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COHOMOLOGY OF FOLIATED FINSLER MANIFOLDS

Adelina MANEA [1]

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Abstract

We consider a foliation $\mathcal{F}$ of a Finsler manifold M. Its tangent manifold TM is Riemannian with respect to the Sasaki-Finsler metric and admits a natural foliation as a fibered manifold, called vertical foliation. Foliation on M determines a foliation $\mathcal{F}_{\mathcal{T}}$ on TM. The bundle TTM has two decompositions with respect to vertical foliation and to $\mathcal{F}_{\mathcal{T}}$, respectively. We define the vertical-tangent, the horizontal-tangent, the vertical-transversal and the horizontal-transversal vector fields on TM. We also define a new type of differential forms on TM. The exterior derivative on TM admits a decomposition into some operators, one of them satisfies a Poincare type lemma, and we established a de Rham theorem for this particular one.

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1 Preliminaries

Finding new topological invariants of differentiable manifolds is still an open problem for geometries. The cohomology groups are such invariants. The Finsler manifolds are interesting models for some physical phenomena, so their properties are also useful to investigate, [1], [2], [3], [5]. The cohomology groups of manifolds, related sometimes to some foliations on the manifolds, have been studied in the last decades, [8], [9]. A study of foliations on a Lagrangian manifolds could be found in [7]. Our present work intends to develop the study of the Finsler manifolds and the foliated structures on the tangent bundle of such a manifold.

For the beginning, we present the verical foliation on the tangent manifold TM of a n-dimensional Finsler manifold (M, F), following [1]. In this paper the indices take the values $i, j, i_1, j_1, ... = \overline{1,n}$. Let (M, F) be a n-dimensional Finsler manifold and G be the Sasaki-Finsler metric on its tangent manifold TM. The vertical bundle VTM of TM is the tangent (structural) bundle to vertical foliation F_V determined by the fibers of $\pi : TM \to M$. If $(x^i, y^i)_{i=\overline{1,n}}$ are local coordinates on TM, then VTM is locally spanned

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: amanea28@yahoo.com

by $\{\frac{\partial}{\partial y^i}\}_i$. A canonical transversal (also called horizontal) distribution is constructed in [1] as follows. We denote by $(g^{ij}(x,y))_{i,j}$ the inverse matrix of $g = (g_{ij}(x,y))_{i,j}$, where

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y), \tag{1.1}$$

and F is the fundamental function of the Finsler manifold. Obviously, we have the equalities $\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j} = \frac{\partial g_{jk}}{\partial y^i}$.

We locally define the functions

$$G^i = \frac{1}{4}g^{ik}\left(\frac{\partial^2 F^2}{\partial y^k \partial x^h}y^h - \frac{\partial F^2}{\partial x^k}\right), \quad G_i^j = \frac{\partial G^j}{\partial y^i}.$$

There exists on TM a n distribution HTM locally spanned by the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}, \quad (\forall) i = \overline{1,n}. \tag{1.2}$$

The Riemannian metric G on TM is satisfying

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g_{ij}, \quad G(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0, \quad (\forall) i, j. \tag{1.3}$$

The local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}_i$ is called *adapted* to vertical foliation F_V and we have the decomposition

$$TTM = HTM \oplus VTM. \tag{1.4}$$

2 Vertical cohomology of TM

Corresponding to decomposition (1.4) we have the decomposition of module of vector fields on TM into horizontal vector fiels $\mathcal{X}^h(TM)$ locally spanned by $\{\frac{\delta}{\delta x^i}\}$ and vertical vector fiels $\mathcal{X}^v(TM)$ locally spanned by $\{\frac{\partial}{\partial y^i}\}$. We also have two distributions, the vertical one being integrable. The Poisson braket of two horizontal vector fields depends on

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R_{ij}^k \frac{\partial}{\partial y^k}, \quad R_{ij}^k = \frac{\delta G_j^i}{\delta x^k} - \frac{\delta G_k^i}{\delta x^j}. \tag{2.1}$$

The horizontal distribution is integrable iff $R_{ij}^k = 0$ for all indices.

We apply theory of differential forms on foliated manifolds, [9], to the Riemannian foliated manifold (TM, G, F_V) and we consider the (p,q)-forms on TM being $(p+q)$-forms on TM which are non-zero only for p-arguments in $\mathcal{X}^h(TM)$ and q-arguments in $\mathcal{X}^v(TM)$. The module of (p,q)- forms is denoted by $\Omega^{p,q}(TM)$ and we have the following decomposition of module of r-forms on TM:

$$\Omega^r(TM) =_{p+q=r} \cup \Omega^{p,q}(TM).$$

Local cobasis adapted to vertical foliation is $\{dx^i, \delta y^i = dy^i + G^i_j dx^j\}$. Locally, $\omega \in \Omega^{p,q}(TM)$ has the following expression:

$$\omega = \alpha_{i_1 i_2 \dots i_p j_1 \dots j_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \wedge \delta y^{j_1} \wedge \dots \wedge \delta y^{j_q}.$$

We compute $d\delta y^i = \sum_{k<j} R^i_{jk} dx^k \wedge dx^j + \sum_{k,j} \frac{\partial G^i_j}{\partial y^k} \delta y^k \wedge dx^j$, hence the exterior derivative on TM admits a decomposition into three operators:

$$d = d_{10} + d_{2,-1} + d_{01},$$

$$d_{10} : \Omega^{p,q}(TM) \to \Omega^{p+1,q}(TM); \quad d_{2,-1} : \Omega^{p,q}(TM) \to \Omega^{p+2,q-1}(TM);$$

$$d_{01} : \Omega^{p,q}(TM) \to \Omega^{p,q+1}(TM),$$

locally given by:

$$d_{10}\omega = \frac{\delta \alpha_{i_1 i_2 \dots i_p j_1 \dots j_q}}{\delta x^i} dx^i \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \wedge \delta y^{j_1} \wedge \dots \wedge \delta y^{j_q} +$$

$$+ \alpha_{i_1 i_2 \dots i_p j_1 \dots j_q} \frac{\partial G^{j_k}_i}{\partial y^l} \delta y^l \wedge dx^i \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \wedge \delta y^{j_1} \wedge \dots \wedge \widehat{\delta y^{j_k}} \wedge \dots \wedge \delta y^{j_q},$$

where the symbol $\widehat{\delta y^{j_k}}$ means that this element is missing.

$$d_{2,-1}\omega = \alpha_{i_1 i_2 \dots i_p j_1 \dots j_q} R^{j_k}_{il} dx^l \wedge dx^i \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \wedge \delta y^{j_1} \wedge \dots \wedge \widehat{\delta y^{j_k}} \wedge \dots \wedge \delta y^{j_q},$$

$$d_{01}\omega = \frac{\partial \alpha_{i_1 i_2 \dots i_p j_1 \dots j_q}}{\partial y^i} \delta y^i \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \wedge \delta y^{j_1} \wedge \dots \wedge \delta y^{j_q}.$$

From $d^2 = 0$ it follows $d^2_{01} = 0$, this operator being exactly the foliated derivative of (TM, F_V) from [9]. It is known that the foliated derivative satisfies a Poincare type lemma. Let us denote by Φ^v the sheaf of germs of basic functions on TM, that means $f \in \Omega^0(TM)$ such that $d_{01} f = 0$. The de Rham cohomology groups of foliated manifold (TM, F_V) are the quotient groups of the following semiexact sequence

$$\Phi^v(TM) \to \Omega^0(TM) \overset{d_{01}}{\to} \Omega^{0,1}(TM) \overset{d_{01}}{\to} \Omega^{0,2}(TM) \overset{d_{01}}{\to} \dots.,$$

$$H^q_v(TM) = \frac{Ker d_{01}}{d_{01}(\Omega^{0,q-1}(TM))},$$

and these groups are so called *v-cohomology* of TM. Here $Ker d_{01} = \{\omega \in \Omega^{0,q}(TM), d_{01}\omega = 0\}$ is the set of d_{01}-closed $(0,q)$-forms which is usually denoted by $Z^{0,q}(TM)$. The set $d_{01}(\Omega^{0,q-1}(TM))$ is usually denoted by $B^{0,q}(TM)$ and it is called the set of d_{01}-exact $(0,q)$-forms.

The sheaves sequence

$$0 \to \Phi^v \to \Omega^0 \overset{d_{01}}{\to} \Omega^{0,1} \overset{d_{01}}{\to} \Omega^{0,2} \overset{d_{01}}{\to} \dots.,$$

is a fine resolution of Φ^v. A de Rham theorem is also true, $H^q_v(TM)$ is isomorphic with the q-dimensional Cech cohomology group of TM with coefficients in Φ^v. All these results come from the theory of foliated manifolds applied to (TM, F_V).

3 Foliated Finsler manifold

In this section we consider the Finsler manifold (M, F) from the previous sections endowed with a m-codimensional foliation $\mathcal{F}$. It follows that there is a partition of M into $n-m$ -dimensional submanifolds, called leaves. In the following, the indices take the values $u, v, u_1, v_1, ... = \overline{m+1, n}$ and $a, b, a_1, b_1, ...= \overline{1, m}$. There is an atlas on M adapted to this foliation with local adapted charts $(U, (x^a, x^u))$ such that the leaves are locally defined by $x^a = constant$, for all $a = \overline{1, m}$.

The local coordinates on the tangent manifold TM are (x^a, x^u, y^a, y^u). Generally, for two local charts $(U, (x^i))$ and $\tilde{U}, (\tilde{x}^{i_1}))$, whose domains overlap, on TM, in $U \cap \tilde{U}$ we have

$$\tilde{y}^{i_1} = \frac{\partial \tilde{x}^{i_1}}{\partial x^i} y^i.$$

Now, the above relations give the following changing coordinates rules on TM:

$$\tilde{x}^{a_1} = \tilde{x}^{a_1}(x^a); \quad \tilde{x}^{u_1} = \tilde{x}^{u_1}(x^a, x^u),$$

$$\tilde{y}^{a_1} = \frac{\partial \tilde{x}^{a_1}}{\partial x^a} y^a; \quad \tilde{y}^{u_1} = \frac{\partial \tilde{x}^{u_1}}{\partial x^a} y^a + \frac{\partial \tilde{x}^{u_1}}{\partial x^u} y^u.$$

Foliation $\mathcal{F}$ on M determine a $2m$-codimensional foliation $\mathcal{F}_{\mathcal{T}}$ on TM, whose leaves are locally defined by

$$x^a = constant, \quad y^a = constant.$$

Taking into account decomposition (1.4) of TTM, the local base $\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^u}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^u}\}$ adapted to vertical foliation satisfies the following relations in $U \cap \tilde{U}$:

$$\frac{\delta}{\delta \tilde{x}^{a_1}} = \frac{\partial x^a}{\partial \tilde{x}^{a_1}} \frac{\delta}{\delta x^a} + \frac{\partial x^u}{\partial \tilde{x}^{a_1}} \frac{\delta}{\delta x^u},$$

$$\frac{\partial}{\partial \tilde{y}^{a_1}} = \frac{\partial x^a}{\partial \tilde{x}^{a_1}} \frac{\partial}{\partial y^a} + \frac{\partial x^u}{\partial \tilde{x}^{a_1}} \frac{\partial}{\partial y^u},$$

$$\frac{\delta}{\delta \tilde{x}^{u_1}} = \frac{\partial x^u}{\partial \tilde{x}^{u_1}} \frac{\delta}{\delta x^u},$$

$$\frac{\partial}{\partial \tilde{y}^{u_1}} = \frac{\partial x^u}{\partial \tilde{x}^{u_1}} \frac{\partial}{\partial y^u},$$

Returning now to the foliation $\mathcal{F}_{\mathcal{T}}$, the tangent bundle $T\mathcal{F}_{\mathcal{T}}$ to leaves, also called the structural bundle of this foliation, is locally spanned by $\{\frac{\delta}{\delta x^u}, \frac{\partial}{\partial y^u}\}$ and it is a subbundle of TTM.

Definition 3.1. *We say that the foliation $\mathcal{F}$ of M is **compatible with the Finsler structure** F on M if, in every local chart around a point $(x, y) \in TM$, the matrix*

$$(g_{uv})_{u,v}, \quad g_{uv}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^u \partial y^v}(x, y),$$

is nondegenerate and the functions G^a are satisfying the relation

$$\frac{\partial G^a}{\partial y^u} = 0,$$

for all $a = \overline{1, m}$, $u = \overline{m+1, n}$.

Remark 3.1. *The condition that a foliation is compatible to F has geometrical meaning. Indeed, in $U \cap \tilde{U}$ we have*

$$\tilde{g}_{u_1, v_1} = g_{uv} \frac{\partial x^u}{\partial \tilde{x}^{u_1}} \frac{\partial x^v}{\partial \tilde{x}^{v_1}},$$

and the matix $(\frac{\partial x^u}{\partial \tilde{x}^{u_1}})_{u,u_1}$ is obviously nondegenerate in every local chart.

Proposition 3.1. *If the foliation $\mathcal{F}$ of M is compatible with the Finsler structure F on M, then the vector fields on TM locally given by*

$$\xi_a = \frac{\delta}{\delta x^a} - t_a^u \frac{\delta}{\delta x^u}, \quad \zeta_a = \frac{\partial}{\partial y^a} - t_a^u \frac{\partial}{\partial y^u}, \tag{3.1}$$

are orthogonal to $\{\frac{\delta}{\delta x^u}\}$, $\{\frac{\partial}{\partial y^u}\}$, with respect to Sasaki-Finsler metric G from (1.3), where $\{t_a^u\}$ are solutions of the system $g_{av} - t_a^u g_{uv} = 0$.

Proof. The compatibility between foliation on M and the Finsler structure of this manifold assures the solvability of the system $g_{av} - t_a^u g_{uv} = 0$. Moreover, computing

$$G(\xi_a, \frac{\delta}{\delta x^u}) = g_{au} - t_v^a g_{vu}, \quad G(\zeta_a, \frac{\partial}{\partial y^u}) = g_{au} - t_v^a g_{vu},$$

it results that these vector fields are orthogonal. □

As a consequence of the above Proposition, for a vector $X \in TM$ we have the followig decomposition:

$$X = X^i \frac{\delta}{\delta x^i} + Y^i \frac{\partial}{\partial y^i} =$$
$$= X^a \frac{\delta}{\delta x^a} + X^u \frac{\delta}{\delta x^u} + Y^a \frac{\partial}{\partial y^a} + Y^u \frac{\partial}{\partial y^u} =$$
$$= X^a \xi_a + (X^u + t_a^u X^a) \frac{\delta}{\delta x^u} + Y^a \zeta_a + (Y^u + t_a^u Y^a) \frac{\partial}{\partial y^u}.$$

The basis

$$\{\xi_a, \frac{\delta}{\delta x^u}, \zeta_a, \frac{\partial}{\partial y^u}\} \tag{3.2}$$

is adapted to $\mathcal{F}_{\mathcal{T}}$ and to vertical foliation, too. We denote by $T^{\perp}\mathcal{F}_{\mathcal{T}}$ the orthogonal bundle to $T\mathcal{F}_{\mathcal{T}}$ with respect to Sasaki-Finsler metric on TM.

Definition 3.2. *The vertical component of a section of $T\mathcal{F}_{\mathcal{T}}$ is called a **vertical-tangent** vector field on TM. Their collection is denoted by $V\mathcal{F}_{\mathcal{T}}$.*

*The horizontal component of a section of $T\mathcal{F}_{\mathcal{T}}$ is called a **horizontal-tangent** vector field on TM. Their collection is denoted by $H\mathcal{F}_{\mathcal{T}}$.*

*The vertical component of a section of $T^{\perp}\mathcal{F}_{\mathcal{T}}$ is called a **vertical-transversal** vector field on TM. Their collection is denoted by $V^{\perp}\mathcal{F}_{\mathcal{T}}$.*

*The horizontal component of a section of $T^{\perp}\mathcal{F}_{\mathcal{T}}$ is called a **horizontal-transversal** vector field on TM. Their collection is denoted by $H^{\perp}\mathcal{F}_{\mathcal{T}}$.*

From the above definition we have the decomposition of module of vector fields on TM into $H^{\perp}\mathcal{F}_{\mathcal{T}} \oplus H\mathcal{F}_{\mathcal{T}} \oplus V^{\perp}\mathcal{F}_{\mathcal{T}} \oplus V\mathcal{F}_{\mathcal{T}}$. Locally, $H^{\perp}\mathcal{F}_{\mathcal{T}}$, $H\mathcal{F}_{\mathcal{T}}$, $V^{\perp}\mathcal{F}_{\mathcal{T}}$, $V\mathcal{F}_{\mathcal{T}}$ are spanned by $\{\xi_a\}$, $\{\frac{\delta}{\delta x^u}\}$, $\{\zeta_a\}$, $\{\frac{\partial}{\partial y^u}\}$, respectively.

The cobasis dual to basis (3.2) is $\{dx^a, \theta^u, \delta y^a, \eta^u\}$, where $\theta^u = dx^u + t^u_a dx^a$, $\eta^u = \delta y^u + t^u_a \delta y^a$.

Definition 3.3. *A $(0,q)$-form ω on TM is called a $(0,s,t)$-form if $s+t=q$ and if it is nonzero only for s-arguments in $V^{\perp}\mathcal{F}_{\mathcal{T}}$ and t-arguments in $V\mathcal{F}_{\mathcal{T}}$.*

We denote the space of $(0,s,t)$-forms by $\Omega^{0,s,t}(TM)$ and we have

$$\Omega^{0,q}(TM) = \cup_{s+t=q}\Omega^{0,s,t}(TM).$$

The foliated derivative d_{01} acts on a $(0,s,t)$-form locally given by

$$\omega = \alpha_{a_1,a_2,\ldots,a_s,u_1,..,u_t}\delta y^{a_1} \wedge \ldots \wedge \delta y^{a_s} \wedge \eta^{u_1} \wedge \ldots \wedge \eta^{u_t}$$

as it follows:

$$d_{01}\omega = \zeta_a(\alpha_{a_1,a_2,\ldots,a_s,u_1,..,u_t})\delta y^a \wedge \delta y^{a_1} \wedge \ldots \wedge \delta y^{a_s} \wedge \eta^{u_1} \wedge \ldots \wedge \eta^{u_t} +$$

$$+(-1)^s \frac{\partial \alpha_{a_1,a_2,\ldots,a_s,u_1,..,u_t}}{\partial y^u}\delta y^{a_1} \wedge \ldots \wedge \delta y^{a_s} \wedge \eta^u \wedge \eta^{u_1} \wedge \ldots \wedge \eta^{u_t}.$$

Hence there are two operators

$$d_{0,1,0} : \Omega^{0,s,t}(TM) \to \Omega^{0,s+1,t}(TM), \quad d_{0,0,1} : \Omega^{0,s,t}(TM) \to \Omega^{0,s,t+1}(TM),$$

with

$$d_{01} = d_{0,1,0} + d_{0,0,1},$$

and

$$d_{0,0,1}\omega = (-1)^s \frac{\partial \alpha_{a_1,a_2,\ldots,a_s,u_1,..,u_t}}{\partial y^u}\delta y^{a_1} \wedge \ldots \wedge \delta y^{a_s} \wedge \eta^u \wedge \eta^{u_1} \wedge \ldots \wedge \eta^{u_t}.$$

The equality $d^2_{01} = 0$ implies $d^2_{0,0,1} = 0$. A function $f \in \Omega^0(TM)$ is called *vertical basic* if

$$d_{0,0,1} f = 0.$$

We denote by $\Phi^{0,0}$ the sheaf of germs of vertical basic functions. The following sequence

$$\Phi^{0,0}(TM) \to \Omega^{0}(TM) \overset{d_{0,0,1}}{\to} \Omega^{0,0,1}(TM) \overset{d_{0,0,1}}{\to} \Omega^{0,0,2}(TM) \overset{d_{0,0,1}}{\to} ....,$$

is semiexact and we call the quotient group

$$H^{q}_{v,t}(TM) = \frac{Z^{0,0,q}(TM)}{B^{0,0,q}(TM)},$$

the *v,t-cohomology* of TM. Here $Z^{0,0,q}(TM) = \{\omega \in \Omega^{0,0,q}(TM), d_{0,0,1}\omega = 0\}$ is the set of $d_{0,0,1}$-closed $(0,0,q)$-forms. The set $B^{0,q}(TM) = d_{0,0,1}(\Omega^{0,0,q-1}(TM))$ is called the set of $d_{0,0,1}$-exact $(0,0,q)$-forms.

Remark 3.2. *We have the following strict inclusions:*

$$Z^{0,q}(TM) \cap \Omega^{0,0,q}(TM) \subset Z^{0,0,q}(TM), \quad B^{0,q}(TM) \cap \Omega^{0,0,q}(TM) \subset B^{0,0,q}(TM).$$

Another property of the operator $d_{0,0,1}$ is a Poincare type lemma, namely

Theorem 3.1. *Let ω be a $d_{0,0,1}$-closed $(0,0,t)$-form. For every open domain $U \in TM$ there is $\gamma \in \Omega^{0,0,t-1}(U)$ such that in U we have the equality $\omega = d_{0,0,1}\gamma$.*

Proof. Since ω is a $(0,0,t)$-form, its local expression is

$$\omega = \alpha_{u_1,u_2,...,u_t}\eta^{u_1} \wedge ... \wedge \eta^{u_t}.$$

From $d_{0,0,1}\omega = 0$ it follows $d_{01}\omega = \zeta_a(\alpha_{u_1,u_2,...,u_t})\delta y^a \wedge \eta^{u_1} \wedge ... \wedge \eta^{u_t}$, so

$$d_{01}\omega = 0 \quad (modulo \quad \delta y^1, \delta y^2, ..., \delta y^{m-n})$$

But the operator d_{01} satisfies a Poincare type lemma, hence in space $\delta y^a = 0$ we have that for every open domain U of TM there is a $(0, t-1)$-form φ such that in U

$$\omega = d_{0,0,1}\varphi \quad (modulo \quad \delta y^1, \delta y^2, ..., \delta y^{m-n}).$$

Let γ be the $(0,0,t-1)$-component of φ. Then we have the following equality in U:

$$\omega = d_{0,0,1}\gamma + \mu_a \wedge \delta y^a,$$

with μ_a $(0,t-1)$-forms on U. By equalizing the terms of the same type in the above equality we obtain $\omega = d_{0,0,1}\gamma$, so every $d_{0,0,1}$-closed form is locally $d_{0,0,1}$-exact. □

Let $\Omega^{0,0,q}$ the sheaf of germs of $(0,0,q)$-forms and $i : \Phi^{0,0} \to \Omega^0$ the natural inclusion. The shaves $\Omega^{0,0,q}$ are fins, so the shaves sequence

$$0 \to \Phi^{0,0} \to \Omega^{0} \overset{d_{0,0,1}}{\to} \Omega^{0,0,1} \overset{d_{0,0,1}}{\to} \Omega^{0,0,2} \overset{d_{0,0,1}}{\to} ....,$$

is a fine resolution of the sheaf $\Phi^{0,0}$. Applying now a well-known theorem of algebraic topology it results a de Rham type theorem for the v,t-cohomology of TM:

Theorem 3.2. *$H^{q}_{v,t}(TM)$ is isomorphic with the q-dimensional Cech cohomology group of TM with coefficients in $\Phi^{0,0}$.*

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