ALMOST CONTACT METRIC STRUCTURES DEFINED BY CONNECTION OVER DISTRIBUTION

Aliya V. BUKUSHEVA[1] and Sergei V. GALAEV[2]

Communicated to:
Finsler Extensions of Relativity Theory, August 29 - September 4, 2011, Braşov, Romania

Abstract

In this paper, the notion of intrinsic geometry of an almost contact metric manifold D is introduced and studied. Using this and the extended connection on D as on the total space of a vector bundle, an almost contact metric structure is defined and investigated.

Key words: almost contact manifold, Sasakian manifold, the intrinsic geometry of almost contact metric manifolds, admissible Finslerian metric.

1 Indroduction

Finslerian vector bundles, which are natural analogs of the tangent bundles of manifolds with Finslerian metrics, were introduced and studied in [9]. A Finslerian vector bundle may be characterized by giving on the total space of a vector bundle a class of linear connections that are associated in a special way with an infinitesimal connection, see e.g. [10]. In [4], the notion of a smooth distribution D with an admissible Finslerian metric, which allows a new viewpoint on the problems of Finslerian vector bundle, is introduced. In the present paper, we define in a natural way an almost contact metric structure on the total space of the vector bundle (D, π, X), where D is a smooth distribution with an admissible Finslerian structure. The properties of this structure are studied by means of the interior geometry of a nonholonomic manifold.

The research of the geometry of manifolds with almost contact metric structures began in the fundamental papers by Chern [3], J. Gray [6] and Sasaki [11]. Almost contact metric structures constitute the odd-dimensional analog of almost Hermitian structures. There are a lot of important interplays between these structures. At the same time, the geometry of almost contact metric structures is appreciably different from the geometry

---

[1]Saratov State University, Chair of Geometry, Astrakhanskaya 83, 410012 Saratov, Russia, e-mail: bukusheva@list.ru

[2]Saratov State University, Chair of Geometry, Astrakhanskaya 83, 410012 Saratov, Russia, e-mail: sgalaev@mail.ru

of almost Hermitian structures and its study requires in principle new tools. One can get the impression of the last achievements in this area and about applications to theoretical physics from the works [7, 1, 12].

In the present paper, the notion of intrinsic geometry of a manifold with an almost contact metric structure is introduced. In the terminology developed by V.V.Wagner [15], the manifold with an almost contact metric structure is a nonholonomic manifold of codimension 1 with additional structures. These structures are called intrinsic by Wagner. The notion of intrinsic geometry of a nonholonomic manifold was defined by Schouten as the properties that depend only on the parallel transport in the nonholonomic manifold and on the closing of the nonholonomic manifold in the ambient manifold. We propose to use the methods of nonholonomic geometry developed by Wagner for the investigation of the geometry of manifolds with almost contact metric structure. The new approach allows us to pick out new types of spaces. For example, we give the definition of a Hermitian almost contact metric space. The already known results obtain new descriptions of the language of intrinsic geometry.

The paper consists of three sections. In Section 2 we provide the basic concepts of the theory of manifolds with almost contact metric structure. We introduce the notion of adapted coordinate system. The adapted coordinates play in the geometry of the nonholonomic manifolds the same role as the holonomic coordinates on a holonomic manifold, see e.g. [15]. Next we introduce the notion of admissible (to a nonholonomic distribution D) tensor structure. An admissible tensor structure is an object of the intrinsic geometry of a nonholonomic manifold [15]. We give some information about the intrinsic connections compatible with admissible tensor structures. Among the connections compatible with the admissible Riemannian metric, we study the connections compatible with an admissible almost complex structure. We discuss the connection over a distribution that was introduced in [13, 8] and applied in [2, 4] to manifolds with an admissible Finslerian metric.

In Section 3 we expound some of the main theses of the geometry of almost contact metric spaces in terms of intrinsic geometry. It is shown that the almost contact metric structure defined in the intrinsic way corresponds to a certain almost contact metric structure defined in the usual way. The intrinsic connection is used for description and characterization of the normal and Sasakian structures.

In Section 4 we prove that the connection over a distribution with a Finslerian metric defines on the total space of the vector bundle D over the manifold X, an almost contact metric structure.

The proofs of some statements are omitted and will be published in more detailed papers elsewhere [4, 5].

2   Admissible tensor structures and intrinsic connection compatible with them

Let X be a smooth manifold of an odd dimension n. Denote by $\Xi(X)$ the $C^\infty(X)$-module of smooth vector fields on X and by d the exterior derivative. All manifolds,

tensors and other geometric objects will be assumed to be smooth of the class C^∞. For simplification, in what follows we refer to tensor fields simply as tensors. An almost contact metric structure on X is an aggregate $(\varphi, \vec{\xi}, \eta, g)$ of tensor fields on X, where φ is a tensor field of type $(1,1)$, which is called the structure endomorphism, $\vec{\xi}$ and η are vector and covector, which are called the structure vector and the contact form, respectively, and g is a (pseudo-)Riemannian metric. Moreover,

$$\eta(\vec{\xi}) = 1, \quad \varphi(\vec{\xi}) = 0, \quad \eta \circ \varphi = 0,$$

$$\varphi^2 \vec{X} = -\vec{X} + \eta(\vec{X})\vec{\xi}, \quad g(\varphi\vec{X}, \varphi\vec{Y}) = g(\vec{X}, \vec{Y}) - \eta(\vec{X})\eta(\vec{Y})$$

for all $\vec{X}, \vec{Y} \in \Xi(X)$. It is easy to check that the tensor $\Omega(\vec{X}, \vec{Y}) = g(\vec{X}, \varphi\vec{Y})$ is skew-symmetric. It is called the fundamental tensor of the structure. A manifold with a fixed almost contact metric structure is called an almost contact metric manifold. If $\Omega = d\eta$ holds, then the almost contact metric structure is called a contact metric structure. An almost contact metric structure is called normal if

$$N_\varphi + 2d\eta \otimes \vec{\xi} = 0,$$

where N_φ is the Nijenhuis torsion defined for the tensor φ. A normal contact metric structure is called a Sasakian structure. A manifold with a given Sasakian structure is called a Sasakian manifold. Let D be the smooth distribution of codimension 1 defined by the form η, and $D^\perp = \text{span}(\vec{\xi})$ be the closing of D. In what follows we assume that the restriction of the 2-form $\omega = d\eta$ to the distribution D is non-degenerate. In this case the vector $\vec{\xi}$ is uniquely defined by the condition

$$\eta(\vec{\xi}) = 1, \quad \ker \omega = \text{span}(\vec{\xi}),$$

and it is called the Reeb vector field. The smooth distribution D we call sometimes a nonholonomic manifold.

For investigation of the intrinsic geometry of a nonholonomic manifold, and generally for the study of almost contact metric structures, it is suitable to use coordinate systems satisfying certain additional conditions. We say that a coordinate map $K(x^\alpha)$ ($\alpha, \beta, \gamma = 1, ..., n$) ($a, b, c, e = 1, ..., n-1$) on a manifold X is adapted to the nonholonomic manifold D if

$$D^\perp = \text{span}\left(\frac{\partial}{\partial x^n}\right)$$

holds. It is easy to show that any two adapted coordinate map are related by a transformation of the form

$$x^a = x^a(x^{\tilde{a}}), \quad x^n = x^n(x^{\tilde{a}}, x^{\tilde{n}}).$$

Such coordinate systems are called by Wagner in [15] gradient coordinate systems.

Let $P : TX \to D$ be the projection map defined by the decomposition $TX = D \oplus D^\perp$ and let $K(x^\alpha)$ be an adapted coordinate map. Vector fields

$$P(\partial_a) = \vec{e}_a = \partial_a - \Gamma_a^n \partial_n$$

are linearly independent, and linearly generate the system D over the domain of the definition of the coordinate map:

$$D = \mathrm{span}(\vec{e}_a).$$

Thus we have on X the nonholonomic field of bases $(\vec{e}_a, \partial_n)$ and the corresponding field of cobases

$$(dx^a, \theta^n = dx^n + \Gamma_a^n dx^a).$$

It can be checked directly that

$$[\vec{e}_a, \vec{e}_b] = M_{ab}^n \partial_n,$$

where the components M_{ab}^n form the so-called tensor of nonholonomicity [15]. Under the assumption that for all adapted coordinate systems it holds $\vec{\xi} = \partial_n$, the following equality takes place

$$[\vec{e}_a, \vec{e}_b] = 2\omega_{ba}\partial_n,$$

where $\omega = d\eta$. In what follows we consider exceptionally adapted coordinate systems that satisfy the condition $\vec{\xi} = \partial_n$. We say also that the basis

$$\vec{e}_a = \partial_a - \Gamma_a^n \partial_n$$

is adapted, as the basis defined by an adapted coordinate map. Under the transformation of the adapted coordinate systems, the vectors of the adapted bases transform in the following way: $\vec{e}_a = \frac{\partial x^{\tilde{a}}}{\partial x^a}\vec{e}_{\tilde{a}}$.

We call a tensor field defined on an almost contact metric manifold admissible (to the distribution D) if it vanishes whenever its vectorial argument belongs to the closing $D^\perp$ and its covectorian argument is proportional to the form η. The coordinate form of an admissible tensor field of type (p, q) in an adapted coordinate map looks like

$$t = t_{b_1, \ldots, b_q}^{a_1, \ldots, a_p} \vec{e}_{a_1} \otimes \ldots \otimes \vec{e}_{a_p} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_q}.$$

In particular, an admissible vector field is a vector field tangent to the distribution D, and an admissible 1-form is a 1-form zero on the closing $D^\perp$. It is clear that any tensor structure defined on the manifold X defines on it a unique admissible tensor structure of the same type. From the definition of an almost contact structure it follows that the field of endomorphisms φ is an admissible tensor field of type $(1, 1)$. The field of endomorphisms φ we call an admissible almost complex structure. The 2-form $\omega = d\eta$ is also an admissible tensor field. In the geometry of the fibered spaces an admissible tensor field is called semi basic.

Theorem 1. *The derivatives $\partial_n t$ of the components of an admissible tensor field t in an adapted coordinate system are components of an admissible tensor field of the same type.*

The proof of the theorem follows from the fact that the components of an admissible tensor field under the change of an admissible coordinate system transform in the following way:

$$t_{b_1, \ldots, b_q}^{a_1, \ldots, a_p} = t_{\tilde{b}_1, \ldots, \tilde{b}_q}^{\tilde{a}_1, \ldots, \tilde{a}_p} A_{\tilde{a}_1}^{a_1} \cdots A_{b_q}^{\tilde{b}_q},$$

where $A^{a_i}_{\tilde{a}_i} = \frac{\partial x^{a^i}}{\partial x^{\tilde{a}^i}}$. $\square$

The invariant character of the above statement is enclosed in the equality

$$L_{\vec{\xi}} t^{a_1,\dots,a_p}_{b_1,\dots,b_q} = \partial_n t^{a_1,\dots,a_p}_{b_1,\dots,b_q},$$

where $L_{\vec{\xi}}$ is the Lie derivative along a vector field $\vec{\xi}$.

We say that an admissible tensor field is integrable if there exists an atlas of adapted coordinate maps such that the components of this tensor in any of these coordinate maps are constant. From Theorem 1 immediately follows that the necessary condition of the integrability of an admissible tensor field t is vanishing of the derivatives $\partial_n t$. We call an admissible tensor structure t quasi-integrable if in adapted coordinates it holds $\partial_n t = 0$. The form $\omega = d\eta$ is an important example of an integrable admissible structure. The following two theorems show the importance of the above given definitions.

Theorem 2. *The field of endomorphism φ is integrable if and only if $P(N_\varphi) = 0$ holds.*

Theorem 3. *An almost contact metric structure is normal if and only if the following conditions hold:*

$$P(N_\varphi) = 0, \quad \omega(\varphi\vec{u}, \varphi\vec{v}) = \omega(\vec{u}, \vec{v}).$$

The next statements show the advisability of notions like adapted coordinate system and integrable tensor field.

Theorem 4. *A contact metric structure is normal if and only if the field of endomorphisms φ is integrable.*

Theorem 4 confirms the importance of introducing of the new type of almost contact metric spaces. Namely, we call an almost contact metric space *a Hermitian almost contact metric space* if the condition $P(N_\varphi) = 0$ holds.

An intrinsic linear connection on a nonholonomic manifold D is defined in [15] as a map

$$\nabla : \Gamma D \times \Gamma D \to \Gamma D$$

that satisfies the following conditions:

$$\begin{aligned} 1) \quad &\nabla_{f_1 \vec{u}_1 + f_2 \vec{u}_2} = f_1 \nabla_{\vec{u}_1} + f_2 \nabla_{\vec{u}_2}; \\ 2) \quad &\nabla_{\vec{u}} f\vec{v} = f\nabla_{\vec{u}}\vec{v} + (\vec{u}f)\vec{v}, \end{aligned}$$

where ΓD is the module of admissible vector fields. The Christoffel symbols are defined by the relation

$$\nabla_{\vec{e}_a} \vec{e}_b = \Gamma^c_{ab} \vec{e}_c.$$

The torsion S of the intrinsic linear connection is defined by the formula

$$S(\vec{X}, \vec{Y}) = \nabla_{\vec{X}} \vec{Y} - \nabla_{\vec{Y}} \vec{X} - P[\vec{X}, \vec{Y}].$$

Thus with respect to an adapted coordinate system it holds

$$S^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}.$$

In the same way as a linear connection on a smooth manifold, an intrinsic connection can be defined by giving a horizontal distribution over a total space of some vector bundle. The role of such bundle plays the distribution D.

In order to define a connection over the distribution D, it is necessary first to introduce a structure of a smooth manifold on D. This structure is defined in the following way. To each adapted coordinate map $K(x^\alpha)$ on the manifold X we put in correspondence the coordinate map $\tilde{K}(x^\alpha, x^{n+\alpha})$ on the manifold D, where $x^{n+\alpha}$ are the coordinates of an admissible vector with respect to the basis $\vec{e}_a = \partial_a - \Gamma_a^n \partial_n$.

The notion of *a connection over a distribution* introduced in [13, 8], was applied later to nonholonomic manifolds with admissible Finsler metrics in [2, 4]. One says that over a distribution D a connection is given if the distribution $\tilde{D} = \pi_*^{-1}(D)$, where $\pi : D \to X$ is the natural projection, can be decomposed into a direct sum of the form

$$\tilde{D} = HD \oplus VD,$$

where VD is the vertical distribution on the total space D. Thus the assignment of a connection over a distribution is equivalent to the assignment of the object $G_b^a(x^a, x^{n+a})$ such that

$$HD = \mathrm{span}(\vec{\epsilon}_a),$$

where $\vec{\epsilon}_a = \partial_a - \Gamma_a^n \partial_n - G_a^b \partial_{n+b}$.

It can be checked in the usual way that the connection over the distribution D coincides with the linear connection in the nonholonomic manifold D if it holds

$$G_b^a(x^a, x^{n+a}) = \Gamma_{bc}^a(x^a) x^{n+c}.$$

In [4], the notion of prolonged connection was introduced. The prolonged connection can be obtained from an intrinsic connection by the equality

$$TD = \tilde{H}D \oplus VD,$$

where $HD \subset \tilde{H}D$. Essentially, the prolonged connection is a connection in a vector bundle.

An important example of a manifold with an admissible tensor structure and compatible with it intrinsic connection considered V.V. Wagner in [15]. In this paper, in a nonholonomic manifold an intrinsic metric is introduced using an admissible tensor field g that satisfies the usual properties of the metric tensor in a Riemannian space.

Similarly to the holonomic case, a metric on a nonholonomic manifold defines an intrinsic linear symmetric connection. The corresponding Christoffel symbols can be derived from the system of equations

$$\nabla_c g_{ab} = \vec{e}_c g_{ab} - \Gamma_{ca}^d g_{db} - \Gamma_{cb}^d g_{ad}.$$

Let φ be an admissible almost complex structure. We will use the following statement.

Theorem 5. *Each nonholonomic manifold with an almost complex structure φ and an intrinsic torsion-free linear connection ∇ admits an intrinsic linear connection $\overset{\smallsmile}{\nabla}$ compatible with the structure φ and having the torsion S such that*

$$S(\vec{u}, \vec{v}) = \frac{1}{4} P N_\varphi(\vec{u}, \vec{v}),$$

where $\vec{u}, \vec{v} \in \Gamma(D)$.

3   Interior characteristics of almost contact metric spaces

Now we introduce the notion of an almost contact metric structure in a new sense. Namely, we will say that a manifold of almost contact metric structure in the new sense is given if on a manifold X with a given contact form η in addition to a pair of admissible tensor structures (φ, g) such that

$$\varphi^2 \vec{u} = -\vec{u}, \quad g(\varphi\vec{u}, \varphi\vec{v}) = g(\vec{u}, \vec{v})$$

is given.

Theorem 6. *The notion of a manifold of almost contact metric structure in the new sense is equivalent to the notion of a manifold of almost contact metric structure in the old sense.*

We say that a Sasakian manifold in the new sense is given if on the manifold X with a given contact metric structure, in addition the equality $P(N) = 0$ holds. Theorems 3 and 6 imply the following statement.

Theorem 7. *The notion of a Sasakian manifold in the new sense is equivalent to the notion of a Sasakian manifold in the old sense.*

In this section we use the following notation. As above, admissible almost complex structure and Riemannian metric will be denoted by φ and g, respectively; the symbol ∇ will denote the intrinsic metric connection, and the symbols $\tilde{g}$ and $\tilde{\nabla}$ will denote the metric tensor in the ambient space and its Levi-Civita connection, respectively.

Theorem 8. *A contact metric structure is normal if and only if the structure φ is quasi-integrable and it holds $\nabla\varphi = 0$, where ∇ is an intrinsic metric connection.*

Note that the equality $\nabla\varphi = 0$ is not true if the connection ∇ and the field of endomorphisms φ are considered as the structures defined on the whole manifold, see e.g. [1].

Next suppose that ∇^1 is the extended connection constructed from the intrinsic connection in the following way:

$$\tilde{H}D = HD \oplus \text{span}(\partial_n),$$

here ∂_n is a vector field on the manifold D. The extended connection allows to formulate the next characteristic feature of the integrability of an almost complex structure φ.

Theorem 9. *An almost complex structure φ is integrable if and only if the equality $\nabla^1\varphi = 0$ holds.*

Finally we formulate a statement concerning K-contact manifolds.

Theorem 10. *An almost contact metric structure is a K-contact structure if and only if the metric g is quasi integrable.*

The theorem follows from the following equivalences:

$$L_{\tilde{\xi}}\tilde{g} = 0 \Leftrightarrow L_{\tilde{\xi}}g = 0 \Leftrightarrow \partial_n g = 0.$$

4 Almost contact metric structure on the total space of the vector bundle (D, π, X).

A coordinate map $K(x^\alpha)$ defines on the total space of the vector bundle (D, π, X) the coordinate map $\tilde{\chi}(\vec{\xi}) = (x^a, x^{n+a})$, where $\vec{\xi} = x^{n+a}\vec{e}_a$. If on the manifold X an admissible Finslerian structure is given, then on the distribution D appears an infinitesimal connection defined by the distribution $HD = span(\vec{\epsilon}_\alpha)$, where

$$\vec{\epsilon}_a = \partial_a - \Gamma_a^n \partial_n - G_{ac}^b x^{n+c}\partial_{n+b}, \quad G_{bc}^a = G_{\cdot b \cdot c}^a = \partial_{n+b}\partial_{n+c}G^a,$$

$$G^a = g^{ab}(\partial_c L_{\cdot b}^2 x^{n+c} - \vec{e}_b L^2), \quad g_{ab} = \frac{1}{2}L_{\cdot a \cdot b}^2,$$

see [4]. We define on the manifold D an admissible field of endomorphisms J to the distribution $HD = span(\vec{\epsilon}_\alpha)$, by putting $J(\vec{\epsilon}_a) = \partial_{n+a}$, $J(\partial_{n+a}) = -\vec{\epsilon}_a$. Using the equalities

$$\tilde{g}(\vec{u}^h, \vec{v}^h) = \tilde{g}(\vec{u}^v, \vec{v}^v) = g(\vec{u}, \vec{v}), \quad \tilde{g}(\vec{u}^h, \vec{v}^v) = 0,$$

where g is an admissible Finslerian structure on the manifold D, an admissible Riemannian structure is defined. The equality

$$\tilde{g}(J(\vec{u}), J(\vec{v})) = \tilde{g}(\vec{u}, \vec{v})$$

implies the following theorem.

Theorem 11. *The pair $(J, \tilde{g})$ of admissible structures defines an almost contact metric structure on the manifold D.*

Let us find the conditions that imply the integrability of an admissible almost complex structure J. The coordinate map $\tilde{\chi}(\vec{\xi})$ defines the nonholonomic basis field $(\vec{\epsilon}_a, \partial_n, \partial_{n+a})$ on the manifold D. The direct computations imply

$$[\vec{\epsilon}_a, \vec{\epsilon}_b] = 2\omega_{ab}\partial_n + R_{ba}^c\partial_{n+c},$$

$$[\vec{\epsilon}_a, \partial_n] = \partial_n\Gamma_a^n\partial_n + \partial_n G_a^b\partial_{n+b},$$

$$[\vec{\epsilon}_a, \partial_{n+b}] = G_{ab}^c\partial_{n+c},$$

where $R_{ba}^c = 2(\vec{e}_{[b}G_{a]}^c - G_{[a}^d G_{b]\cdot d}^c)$. The last equality and Theorem 3 imply the following.

Theorem 12. *An almost complex structure J is integrable if and only if it holds*

$$R_{ab}^c = 0, \quad \partial_n G_a^b = 0.$$

References

[1] Boyer, Ch.P., Nakamaye, M., *On Sasaki-Einstein manifolds in dimension five*, Geom. Dedicata **144** (2010), 141–156.

[2] Bukusheva, A.V., Galaev, S.V. *About admissible Kählerian structure on the tangent space to a nonholonomic manifold*, Math. Mech. Col. Sci. Works, Saratov, SGU **7** (2005), 12–14.

[3] Chern, S.-S., *Pseudo-groupes continus infinis*, Géométrie différentielle. pp. 119–136 Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, Centre National de la Recherche Scientifique, Paris, 1953.

[4] Galaev, S.V., *Extension of the interior connection of a nonholonomic manifold with a Finsler metric*, (arXiv: 1103.4337), 2011.

[5] Galaev, S.V., *The Intrinsic Geometry of Almost Contact Metric Manifolds*, arXiv:1107.5532.

[6] Gray, J.W., *Some global properties of contact structures*, Ann. of Math. **69** (1959), 421–450.

[7] Kirichenko, V.F., *Methods of generalized Hermitian geometry in the theory of almost contact manifolds*, J. Soviet Math. **42** (1988), no. 5, 1885–1919.

[8] Manin, Yu.I., *Gauge field theory and complex geometry,* Grundlehren der Mathematischen Wissenschaften, 289. Springer-Verlag, Berlin, 1997. xii+346 pp.

[9] Miron, R., *Techniques of Finsler geometry in the theory of vector bundles*, Acta Sci. Math. **49** (1985), 119 – 129.

[10] Sai Prasad, K. L., *Quarter Symmetric Metric Finsler Connections on Kenmotsu and P-Kenmotsu Vector Bundles.* International Mathematical Forum **3** (2008), no. 18, 847 – 855.

[11] Sasaki, S., *On differentiable manifolds with certain structures which are closely related to almost contact structure*, I. Tohoku Math. J. **12** (1960), no. 2, 459–476.

[12] Stamin, C., Udriste, C., *Nonholonomic geometry of Gibbs contact structure*, A Appl. Math. Phys. – Politehn. Univ. Bucharest Sci. Bull. Ser. **72** (2010), no. 1, 153–170.

[13] Vershik, A.M., Gershkovich, V.Ya., *Nonholonomic dynamical systems. Geometry of distributions and variational problems*, Current problems in mathematics. Fundamental directions, Vol. 16, 5—85, Moscow, 1987.

[14] Wagner, V.V., *Differential geometry of nonholonomic manifolds*, VIII international competition on the prize of N.I. Lobachevsky (1937). Report. Kazan: Kazan Phys. Math. Soc. 1940.

[15] Wagner, V.V., *Geometry of $(n-1)$-dimensional nonholonomic manifold in an n-dimensional space*, Proc. Sem. on vect. and tens. analysis (Moscow Univ.). **5** (1941), 173–255.