Bulletin of the *Transilvania* University of Braşov • Vol 5(54), No. 2 - 2012
Series III: Mathematics, Informatics, Physics, 113-116

SOME CRITERIA FOR UNIVALENT FUNCTIONS

Horiana TUDOR[1]

Abstract

In this paper we establish some very simple and useful univalence criteria for a class of functions defined by an integral operator.

2000 *Mathematics Subject Classification:* 30C55 .
Key words: analytic functions, univalent functions, univalence criteria.

1 Introduction

Let A be the class of analytic functions f in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0, f'(0) = 1$. We denote by $U_r = \{ z \in \mathbb{C} : |z| < r\}$ the disk of z-plane, where $r \in (0, 1]$, $U_1 = U$ and $I = [0, \infty)$.

Our considerations are based on the theory of Löewner chains; we recall the basic results of this theory, from Pommerenke.

A family of functions $\{L(z, t)\}$, $z \in U$, $t \in I$, is a Löewner chain if $L(z, t)$ is analytic and univalent in U for all $t \in I$, and $L(z, t)$ is subordinate to $L(z, s)$ for all $0 \leq t \leq s$.

Theorem 1. ([3]). *Let $L(z, t) = a_1(t)z + a_2(t)z^2 + \dots,\ \ a_1(t) \neq 0$ be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and locally uniformly with respect to U_r. For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z, t)$ is analytic in U_r and satisfies the condition Re $p(z, t) > 0$, for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z, t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, function $L(z, t)$ has an analytic and univalent extension to the whole disk U.

2 Main results

Theorem 2. *Let $f \in A$ and let α be a complex number, $\Re\alpha > 0$. If there exists an analytic function g in U, $g(z) = 1 + a_1 z + \dots$ such that the inequalities*

$$\left| \frac{f'(z)}{g(z)} - (\alpha + 1) \right| < |\alpha + 1| \tag{1}$$

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: htudor@unitbv.ro

and

$$\left| \frac{1}{\alpha+1}\left(\frac{f'(z)}{g(z)}-(\alpha+1)\right)|z|^{4\alpha}+\frac{1-|z|^{4\alpha}}{2\alpha}\left(\frac{zg'(z)}{g(z)}-\alpha\right)\right| \le 1 \tag{2}$$

are true for all $z \in U \setminus \{0\}$*, then function*

$$F_\alpha(z)=\left(\alpha\int_0^z u^{\alpha-1}f'(u)du\right)^{1/\alpha} \tag{3}$$

is analytic and univalent in U*, where the principal branch is intended.*

Proof. Let us consider function $h_1(z,t)$ given by

$$h_1(z,t)=2\alpha\int_0^{e^{-t}z} u^{\alpha-1}f'(u)du.$$

We have $h_1(z,t)=z^\alpha h_2(z,t)$, where it is easy to see that function h_2 is analytic in U for all $t\in I$ and $h_2(0,t)=2e^{-\alpha t}$. From the analyticity of g in U it follows that the function

$$h_3(z,t)=h_2(z,t)+(\alpha+1)(e^{4\alpha t}-1)e^{-\alpha t}g(e^{-t}z)$$

is also analytic in U and $h_3(0,t)=(\alpha+1)e^{3\alpha t}+(1-\alpha)e^{-\alpha t}$. We will prove that $h_3(0,t)\neq 0$ for all $t\in I$. We have $h_3(0,0)=2$. Assume that there exists $t_0>0$ such that $h_3(0,t_0)=0$. Then $e^{4\alpha t_0}=(\alpha-1)/(\alpha+1)$. Since $\Re\alpha>0$ is equivalent to $|(\alpha-1)/(\alpha+1)|<1$ and for $t_0>0$ we have $|e^{4\alpha t_0}|=e^{4\Re\alpha t_0}>1$, we conclude that $h_3(0,t)\neq 0$ for all $t\in I$. Therefore, there is a disk U_{r_1}, $0<r_1\le 1$, in which $h_3(z,t)\neq 0$ for all $t\in I$. Then we can choose an analytic branch of $[h_3(z,t)]^{1/\alpha}$ denoted by $h(z,t)$ which, at the origin, is equal to

$$a_1(t)=e^{3t}[(\alpha+1)+(1-\alpha)e^{-4\alpha t}]^{1/\alpha}. \tag{4}$$

From these considerations, it follows that the function

$$L(z,t)=z\cdot h(z,t)=a_1(t)z+a_2(t)z^2+\dots,$$

where $a_1(t)$ is given by (4), is analytic in U_{r_1} for all $t\in I$ and can be written as follows

$$L(z,t)=\left[2\alpha\int_0^{e^{-t}z}u^{\alpha-1}f'(u)du+(\alpha+1)(e^{4\alpha t}-1)e^{-\alpha t}z^\alpha g(e^{-t}z)\right]^{1/\alpha}. \tag{5}$$

Under the assumption of the theorem we have $a_1(t)\neq 0$ and $\lim_{t\to\infty}|a_1(t)|=\infty$. From the analyticity of $L(z,t)$ in U_{r_1}, it follows that there exists a number r_2, $0<r_2\le r_1$, and a constant $K=K(r_2)$ such that

$$|\ L(z,t)/a_1(t)\ | < K, \qquad \forall z\in U_{r_2},\quad t\ge 0,$$

and hence $\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_2}. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T>0$ and r_3, $0<r_3\le r_2$, there exists a constant $K_1>0$ (that depends on T and r_3) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1, \qquad \forall z\in U_{r_3},\quad t\in[0,T].$$

It follows that function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to $z \in \mathcal{U}_{r_3}$.

Let us set

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t} \qquad \text{and} \qquad w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}.$$

For all $t \in I$, function $p(z,t)$ is analytic in a disk U_r, $0 < r \le r_3$ and so is $w(z,t)$.

Function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$, if function $w(z,t)$ can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

By simple calculation, we obtain

$$w(z,t) = \frac{1}{\alpha+1}\left(\frac{f'(e^{-t}z)}{g(e^{-t}z)} - (\alpha+1)\right)e^{-4\alpha t} + \frac{1-e^{-4\alpha t}}{2\alpha}\left(\frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)} - \alpha\right). \tag{6}$$

From (1) and (2) we deduce that function $w(z,t)$ is analytic in the unit disk U. In view of (1), from (6) we have

$$|\, w(z,0)\,| = \left|\, \frac{1}{\alpha+1}\left(\, \frac{f'(z)}{g(z)} - (\alpha+1)\,\right)\right| \;<\; 1\,, \tag{7}$$

and also

$$|\, w(0,t)\,| = \left|\frac{1-\alpha}{2(\alpha+1)}e^{-4\alpha t} - \frac{1}{2}\right| \le \frac{1}{2}\left|\frac{\alpha-1}{\alpha+1}\right| e^{-4\Re\alpha t} + \frac{1}{2} < 1. \tag{8}$$

Let t be a fixed positive number, $z \in U$, $z \neq 0$. Since $|e^{-t}z| \le e^{-t} < 1$ for all $z \in \overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ we conclude that function $w(z,t)$ is analytic in $\overline{U}$. Using the maximum modulus principle it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|w(z,t)| < \max_{|\xi|=1} |w(\xi,t)| = |w(e^{i\theta},t)|. \tag{9}$$

We denote $u = e^{-t}\cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (6) we get

$$w(e^{i\theta},t) = \left|\, \frac{1}{\alpha+1}\left(\frac{f'(u)}{g(u)} - (\alpha+1)\right)|u|^{4\alpha} + \frac{1-|u|^{4\alpha}}{2\alpha}\left(\frac{ug'(u)}{g(u)} - \alpha\right)\right|.$$

Since $u \in U$, the inequality (2) implies $|w(e^{i\theta},t)| \le 1$ and from (7), (8) and (9) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \ge 0$.

From Theorem 1 it results that function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t \in I$, in particular

$$L(z,0) = \left(2\alpha\int_0^z u^{\alpha-1}f'(u)du\right)^{1/\alpha}.$$

Therefore function $F_\alpha(z)$ defined by (3) is analytic and univalent in U. □

Corollary 1. *Let $f \in A$ and let α be a complex number, $\Re\alpha > 0$, $|\alpha+1| \leq 2\Re\alpha$. If the inequality*

$$\left| \frac{zf'(z)}{f(z)} - (\alpha+1) \right| < |\alpha+1| \tag{10}$$

is true for all $z \in U$, then function F_α defined by (3) is analytic and univalent in U.

Proof. In the particular case $g(z) \equiv \frac{f(z)}{z}$, from (1) we get inequality (10). Also we observe that both terms of the sum which appear in inequality (2) contain the same expression $\frac{zf'(z)}{f(z)} - (\alpha+1)$ and then we try to obtain from Theorem 2 a very simple and useful univalence criterion. It is known (see [2]) that, for all $z \in U$, $z \neq 0$ and $\Re\alpha > 0$, we have

$$\left| \frac{1-|z|^{2\alpha}}{\alpha} \right| \leq \frac{1-|z|^{2\Re\alpha}}{\Re\alpha}. \tag{11}$$

In view of (10) and (11), inequality (2) is satisfied and hence function F_α defined by (3) is analytic and univalent in U. □

Example. Let α be a real number, $\alpha \geq 1$. For function $f(z) = z \cdot e^z$, the inequality (10) is verified and then function $F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1}(1+u)e^u du\right)^{1/\alpha}$ is analytic and univalent in U.

Corollary 2. *Let $f \in A$ and let α be a complex number, $\Re\alpha > 0$, $|\alpha| \leq 2\Re\alpha$. If the inequality*

$$\left| f'(z) - (\alpha+1) \right| < |\alpha+1| \tag{12}$$

is true for all $z \in U$, then function F_α defined by (3) is analytic and univalent in U.

Proof. For function $g(z) \equiv 1$, from (1) we get inequality (12). Using (11) and (12), we see that inequality (2) is verified

$$\left| \frac{1}{\alpha+1}\left(f'(z)-(\alpha+1)\right)|z|^{4\alpha} + \frac{1-|z|^{4\alpha}}{2\alpha}(-\alpha) \right| \leq |z|^{4\Re\alpha} + \frac{1-|z|^{4\Re\alpha}}{2\Re\alpha}|\alpha| \leq 1.$$

□

References

[1] Ovesea-Tudor, H., *On univalence condition* , Matematyka Zeszyty Naukowe Rzeszów **27** (2004), 39-42.

[2] Pascu, N.N., *An improvement of Becker's univalence criterion*, Ses. com. S. Stoilow, Univ.Braşov, 1987, 43-48.

[3] Pommerenke, Ch., *Univalent Functions*, Vandenhoech Ruprecht in Göttingen, 1975.