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A DIRECT APPROACH FOR TIME OPTIMAL CONTROL PROBLEM WITH LINEAR DIFFERENTIAL SYSTEM

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Abstract

The purpose of this paper is to present an approach to solve the time-optimal control problem. While searching the control as a piecewise constant function the optimal control problem is reduced to a nonlinear programming problem. Two examples are presented, in which cases the computation is carried out with the *Mathematica* software.

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Key words: time optimal control problem, nonlinear programming, computer algebra system.

1 Introduction

The control of a time optimal control problem with linear differential system is of bang-bang type with unknown switching times.

The time optimal control problem considered in this paper is

$$\text{minimize } T \tag{1}$$

subject to the constrains:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) && (2)\\ x(0) &= x_0 && (3)\\ x(T) &= x_T && (4)\\ u(t) &\in U && (5) \end{aligned}$$

where $x(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^q$. The elements of the matrix $A(t) \in M_{m,m}(\mathbb{R})$ and $B(t) \in M_{m,q}(\mathbb{R})$ are supposed to be continuous. U is a convex subset of $\mathbb{R}^q$.

Several computational techniques are derived to solve the time optimal control problems. These techniques are based on transforming into an optimization problem [1], [3] (i.e. the control functions and/or state functions are discretized) and/or transforming into

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a two boundary value problem (the transformation is based on the necessary optimality conditions), [2].

Here we shall discretize only the control functions as in [2]. A control function will be searched as a piecewise constant function. In the given examples, using the *Mathematica* Computer Algebra System, the ordinary differential equations are integrated symbolically.

The main component of this approach is the minimization procedure.

2 The transformation of optimal control into nonlinear programming problem

Let $X(t)$ be a fundamental system of the linear differential system (2) (i.e. the columns of $X(t)$ are m linear independent solutions of the homogeneous linear differential system $\dot{x}(t) = A(t)x(t)$). Denoting $H(t,s) = X(t)X^{-1}(s)$ the solution of the initial value problem (2)-(3) is

$$x(t) = H(t,0)x_0 + \int_0^t H(t,s)B(s)u(s)\mathrm{d}s.$$

The constraint (4) will be

$$x_T = H(T,0)x_0 + \int_0^T H(T,s)B(s)u(s)\mathrm{d}s.$$

or

$$\int_0^T X^{-1}(s)B(s)u(s)\mathrm{d}s = X^{-1}(T)x_T - X^{-1}(0)x_0. \qquad (6)$$

We search an approximation the optimal control as a piecewise constant function: for a mesh

$$0 = t_0 < t_1 < \ldots < t_n = T$$

let be $\tilde{u}(t) = u_i,\ t \in (t_{i-1}, t_i],\ u_i \in U,\ i \in \{1, 2, \ldots, n\}$.

Substituting $u = \tilde{u}$ in (6) we obtain

$$\sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} X^{-1}(s)B(s)\mathrm{d}s \right) u_i = X^{-1}(T)x_T - X^{-1}(0)x_0.$$

Denoting

$$C_i = \int_{t_{i-1}}^{t_i} X^{-1}(s)B(s)\mathrm{d}s \in M_{m,q}(\mathbb{R}), \quad i \in \{1, 2, \ldots, n\}$$

and

$$d = X^{-1}(T)x_T - X^{-1}(0)x_0 \in \mathbb{R}^m,$$

the following nonlinear programming problem

$$\text{minimize } g_0(T, u_1, \ldots, u_n) = T \qquad (7)$$

subject to

$$g(T, u_1, \ldots, u_n) = \sum_{i=1}^{n} C_i u_i = d; \tag{8}$$

$$u_i \in U \qquad i \in \{1, 2, \ldots, n\}. \tag{9}$$

Usually, the nodes t_i are equidistant, $t_i = \frac{T}{n} i,\ i \in \{1, 2, \ldots, n\}$, for some prescribed $n \in \mathbb{N}^*$. As a consequence matrix C_i depends on T and thus the minimization problem is nonlinear.

3 Examples

The computation was carried out with *Mathematica*. *Mathematica* allows a simple and nice way to generate the constraints for any $n \in \mathbb{N}^*$. The minimization is realized calling the `NMinimize` *Mathematica* function.

1. minimize T

subject to

$$\begin{array}{lll} \dot{x}_1 = x_2 & x_1(0) = x_1^0 & x_1(T) = 0 \\ \dot{x}_2 = u & x_2(0) = x_2^0 & x_2(T) = 0 \\ |u| \leq 1 & & \end{array}$$

The solution may be easily computed using the Pontryagin's maximum principle, [4]. Almost any introductory tutorial of optimal control presents this example, but here we are interested in a solution obtained by a computer for arbitrary x_1^0, x_2^0. The plot of the possible optimal trajectories a given in Fig.1.

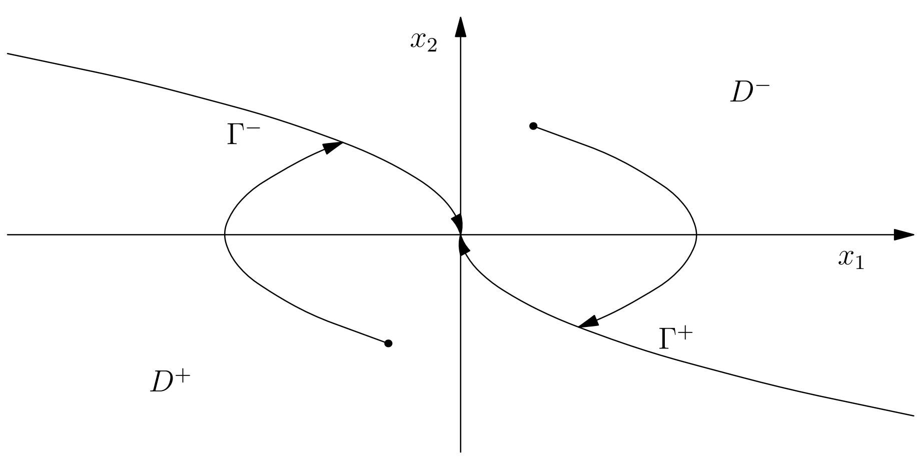

Fig. 1: The shape of the optimal state trajectories.

A fundamental matrix of the corresponding homogeneous system is

$$X(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The constraints (8) are

$$\begin{aligned} \sum_{i=1}^{n} u_i \int_{t_{i-1}}^{t_i} s\mathrm{d}s &= x_1^0, \\ \sum_{i=1}^{n} u_i \int_{t_{i-1}}^{t_i} \mathrm{d}s &= -x_2^0. \end{aligned}$$

For equidistant nodes, the above equations become

$$\frac{T^2}{n^2}\sum_{i=1}^{n}(i-\frac{1}{2})u_i = x_1^0 \quad \Leftrightarrow \quad \frac{T^2}{n^2}\sum_{i=1}^{n} iu_i = x_1^0 - \frac{T}{2n}x_2^0$$

$$\frac{T}{n}\sum_{i=1}^{n} u_i = -x_2^0$$

The optimization problem is: minimize T subject to the constraints:

$$\frac{T^2}{n^2}\sum_{i=1}^{n} iu_i - x_1^0 + \frac{T}{2n}x_2^0 = 0,$$

$$\frac{T}{n}\sum_{i=1}^{n} u_i + x_2^0 = 0,$$

$$|u_i| \le 1,\ \forall i \in \{1, 2, \ldots, n\}.$$

The *Mathematica* code to solve this nonlinear programming problem is

```
OCP[n_, x10_, x20_] :=
 NMinimize[
  Join[{T, x10 - T x20/(2 n) -
     T^2/n^2 Sum[
        i ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}] == 0,
     x20 + T/n Sum[
        ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}] == 0,
     T > 0}, Table[
     ToExpression[StringJoin["-1<=u", ToString[i]]], {i, 1, n}],
     Table[ToExpression[StringJoin["1>=u", ToString[i]]], {i, 1, n}]],
  Join[{T},
   Table[ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}]]]
```

To obtain a valid solution, there is required the additional constraint $T > 0$.

For $n = 64, (x_1^0, x_2^0) = (3, 2) \in D^-$ we have obtained $T = 6.47259$ in concordance with the theoretical value $T = x_2^0 + 2\sqrt{\frac{1}{2}(x_2^0)^2 + x_1^0}$. The obtained results are

```
{6.47259, {T -> 6.47259, u1 -> -1., u2 -> -1., u3 -> -1., u4 -> -1.,
  u5 -> -1., u6 -> -1., u7 -> -1., u8 -> -1., u9 -> -1., u10 -> -1.,
  u11 -> -1., u12 -> -1., u13 -> -1., u14 -> -1., u15 -> -1.,
  u16 -> -1., u17 -> -1., u18 -> -1., u19 -> -1., u20 -> -1.,
  u21 -> -1., u22 -> -1., u23 -> -1., u24 -> -1., u25 -> -1.,
```

```
u26 -> -1., u27 -> -1., u28 -> -1., u29 -> -1., u30 -> -1.,
u31 -> -0.999999, u32 -> -0.999999, u33 -> -0.999999,
u34 -> -0.999999, u35 -> -0.999999, u36 -> -0.999999,
u37 -> -0.999999, u38 -> -0.999999, u39 -> -0.999997,
u40 -> -0.999997, u41 -> -0.999993, u42 -> -0.775689,
u43 -> 0.999989, u44 -> 0.999996, u45 -> 0.999998, u46 -> 0.999998,
u47 -> 0.999999, u48 -> 0.999999, u49 -> 0.999999, u50 -> 0.999999,
u51 -> 0.999999, u52 -> 0.999999, u53 -> 0.999999, u54 -> 1.,
u55 -> 1., u56 -> 1., u57 -> 1., u58 -> 1., u59 -> 1., u60 -> 1.,
u61 -> 1., u62 -> 1., u63 -> 1., u64 -> 1.}}
```

The plot of the corresponding control is given in Fig. 2.

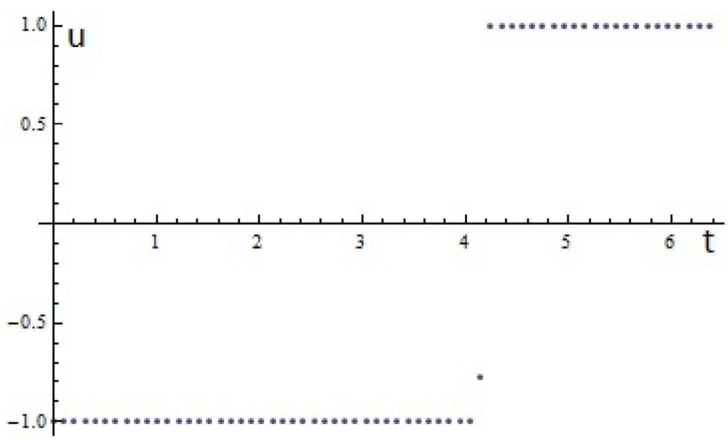

Fig. 2: Plot of the control for Example 1.

An approximation of the switching time is $\xi_1 \approx t_{42} = 4.24764$.

2. The second problem is chosen from [2] (with several included references):

$$\text{minimize } T$$

subject to

$$\begin{array}{lll}
\dot{x}_1 = x_3 & x_1(0) = 0 & x_1(T) = 1 \\
\dot{x}_2 = x_4 & x_2(0) = 0 & x_2(T) = 1 \\
\dot{x}_3 = \frac{u}{m_1} - \frac{k}{m_1}(x_1 - x_2) & x_3(0) = 0 & x_3(T) = 0 \\
\dot{x}_4 = \frac{k}{m_2}(x_1 - x_2) & x_4(0) = 0 & x_4(T) = 0 \\
|u| \le 1 & &
\end{array}$$

For $m_1 = m_2 = k = 1$, using *Mathematica* we have found the fundamental matrix

$$X(t) =$$

$$\begin{pmatrix}
\text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 & \frac{1}{4}\left(2t + \sqrt{2}\text{Sin}\left[\sqrt{2}t\right]\right) & \frac{1}{4}\left(2t - \sqrt{2}\text{Sin}\left[\sqrt{2}t\right]\right) \\
\text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 & \frac{1}{4}\left(2t - \sqrt{2}\text{Sin}\left[\sqrt{2}t\right]\right) & \frac{1}{4}\left(2t + \sqrt{2}\text{Sin}\left[\sqrt{2}t\right]\right) \\
-\frac{\text{Sin}\left[\sqrt{2}t\right]}{\sqrt{2}} & \frac{\text{Sin}\left[\sqrt{2}t\right]}{\sqrt{2}} & \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 \\
\frac{\text{Sin}\left[\sqrt{2}t\right]}{\sqrt{2}} & -\frac{\text{Sin}\left[\sqrt{2}t\right]}{\sqrt{2}} & \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2
\end{pmatrix}$$

with the inverse

$$X^{-1}(t) =$$

$$\begin{pmatrix} \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 & \frac{1}{4}\left(-2t-\sqrt{2}\text{Sin}\left[\sqrt{2}t\right]\right) & \frac{1}{4}\left(-2t+\sqrt{2}\text{Sin}\left[\sqrt{2}t\right]\right) \\ \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 & \frac{1}{4}\left(-2t+\sqrt{2}\text{Sin}\left[\sqrt{2}t\right]\right) & \frac{1}{4}\left(-2t-\sqrt{2}\text{Sin}\left[\sqrt{2}t\right]\right) \\ \frac{\text{Sin}\left[\sqrt{2}t\right]}{\sqrt{2}} & -\frac{\text{Sin}\left[\sqrt{2}t\right]}{\sqrt{2}} & \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 \\ -\frac{\text{Sin}\left[\sqrt{2}t\right]}{\sqrt{2}} & \frac{\text{Sin}\left[\sqrt{2}t\right]}{\sqrt{2}} & \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 \end{pmatrix}$$

Because $\int X^{-1}(t)B(s)\mathrm{d}s =$

$$\left\{\frac{1}{4}\left(-s^2+\text{Cos}\left[\sqrt{2}s\right]\right), \frac{1}{4}\left(-s^2-\text{Cos}\left[\sqrt{2}s\right]\right), \frac{s}{2}+\frac{\text{Sin}\left[\sqrt{2}s\right]}{2\sqrt{2}}, \frac{s}{2}-\frac{\text{Sin}\left[\sqrt{2}s\right]}{2\sqrt{2}}\right\}$$

and $d = \{1,1,0,0\}$, after some simple algebraic processing, the constraints (8) become

$$\frac{T^2}{n^2}\sum_{i=1}^n iu_i = -1, \qquad \sum_{i=1}^n u_i \sin\frac{T(i-\frac{1}{2})\sqrt{2}}{n} = 0,$$

$$\sum_{i=1}^n u_i \cos\frac{T(i-\frac{1}{2})\sqrt{2}}{n} = 0, \qquad \sum_{i=1}^n u_i = 0.$$

The *Mathematica* code for the minimization function is

```
OCP[n_] :=
  NMinimize[
   Join[{T, T^2/(2 n^2) Sum[
        i ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}] == -1,
     Sum[ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}] == 0,
     Sum[ToExpression[StringJoin["u", ToString[i]]] Cos[
         T (i - 0.5) Sqrt[2]/n], {i, 1, n}] == 0,
     Sum[ToExpression[StringJoin["u", ToString[i]]] Sin[
         T (i - 0.5) Sqrt[2]/n], {i, 1, n}] == 0, T > 3},
    Table[ToExpression[StringJoin["-1<=u", ToString[i]]], {i, 1, n}],
    Table[ToExpression[StringJoin["1>=u", ToString[i]]], {i, 1, n}]],
   Join[{T},
    Table[ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}]]]
```

Here the additional constraint is $T > 3$.

For $n = 32$ we obtained

```
{4.22218, {T -> 4.22218, u1 -> 1., u2 -> 1., u3 -> 1., u4 -> 1.,
  u5 -> 1., u6 -> 1., u7 -> 1., u8 -> 0.228417, u9 -> -1., u10 -> -1.,
   u11 -> -1., u12 -> -1., u13 -> -1., u14 -> -1., u15 -> -1.,
  u16 -> -1., u17 -> 1., u18 -> 1., u19 -> 1., u20 -> 1., u21 -> 1.,
  u22 -> 1., u23 -> 1., u24 -> 1., u25 -> -0.228417, u26 -> -1.,
  u27 -> -1., u28 -> -1., u29 -> -1., u30 -> -1., u31 -> -1.,
  u32 -> -1.}}
```

Thus $T = 4.22218$. The plot of the corresponding control is given in Fig. 3.

The approximation of the switching times is $\xi_1 \approx t_8 = 1.05555, \xi_2 \approx t_{16+\frac{1}{2}} = 2.17706, \xi_3 \approx t_{25} = 3.29858$. These results agree with the results reported in [2].

The drawback of this approach is that an additional constraint is required and that the time to evaluate the minimization function is frustrating. On a two cores computer, the duration to solve the two examples, for $n = 64$ and respectively $n = 32$, is a few minutes.

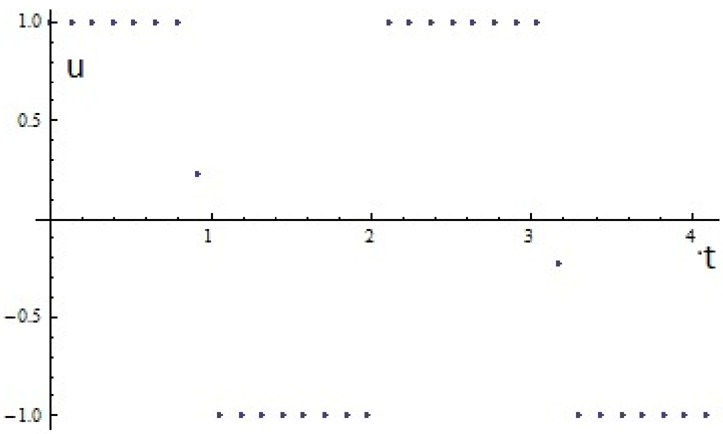

Fig. 3: Plot of the control for Example 2.

4 Conclusions

If the transformation of the optimal control problem into a mathematical programming problem is straightforward, the contribution of this paper is the *Mathematica* coding to generate that mathematical programming problem. It results a simple method to solve a class of time optimal control problems. The method requires only the general-purpose *Mathematica* software.

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