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A NOTE ON FUNDAMENTAL GROUP LATTICES

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Abstract

The main goal of this note is to provide a new proof of a classical result about projectivities between finite abelian groups. It is based on the concept of fundamental group lattice, studied in our previous papers [8] and [9]. A generalization of this result is also given.

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1 Introduction

The relation between the structure of a group and the structure of its subgroup lattice constitutes an important domain of research in group theory. One of the most interesting problems concerning it is to study whether a group G is determined by the subgroup lattice of the n-th direct power G^n, $n \in \mathbb{N}^*$. In other words, if the n-th direct powers of two groups have isomorphic subgroup lattices, are these groups isomorphic? For $n = 1$ it is well-known that this problem has a negative answer (see [4]). The same thing can be also said for $n = 2$, except for some particular classes of groups, as simple groups (see [5]), finite abelian groups (see [3]) or abelian groups with the square root property (see [2]). In the general case (when $n \geq 2$ is arbitrary) we recall Remark 1 of [2], which states that an abelian group is determined by the subgroup lattice of its n-th direct power if and only if it has the n-th root property. This follows from some classical results of Baer [1].

The starting point of our discussion is given by papers [8] and [9] (see also Section I.2.1 of [7]), where the concept of fundamental group lattice is introduced and studied. It gives an arithmetic description of the subgroup lattice of a finite abelian group and has many applications. Fundamental group lattices were successfully used in [8] to solve the problem of existence and uniqueness of a finite abelian group whose subgroup lattice is isomorphic to a fixed lattice and in [9] to count some types of subgroups of a finite abelian group. In this paper they will be used to prove that the finite abelian groups are determined by the subgroup lattices of their direct n-powers, for any $n \geq 2$. Notice that our proof is more simple than the original one. A more general result will be also inferred.

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Most of our notation is standard and will usually not be repeated here. Basic definitions and results on groups can be found in [6]. For subgroup lattice notions we refer the reader to [4] and [7].

In the following we recall the concept of fundamental group lattice and two related theorems. Let G be a finite abelian group and $L(G)$ be the subgroup lattice of G. Then, by the fundamental theorem of finitely generated abelian groups, exist (uniquely determined by G) numbers $k \in \mathbb{N}^*$ and $d_1, d_2, ..., d_k \in \mathbb{N} \setminus \{0, 1\}$ satisfying $d_1 | d_2 | ... | d_k$ such that

$$(*) \qquad G \cong \bigtimes_{i=1}^{k} \mathbb{Z}_{d_i}.$$

This decomposition of a G into a direct product of cyclic groups together with the form of subgroups of $\mathbb{Z}^k$ (see Lemma 2.1 of [8]) leads us to the following construction:

Let $k \geq 1$ be an integer. Then, for every $(d_1, d_2, ..., d_k) \in (\mathbb{N}\setminus\{0,1\})^k$, we consider the set $L_{(k;d_1,d_2,...,d_k)}$ consisting of all matrices $A = (a_{ij}) \in \mathcal{M}_k(\mathbb{Z})$ which satisfy the conditions:

I. $a_{ij} = 0$, for any $i > j$,

II. $0 \leq a_{1j}, a_{2j}, ..., a_{j-1j} < a_{jj}$, for any $j = \overline{1, k}$,

III. 1) $a_{11} | d_1$,

2) $a_{22} | \left(d_2, d_1 \dfrac{a_{12}}{a_{11}}\right)$,

3) $a_{33} | \left(d_3, d_2 \dfrac{a_{23}}{a_{22}}, d_1 \dfrac{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}{a_{22}a_{11}}\right)$,

$\vdots$

k) $a_{kk} | \left(d_k, d_{k-1} \dfrac{a_{k-1k}}{a_{k-1\,k-1}}, d_{k-2} \dfrac{\begin{vmatrix} a_{k-2\,k-1} & a_{k-2k} \\ a_{k-1\,k-1} & a_{k-1k} \end{vmatrix}}{a_{k-1\,k-1}a_{k-2\,k-2}}, ..., \right.$

$$\left. d_1 \frac{\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1k} \\ a_{22} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{k-1\,k} \end{vmatrix}}{a_{k-1\,k-1}a_{k-2\,k-2}...a_{11}}\right),$$

where by $(x_1, x_2, ..., x_m)$ we denote the greatest common divisor of numbers $x_1, x_2, ..., x_m \in \mathbb{Z}$. On the set $L_{(k;d_1,d_2,...,d_k)}$ we introduce the ordering relation " $\leq$ ", defined as follows: for $A = (a_{ij})$, $B = (b_{ij}) \in L_{(k;d_1,d_2,...,d_k)}$, put $A \leq B$ if and only if we have

1)′ $b_{11} | a_{11}$,

2)′ $b_{22} | \left(a_{22}, \dfrac{\begin{vmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{vmatrix}}{b_{11}}\right)$,

$$3)' \quad b_{33} \Big| \left(a_{33}, \frac{\begin{vmatrix} a_{22} & a_{23} \\ b_{22} & b_{23} \end{vmatrix}}{b_{22}}, \frac{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \end{vmatrix}}{b_{22} b_{11}} \right),$$

$$\vdots$$

$$k)' \quad b_{kk} \Big| \left(a_{kk}, \frac{\begin{vmatrix} a_{k-1\,k-1} & a_{k-1\,k} \\ b_{k-1\,k-1} & b_{k-1\,k} \end{vmatrix}}{b_{k-1\,k-1}}, \frac{\begin{vmatrix} a_{k-2\,k-2} & a_{k-2\,k-1} & a_{k-2\,k} \\ b_{k-2\,k-2} & b_{k-2\,k-1} & b_{k-2\,k} \\ 0 & b_{k-1\,k-1} & b_{k-1\,k} \end{vmatrix}}{b_{k-1\,k-1} b_{k-2\,k-2}}, ..., \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ b_{11} & b_{12} & \cdots & b_{1k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{k-1\,k} \end{vmatrix}}{b_{k-1\,k-1} b_{k-2\,k-2} ... b_{11}} \right).$$

Then $L_{(k;d_1,d_2,...,d_k)}$ forms a complete modular lattice with respect to $\leq$, called a *fundamental group lattice of degree* k. A powerful connection between this lattice and $L(G)$ has been established in [8].

Theorem A. *If G is a finite abelian group with the decomposition* $(*)$*, then its subgroup lattice $L(G)$ is isomorphic to the fundamental group lattice $L_{(k;d_1,d_2,...,d_k)}$.*

In order to study when two fundamental group lattices are isomorphic (that is, when two finite abelian groups are lattice-isomorphic), the following notation is useful. For every integer $n \geq 2$, we denote by $\pi(n)$ the set consisting of all primes dividing n. Let $d_i, d'_{i'} \in \mathbb{N} \setminus \{0,1\}$, $i = \overline{1,k}$, $i' = \overline{1,k'}$, such that $d_1 | d_2 | ... | d_k$ and $d'_1 | d'_2 | ... | d'_{k'}$. Then we shall write

$$(d_1, d_2, ..., d_k) \sim (d'_1, d'_2, ..., d'_{k'})$$

whenever the next three conditions are satisfied:

a) $k = k'$.

b) $d_i = d'_i, \ i = \overline{1, k-1}$.

c) The sets $\pi(d_k) \setminus \pi\left(\prod_{i=1}^{k-1} d_i\right)$ and $\pi(d'_k) \setminus \pi\left(\prod_{i=1}^{k-1} d'_i\right)$ have the same number of elements, say r. Moreover, for $r = 0$ we have $d_k = d'_k$ and for $r \geq 1$, by denoting $\pi(d_k) \setminus \pi\left(\prod_{i=1}^{k-1} d_i\right) = \{p_1, p_2, ..., p_r\}$, $\pi(d'_k) \setminus \pi\left(\prod_{i=1}^{k-1} d'_i\right) = \{q_1, q_2, ..., q_r\}$, we have

$$\frac{d_k}{d'_k} = \prod_{j=1}^{r} \left(\frac{p_j}{q_j}\right)^{s_j},$$

where $s_j \in \mathbb{N}^*$, $j = \overline{1, r}$.

The following theorem of [8] will play an essential role in proving our main results.

Theorem B. *Two fundamental group lattices* $L_{(k;d_1,d_2,...,d_k)}$ *and* $L_{(k';d'_1,d'_2,...,d'_{k'})}$ *are isomorphic if and only if* $(d_1, d_2, ..., d_k) \sim (d'_1, d'_2, ..., d'_{k'})$.

2 Main results

As we have already mentioned, large classes of non-isomorphic finite abelian groups exist whose lattices of subgroups are isomorphic. Simple examples of such groups are easily obtained by using Theorem B:

1. $G = \mathbb{Z}_6$ and $H = \mathbb{Z}_{10}$ (cyclic groups),
2. $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $H = \mathbb{Z}_2 \times \mathbb{Z}_{10}$ (non-cyclic groups).

Moreover, Theorem B allows us to find a subclass of finite abelian groups which are determined by their lattices of subgroups (see also Proposition 2.8 of [8]).

Theorem 2.1. *Let G and H be two finite abelian groups such that one of them possesses a decomposition of type* $(*)$ *with* $\pi(d_k) = \pi\left(\prod_{i=1}^{k-1} d_i\right)$. *Then* $G \cong H$ *if and only if* $L(G) \cong L(H)$.

Next we shall focus on isomorphisms between the subgroup lattices of the direct n-powers of two finite abelian groups, for $n \geq 2$. An alternative proof of the following well-known result can be also inferred from Theorem B.

Theorem 2.2. *Let G and H be two finite abelian groups. Then* $G \cong H$ *if and only if* $L(G^n) \cong L(H^n)$ *for some integer* $n \geq 2$.

Proof. Let $G \cong \bigtimes_{i=1}^{k} \mathbb{Z}_{d_i}$ and $H \cong \bigtimes_{i=1}^{k'} \mathbb{Z}_{d'_i}$ be the corresponding decompositions $(*)$ of G and H, respectively, and assume that $L(G^n) \cong L(H^n)$ for some integer $n \geq 2$. Then the fundamental group lattices

$$L_{(k;\, \underbrace{d_1, d_1, ..., d_1}_{n \text{ factors}}, ..., \underbrace{d_k, d_k, ..., d_k}_{n \text{ factors}})} \text{ and } L_{(k';\, \underbrace{d'_1, d'_1, ..., d'_1}_{n \text{ factors}}, ..., \underbrace{d'_{k'}, d'_{k'}, ..., d'_{k'}}_{n \text{ factors}})}$$

are isomorphic. By Theorem B, one obtains

$$(\underbrace{d_1, d_1, ..., d_1}_{n \text{ factors}}, ..., \underbrace{d_k, d_k, ..., d_k}_{n \text{ factors}}) \sim (\underbrace{d'_1, d'_1, ..., d'_1}_{n \text{ factors}}, ..., \underbrace{d'_{k'}, d'_{k'}, ..., d'_{k'}}_{n \text{ factors}})$$

and therefore $k = k'$ and $d_i = d'_i$, for all $i = \overline{1, k}$. These equalities show that $G \cong H$, which completes the proof. $\square$

Clearly, two finite abelian groups G and H satisfying $L(G^m) \cong L(H^n)$ for some (possibly different) integers $m, n \geq 2$ are not necessarily isomorphic. Nevertheless, a lot of conditions of this type can lead to $G \cong H$, as the following theorem shows.

Theorem 2.3. *Let G and H be two finite abelian groups. Then $G \cong H$ if and only if there are the integers $r \geq 1$ and $m_1, m_2, ..., m_r, n_1, n_2, ..., n_r \geq 2$ such that $(m_1, m_2, ..., m_r) = (n_1, n_2, ..., n_r)$ and $L(G^{m_i}) \cong L(H^{n_i})$, for all $i = \overline{1, r}$.*

Proof. Suppose that G and H have the decompositions in the proof of Theorem 2. For every $i = 1, 2, ..., r$, the lattice isomorphism $L(G^{m_i}) \cong L(H^{n_i})$ implies that $km_i = k'n_i$, in view of Theorem B. Set $d = (m_1, m_2, ..., m_r)$. Then $d = \sum_{i=1}^{r} \alpha_i m_i$ for some integers $\alpha_1, \alpha_2, ..., \alpha_r$, which leads to

$$kd = k \sum_{i=1}^{r} \alpha_i m_i = \sum_{i=1}^{r} \alpha_i k m_i = \sum_{i=1}^{r} \alpha_i k' n_i = k' \sum_{i=1}^{r} \alpha_i n_i.$$

Since $d \,|\, n_i$, for all $i = \overline{1, r}$, we infer that $k' \,|\, k$. In a similar manner one obtains $k \,|\, k'$, and thus $k = k'$. Hence $m_i = n_i$ and the group isomorphism $G \cong H$ is obtained from Theorem 2.2 □

Finally, we indicate an open problem concerning the above results.

Open problem. In Theorem 3 replace condition $(m_1, m_2, ..., m_r) = (n_1, n_2, ..., n_r)$ with other connections between numbers m_i and n_i, $i = 1, 2, ..., r$, such that the respective equivalence be also true.

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