}^l}{\partial x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l + C_{i(r)}^{l(1)} R_{(1)jk}^{(r)}, \quad P_{ij(k)}^{l\ (1)} = -C_{i(k)|j}^{l(1)},$$

$$S_{i(j)(k)}^{l(1)(1)} \stackrel{def}{=} \frac{\partial C_{i(j)}^{l(1)}}{\partial y_1^k} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y_1^j} + C_{i(j)}^{r(1)} C_{r(k)}^{l(1)} - C_{i(k)}^{r(1)} C_{r(j)}^{l(1)}.$$

Proof. Generally, an *h-normal* Γ-*linear connection* on the 1-jet space $J^1(\mathbb{R}, M^4)$ has *five* effective local d-tensors of curvature (for more details, see [4]). For the Cartan canonical connection (10) these reduce only to *three* (the other two are zero); these are $S^{l(1)(1)}_{i(j)(k)}$ and

$$R^l_{ijk} \overset{def}{=} \frac{\delta L^l_{ij}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta x^j} + L^r_{ij}L^l_{rk} - L^r_{ik}L^l_{rj} + C^{l(1)}_{i(r)}R^{(r)}_{(1)jk},$$

$$P^{l\ (1)}_{ij(k)} \overset{def}{=} \frac{\partial L^l_{ij}}{\partial y^k_1} - C^{l(1)}_{i(k)|j} + C^{l(1)}_{i(r)}P^{(r)\ (1)}_{(1)j(k)} = -C^{l(1)}_{i(k)|j},$$

where

$$P^{(r)\ (1)}_{(1)j(k)} \overset{def}{=} \frac{\partial N^{(r)}_{(1)j}}{\partial y^k_1} - L^r_{jk} = 0.$$

$\square$

4   From x-conformal deformations of the rheonomic Berwald-Moór metric of order four to field-like geometrical models

4.1   Gravitational-like geometrical model

The x-conformal deformed rheonomic Berwald-Moór metric (1) produces on the 1-jet space $J^1(\mathbb{R}, M^4)$ the adapted metrical d-tensor (sum by i and j)

$$\mathbf{G} = h_{11}dt \otimes dt + \overset{*}{g}_{ij}dx^i \otimes dx^j + h^{11}\overset{*}{g}_{ij}\delta y^i_1 \otimes \delta y^j_1, \tag{12}$$

where $\overset{*}{g}_{ij}$ is given by (3) and we have $\delta y^i_1 = dy^i_1 - \kappa^1_{11}y^i_1 dt + 4\sigma_i y^i_1 dx^i$ (no sum by i). From a physical point of view, the metrical d-tensor (12) may be regarded as a *"non-isotropic gravitational potential"*. In our geometric-physical approach, one postulates that the non-isotropic gravitational potential $\mathbf{G}$ is governed by the *geometrical Einstein equations*

$$\mathrm{Ric}\,(C\overset{*}{\Gamma}) - \frac{\mathrm{Sc}\,(C\overset{*}{\Gamma})}{2}\mathbf{G} = \mathcal{K}\mathcal{T}, \tag{13}$$

where

- $\mathrm{Ric}\,(C\overset{*}{\Gamma})$ is the *Ricci d-tensor* associated to the Cartan canonical connection (10);

- $\mathrm{Sc}\,(C\overset{*}{\Gamma})$ is the *scalar curvature*;

- $\mathcal{K}$ is the *Einstein constant* and $\mathcal{T}$ is the intrinsic *stress-energy* d-tensor of matter.

Therefore, using the adapted basis of vector fields (8), we can locally describe the global geometrical Einstein equations (13). Consequently, by some direct computations we find:

Lemma 1. *The Ricci d-tensor of the Cartan canonical connection $C\overset{*}{\Gamma}$ of the x-conformal deformed rheonomic Berwald-Moór metric of order four (1) has the following **two** effective local Ricci d-tensors (no sum by i, j, k or l):*

$$
R_{ij} = \begin{cases} -2\sigma_{ij} - \sigma_{jk}\dfrac{y_1^k}{y_1^i} - \sigma_{jl}\dfrac{y_1^l}{y_1^i}, & i \neq j, \ \{i,j,k,l\} = \{1,2,3,4\} \\[2ex] 0, & i = j, \end{cases}
\tag{14}
$$

$$
S_{(i)(j)}^{(1)(1)} = \frac{4\delta_{ij} - 1}{8} \frac{1}{y_1^i y_1^j}.
$$

Proof. Generally, the Ricci d-tensor of a Cartan canonical connection $C\Gamma$, produced by an arbitrary jet Lagrangian function, is determined by *six* effective local Ricci d-tensors (for more details, see [4]). For the Cartan canonical connection (10) these reduce only to the following *two* (the other four are zero):

$$
R_{ij} \overset{def}{=} R_{ijm}^m = \frac{\partial L_{ij}^m}{\partial x^m} - \frac{\partial L_{im}^m}{\partial x^j} + L_{ij}^r L_{rm}^m - L_{im}^r L_{rj}^m + C_{i(r)}^{m(1)} R_{(1)jm}^{(r)},
$$

$$
S_{(i)(j)}^{(1)(1)} \overset{def}{=} S_{(i)(j)(m)}^{m(1)(1)} = \frac{\partial C_{i(j)}^{m(1)}}{\partial y_1^m} - \frac{\partial C_{i(m)}^{m(1)}}{\partial y_1^j} + C_{i(j)}^{r(1)} C_{r(m)}^{m(1)} - C_{i(m)}^{r(1)} C_{r(j)}^{m(1)} =
$$

$$
= \frac{\partial C_{i(j)}^{m(1)}}{\partial y_1^m} - C_{i(m)}^{r(1)} C_{r(j)}^{m(1)},
$$

with sum by r and m. $\qquad\square$

Lemma 2. *The scalar curvature of the Cartan canonical connection $C\overset{*}{\Gamma}$ of the x-conformal deformed rheonomic Berwald-Moór metric of order four (1) has the value*

$$
\mathrm{Sc}\,(C\overset{*}{\Gamma}) = -2e^{-2\sigma}G_{1111}^{-1/2}\left(3h_{11} + 8Y_{11}\right),
$$

where

$$
Y_{11} = \sigma_{12}y_1^1 y_1^2 + \sigma_{13}y_1^1 y_1^3 + \sigma_{14}y_1^1 y_1^4 + \sigma_{23}y_1^2 y_1^3 + \sigma_{24}y_1^2 y_1^4 + \sigma_{34}y_1^3 y_1^4.
$$

Proof. The scalar curvature of the Cartan canonical connection (10) is given by (for more details, see [4])

$$
\mathrm{Sc}\,(C\overset{*}{\Gamma}) = \overset{*}{g}^{pq} R_{pq} + h_{11}\overset{*}{g}^{pq} S_{(p)(q)}^{(1)(1)} = -16e^{-2\sigma}G_{1111}^{-1/2}Y_{11} - 6e^{-2\sigma}G_{1111}^{-1/2}h_{11}.
$$

$\qquad\square$

The local description in the adapted basis of vector fields (8) of the global geometrical Einstein equations (13) is given by (for more details, see [4]):

Proposition 5. *The **geometrical Einstein equations** produced by the x-conformal deformed rheonomic Berwald-Moór metric of order four (1) are locally described by:*

$$
\begin{cases}
e^{-2\sigma}G_{1111}^{-1/2}\left(3h_{11}+8Y_{11}\right)h_{11} = \mathcal{K}\mathcal{T}_{11} \\[2mm]
R_{ij}+e^{-2\sigma}G_{1111}^{-1/2}\left(3h_{11}+8Y_{11}\right)\overset{*}{g}_{ij} = \mathcal{K}\mathcal{T}_{ij} \\[2mm]
S_{(i)(j)}^{(1)(1)}+e^{-2\sigma}G_{1111}^{-1/2}\left(3h_{11}+8Y_{11}\right)h^{11}\overset{*}{g}_{ij} = \mathcal{K}\mathcal{T}_{(i)(j)}^{(1)(1)} \\[2mm]
0 = \mathcal{T}_{1i}, \qquad 0 = \mathcal{T}_{i1}, \qquad 0 = \mathcal{T}_{(i)1}^{(1)} \\[2mm]
0 = \mathcal{T}_{1(i)}^{\;(1)}, \quad 0 = \mathcal{T}_{i(j)}^{\;(1)}, \quad 0 = \mathcal{T}_{(i)j}^{(1)}.
\end{cases}
\tag{15}
$$

Corollary 1. *The stress-energy d-tensor of matter $\mathcal{T}$ satisfies the following **geometrical conservation laws** (sum by m):*

$$
\begin{cases}
\mathcal{T}_{1/1}^{1}+\mathcal{T}_{1|m}^{m}+\mathcal{T}_{(1)1}^{(m)}|_{(m)}^{(1)} = 0 \\[3mm]
\mathcal{T}_{i/1}^{1}+\mathcal{T}_{i|m}^{m}+\mathcal{T}_{(1)i}^{(m)}|_{(m)}^{(1)} = E_{i|m}^{m} \\[3mm]
\mathcal{T}_{(i)/1}^{1(1)}+\mathcal{T}_{(i)|m}^{m(1)}+\mathcal{T}_{(1)(i)}^{(m)(1)}|_{(m)}^{(1)} = \dfrac{4e^{-2\sigma}G_{1111}^{-1/2}}{\mathcal{K}}\cdot\left[2\dfrac{\partial Y_{11}}{\partial y_1^i}-\dfrac{Y_{11}}{y_1^i}\right],
\end{cases}
$$

where (sum by r):

$$\mathcal{T}_{1}^{1}\overset{def}{=}h^{11}\mathcal{T}_{11} = \mathcal{K}^{-1}e^{-2\sigma}G_{1111}^{-1/2}\left(3h_{11}+8Y_{11}\right),\quad \mathcal{T}_{1}^{m}\overset{def}{=}\overset{*}{g}^{mr}\mathcal{T}_{r1} = 0,$$

$$\mathcal{T}_{(1)1}^{(m)}\overset{def}{=}h_{11}\overset{*}{g}^{mr}\mathcal{T}_{(r)1}^{(1)} = 0,\quad \mathcal{T}_{i}^{1}\overset{def}{=}h^{11}\mathcal{T}_{1i} = 0,$$

$$\mathcal{T}_{i}^{m}\overset{def}{=}\overset{*}{g}^{mr}\mathcal{T}_{ri} = E_{i}^{m} := \mathcal{K}^{-1}\left[\overset{*}{g}^{mr}R_{ri}+e^{-2\sigma}G_{1111}^{-1/2}\left(3h_{11}+8Y_{11}\right)\delta_i^m\right],$$

$$\mathcal{T}_{(1)i}^{(m)}\overset{def}{=}h_{11}\overset{*}{g}^{mr}\mathcal{T}_{(r)i}^{(1)} = 0,\; \mathcal{T}_{(i)}^{1(1)}\overset{def}{=}h^{11}\mathcal{T}_{1(i)}^{\;(1)} = 0,\; \mathcal{T}_{(i)}^{m(1)}\overset{def}{=}\overset{*}{g}^{mr}\mathcal{T}_{r(i)}^{\;(1)} = 0,$$

$$\mathcal{T}_{(1)(i)}^{(m)(1)}\overset{def}{=}h_{11}\overset{*}{g}^{mr}\mathcal{T}_{(r)(i)}^{(1)(1)} = \dfrac{e^{-2\sigma}G_{1111}^{-1/2}}{\mathcal{K}}\cdot\left[\dfrac{h_{11}}{2}\dfrac{y_1^m}{y_1^i}+\left(h_{11}+8Y_{11}\right)\delta_i^m\right],\text{ and we also have}$$

(summation by m and r, but no sum by i)

$$\mathcal{T}_{1/1}^{1} = \dfrac{\delta\mathcal{T}_{1}^{1}}{\delta t},\quad \mathcal{T}_{1|m}^{m}\overset{def}{=}\dfrac{\delta\mathcal{T}_{1}^{m}}{\delta x^m}+\mathcal{T}_{1}^{r}L_{rm}^{m},$$

$$\mathcal{T}_{(1)1}^{(m)}|_{(m)}^{(1)}\overset{def}{=}\dfrac{\partial\mathcal{T}_{(1)1}^{(m)}}{\partial y_1^m}+\mathcal{T}_{(1)1}^{(r)}C_{r(m)}^{m(1)} = \dfrac{\partial\mathcal{T}_{(1)1}^{(m)}}{\partial y_1^m},$$

$$\mathcal{T}_{i/1}^{1}\overset{def}{=}\dfrac{\delta\mathcal{T}_{i}^{1}}{\delta t}+\mathcal{T}_{i}^{1}\mathrm{K}_{11}^{1}-\mathcal{T}_{r}^{1}G_{i1}^{r} = \dfrac{\delta\mathcal{T}_{i}^{1}}{\delta t}+\mathcal{T}_{i}^{1}\mathrm{K}_{11}^{1},$$

$$\mathcal{T}_{i|m}^{m} \overset{def}{=} \frac{\delta \mathcal{T}_i^m}{\delta x^m} + \mathcal{T}_i^r L_{rm}^m - \mathcal{T}_r^m L_{im}^r = E_{i|m}^m := \frac{\delta E_i^m}{\delta x^m} + 4E_i^m \sigma_m - 4E_i^i \sigma_i,$$

$$\mathcal{T}_{(1)i}^{(m)}|_{(m)}^{(1)} \overset{def}{=} \frac{\partial \mathcal{T}_{(1)i}^{(m)}}{\partial y_1^m} + \mathcal{T}_{(1)i}^{(r)} C_{r(m)}^{m(1)} - \mathcal{T}_{(1)r}^{(m)} C_{i(m)}^{r(1)} = \frac{\partial \mathcal{T}_{(1)i}^{(m)}}{\partial y_1^m} - \mathcal{T}_{(1)r}^{(m)} C_{i(m)}^{r(1)},$$

$$\mathcal{T}_{(i)/1}^{1(1)} \overset{def}{=} \frac{\delta \mathcal{T}_{(i)}^{1(1)}}{\delta t} + 2\mathcal{T}_{(i)}^{1(1)} \mathrm{K}_{11}^1, \quad \mathcal{T}_{(i)|m}^{m(1)} \overset{def}{=} \frac{\delta \mathcal{T}_{(i)}^{m(1)}}{\delta x^m} + \mathcal{T}_{(i)}^{r(1)} L_{rm}^m - \mathcal{T}_{(r)}^{m(1)} L_{im}^r,$$

$$\mathcal{T}_{(1)(i)}^{(m)(1)}|_{(m)}^{(1)} \overset{def}{=} \frac{\partial \mathcal{T}_{(1)(i)}^{(m)(1)}}{\partial y_1^m} + \mathcal{T}_{(1)(i)}^{(r)(1)} C_{r(m)}^{m(1)} - \mathcal{T}_{(1)(r)}^{(m)(1)} C_{i(m)}^{r(1)} = \frac{\partial \mathcal{T}_{(1)(i)}^{(m)(1)}}{\partial y_1^m}.$$

Proof. The local Einstein equations (15), together with some direct computations, lead us to what we were looking for. □

4.2 Electromagnetic-like geometrical model

In book [4], a geometrical theory for electromagnetism was also created, using only a given Lagrangian function L on the 1-jet space $J^1(\mathbb{R}, M^4)$. In the background of the jet relativistic rheonomic Lagrange geometry from [4], one works with the *electromagnetic distinguished 2-form* (sum by i and j)

$$\mathbf{F} = F_{(i)j}^{(1)} \delta y_1^i \wedge dx^j,$$

where (sum by m and r)

$$F_{(i)j}^{(1)} = \frac{h^{11}}{2} \left[\overset{*}{g}_{jm} N_{(1)i}^{(m)} - \overset{*}{g}_{im} N_{(1)j}^{(m)} + \left(\overset{*}{g}_{ir} L_{jm}^r - \overset{*}{g}_{jr} L_{im}^r \right) y_1^m \right].$$

This is characterized by some natural *geometrical Maxwell equations* (for more details, see Miron and Anastasiei [11] and Balan and Neagu [4]).

By a direct calculation, we observe that the x-conformal deformed rheonomic Berwald-Moór metric of order four (1) produces null electromagnetic components:

$$F_{(i)j}^{(1)} = 0.$$

Consequently, our x-conformal deformed jet Berwald-Moór geometrical electromagnetic theory is trivial. Probably, this fact suggests that the x-conformal deformed rheonomic Berwald-Moór structure (1) has rather strong gravitational connotations than electromagnetic ones.

As a conclusion, it is possible for the new Voicu-Siparov approach to electromagnetism in spaces with anisotropic metrics (this electromagnetic approach is different from the electromagnetic theory exposed above, and it is developed by Voicu and Siparov in the paper [19]) to give other interesting electromagnetic-geometrical results.

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