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# HOMOGENOUS STELLAR MODEL HAVING THE CHEMICAL COMPOSITION: $X = 0.7405$ AND $Z = 0.0135$

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**Abstract**

A mathematical model having the mass equal with one solar mass, the abundance of the hydrogen $X = 0.7405$ the abundance of the helium $Y = 0.246$ and the abundance of the metals $Z = 0.0135$ is presented. This model corresponds to the old stars of Population II or to the stars of sequence of the subdwarfish stars with the deficiency of the metals.

In this paper the differential equations for the radiativ nucleus, their numerical solution and the numerical results of the model are presented.

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*Key words:* convective cover

## 1 Problem formulation

We consider a model of a star with a radiative nucleus and a convective cover. Solving the problem integration must be performed both from the centre and from the surface and the solutions thus obtained have to be connected, so that the continuity of the considered parameters should be ensured. To give a model of the interior of a star means to determine the variations of pressure, temperature, mass and luminosity along the ray. The following equations of hydrostatic equilibrium, mass distributions, luminosity and temperature are valid for the radiative nucleus (see, e.g., Menzel and others, 1963; Aller and McLaughlin, 1965; Cox and Giuli, 1968):

$$\begin{aligned}
\frac{dP(r)}{dr} &= \frac{GM(r)}{r^2}\rho(r) \qquad (1)\\
\frac{dM(r)}{dr} &= 4\pi r^2\rho(r)\\
\frac{dL(r)}{dr} &= 4\pi r^2\rho(r)\varepsilon(r)\\
\frac{dT(r)}{dr} &= -\frac{3}{4ac}\frac{\kappa(r)\rho(r)L(r)}{T^3(r)}\frac{L(r)}{4\pi r^2}
\end{aligned}$$

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$\rho(r)$ is the density at the $r$ distance from the centre, $\varepsilon(\rho, T, X, Y)$ is the energy generation per gram per second, $\kappa(\rho, T, X, Y)$ is the opacity corresponding to the mass unity, and $X, Y$ are the proportions of hydrogen and helium. The system (1) has the following limit conditions in the centre of the star:

$$M(0) = 0,\ L(0) = 0,\ P(0) = P(c) = ?,\ T(0) = T(c) = ? \text{ at } r = 0. \tag{2}$$

The law of the gas $P(r) = \frac{1}{\mu}\frac{\kappa}{H}\rho(r)\,T(r)$ is valid for the whole interior.

The hydrostatic equilibrium equation as well as the mass distribution and the adiabatic equations (Menzel and others, 1963):

$$\begin{aligned} \frac{dP(r)}{dr} &= \frac{GM(r)}{r^2}\rho(r) \\ \frac{dM(r)}{dr} &= 4\pi r^2 \rho(r) \\ P(r) &= K\rho(r)^{\frac{5}{3}} \text{ or } P(r) = K_1 T(r)^{2.5} \end{aligned} \tag{3}$$

are valid for the whole convective zone.

System (3) has the following boundary conditions at the star surface:

$$M = M_0,\ L = L_0,\ T = 0,\ P = 0, \text{ at } r = R_0. \tag{4}$$

Schwarzschild's transformations are applied to systems (1) and (3) (Schwarzschild,1958):

$$\begin{aligned} P(r) &= \frac{pGM^2}{4\pi R^4} \\ T(r) &= t\frac{\mu H}{\kappa}\frac{GM}{R} \\ M(r) &= q \cdot M \\ L(r) &= f \cdot L \\ r &= R \cdot x \end{aligned} \tag{5}$$

where henceforth $p, t, q, x, f$ are dimensionless variables. To produce the energy we consider the following formula :

$$\varepsilon = \varepsilon_0 \rho(r)\, T^{4.5}(r) \text{ where } \varepsilon_0 = 2.8 \cdot 10^{-33} X^2 \tag{6}$$

and for opacity :

$$\kappa = \kappa_0 \rho_{0.75}(r) T^{-3.5}(r) \text{ where } \kappa_0 = 6.52 \cdot 10^{24}(Z + \frac{X+Y}{59.3})(1+X)^{0.75} \tag{7}$$

Using Schwarzschild's transformations and laws (6) and (7) in systems (1) and (3), they become :

$$\frac{dp}{dx} = -\frac{pq}{tx^2},\ \frac{dq}{dx} = \frac{px^2}{t},\ \frac{df}{dx} = Dp^2x^2t^{2.5},\ \frac{dt}{dx} = -C\frac{p^{1.75}}{x^2 t^{8.25}} \tag{8}$$

respectively

$$\frac{dp}{dx} = -\frac{pq}{tx^2},\ \frac{dq}{dx} = \frac{px^2}{t},\ p = Et^{2.5} \tag{9}$$

where

$$\begin{aligned} E &= 4\pi K_1 \left(\frac{H}{k}\right)^{2.5} G^{1.5} M^{0.5} R^{1.5} \mu^{2.5} \\ C &= \frac{3\kappa_0}{4ac} \frac{1}{(4\pi)^{2.75}} \left(\frac{k}{HG}\right)^{7.5} \frac{LR^{1.25}}{M^{5.75}\mu^{7.5}} \\ D &= \frac{\varepsilon_0}{4} \left(\frac{GH}{k}\right)^{4.5} \frac{M^{6.5}}{LR^{7.5}} \mu^{4.5} \end{aligned} \tag{10}$$

The boundaries become as follows :

$$\begin{aligned} \text{at the centre} &: \quad x = 0, f = 0, q = 0, t =?, p =? \\ \text{and at the surface} &: \quad x = 1, f = 1, q = 1, t = 0, p = 0. \end{aligned} \tag{11}$$

If we start the integration of system (8), we obtain two infinite assemblies of solutions for the nucleus, due to the possibility of choosing the values of pressure and temperature in the centre. We perform another variable transformation, which will remove an infinite assembly of solutions for the radiative nucleus.

We consider :

$$\begin{aligned} x &= x_0 x^* \text{ and } t = t_0 t^*, \\ f &= f_0 f^*, \\ p &= p_0 p^* \text{ and } q = q_0 q^* \end{aligned} \tag{12}$$

where $x_0, t_0, f_0, p_0, q_0$ are indefinite constants. We impose the following form to the system (8) :

$$\begin{aligned} \frac{dp^*}{dx^*} &= -\frac{p^* q^*}{t^* x^{*2}} \\ \frac{dq^*}{dx^*} &= \frac{p^* x^{*2}}{t^*} \\ \frac{df^*}{dx^*} &= p^{*2} x^{*2} t^{*2.5} \\ \frac{dt^*}{dx^*} &= -\frac{p^{*1.75} f^*}{t^{*8.25} x^{*2}} \end{aligned} \tag{13}$$

and thus $x_0, t_0, p_0, f_0, q_0, C, D$ verify system (14) :

$$\frac{q_0}{t_0 x_0} = 1,\ \frac{p_0 x_0^3}{t_0 q_0} = 1,\ C\frac{p_0^{1.75} f_0}{t_0^{9.25} x_0} = 1,\ D\frac{p_0^2 t_0^{2.5} x_0^3 = 1}{f_0} \tag{14}$$

If we consider an already known chemical composition, we may calculate the value of $C$ and $D$, but beside them system (14) contains five unknown quantities, so one of them may be chosen. We have chosen $t_0 = t_c$, so $t_c^* = 1$.

Both system (9) and system (13) present singularities in the points where the boundary conditions are given. A difficult problem, the one of connecting the solutions should be elucidated. We have to ensure the continuity of parameters $P(r), T(r), M(r)$ and $L(r)$. The relations introduce three new parameters :

$$\begin{aligned} U &= \frac{d\log M(r)}{d\log r} \\ V &= \frac{d\log P(r)}{d\log r} \\ (n+1) &= \frac{d\log P(r)}{d\log T(r)} \end{aligned} \tag{15}$$

We perform the calculations in (15) and we obtain:

$$\begin{aligned} U &= 4\pi r^3 \frac{\rho(r)}{M(r)} = \frac{px^3}{qt} = \frac{p^* x^{*3}}{q^* t^*} \\ V &= \frac{\rho(r)}{P(r)} \frac{GM(r)}{r} = \frac{q}{tx} = \frac{q^*}{t^* x^*} \end{aligned} \tag{16}$$

and $(n+1)$ corresponding to the radiative nucleus from (15) will become:

$$(n+1)_{rad} = \frac{16\pi ac}{3} \frac{GM(r)T^4(r)}{P(r)\kappa(r)L(r)} = \frac{1}{C}\frac{qt^{8.25}}{fp^{1.75}} = \frac{q^* t^{*8.25}}{f^* p^{*1.75}} \tag{17}$$

We obtain $(n+1)$ corresponding to the convective zone and we get :

$$(n+1)_{conv} = 2.5 \tag{18}$$

Pressure and temperature being continuous functions, $(n+1)$ should be a continuous function. The convective zone begins in point $x^*$ where $(n+1) = 2.5$. Starting with a certain value for $p_c^*$, within the plan $(U,V)$ we obtain a corresponding curve with a final corresponding value $(U_i, V_i)$ where the radiative zone ceases to exist. Starting with a certain $E$ we can integrate system (9) and set out plot a corresponding curve in the plan $(U,V)$. But the continuity of the functions corresponding to mass and pressure asks a continuous curve in the plan $(U,V)$. Thus, if we choose a certain $E$, then we may choose a value for $p_c^*$ so that continuity within the plan $(U,V)$ should be obtained, but we may consider the problem the other way as well, that is to start by choosing $p_c^*$ and then to interpolate as against $E$.

We suppose that a connection for a certain $E$ and for a $p_c^*$ has been achieved, then we determine constants $x_0, p_0, f_0, q_0, t_0, C$ and $D$. The assumption that a connection has been achieved gives us the value of parameters $q, p, f, t$ at the overlapping both from the surface and from the centre, thus we know: $x_{is}, q_{is}, t_{is}, p_{is}$ and $x_{ic}, p_{ic}, t_{ic}, f_{ic}, q_{ic}$ and, as there is no energy produced within the convective zone, it follows that $f_{is} = 1$, where "*is*" shows that there is a value at the inference considered from surface, and "*ic*" shows that there is a value of a parameter, considered from the centre. Using (12), we have:

$$x_{is} = x_0 x_{ic}^*,\ p_{is} = p_0 p_{ic}^*,\ f_{is} = 1 = f_0 f_{ic}^*,\ q_{is} = q_0 q_{ic}^*,\ t_{is} = t_0 t_{ic}^*, \tag{19}$$

which give us values $x_0, f_0, t_0, q_0$. The system (14) gives us the values of C and D. We suppose that the values of $C$ and $D$ are calculated for a certain $E$ and $p_c^*$ for which a connection of the solutions has been achieved. Using the formulae of $C$ and $D$ given by (10), where $M, R, L$ which stand for mass, ray, luminosity corresponding to Sun at the present time are considered as known data and testing with different chemical compositions, we try to obtain values for $C$ and $D$, equal to those resulting from the calculation. Thus, once the calculus achieved, that is a chemical composition which has been determined, it should be reconsidered until a chemical composition as close as possible to the one determined in spectroscopy is obtained.

The formulas (1) – (19) are given in (Menzel, 1963).

## 2 The problem solved numerically

System (13) has the following limit conditions:

$$x^* = 0, f^* = 0, q^* = 0, t^* = 1 \text{ and } p^* \text{ chosen} \tag{20}$$

This system has a singularity in $x^* = 0$, but system (13) admits solutions in analytic form for each and every neighbourhood of this singularity point. These analytic solutions are prolonged by continuity in point $x^* = 0$ as well. We note $p_c^* = p_0$, considering the $\Sigma a_n x^n$ solutions and imposing the condition that these series should verify (13), we obtain:

$$\begin{aligned}
p(x) &= p_0 - \frac{1}{6} p_0^2 x^2 + \frac{1}{45}\left(p_0^3 - p_0^{5.75}\right)^4 + 0x^5 + A_6 x^6 + ... \qquad (21)\\
q(x) &= \frac{1}{3} p_0 x^3 + \frac{1}{30}\left(p_0^{4.75} - p_0^2\right) x^5 + 0x^6 + B_7 x^7 + ...\\
f(x) &= \frac{1}{3} p_0^2 x^3 - \left(\frac{1}{15} p_0^3 + \frac{1}{12} p_0^{5.75}\right) x^5 + 0x^6 + C_7 x^7 + ...\\
t(x) &= 1 - \frac{1}{6} p_0^{3.75} x^2 + \left(\frac{59}{1440} p_0^{7.75} - \frac{3}{32} p_0^{7.5}\right) x^4 + 0x^5 + D_6 x^6 + ...
\end{aligned}$$

Series (21) will help us in calculating the values of the solutions in four points contiguous to the origin and to the integration pass $h = 0.01$. In order to obtain the value of the solutions for the following point, we use Adams-Bashforth's extrapolation formula of the forth order (Mozynski, 1973):

$$V_{k+1} = V_k + h\left(\frac{55}{24}f_k - \frac{59}{24}f_{k-1} + \frac{37}{24}f_{k-2} - \frac{9}{24}f_{k-3}\right) \tag{22}$$

which allows us to calculate the solution in a certain point, if we know the values in four previous points.

Adams-Moulton's interpolation formula :

$$V_{k+1} = V_k + h\left[b_{-1}f_{k+1} + ... + b_3 f_{k-3}\right] \tag{23}$$

contains solution $V_{k+1}$ within the right term in the item $f_{k+1}$. From (22) we obtain a $V_{k+1}{}^{(0)}$ which substituted in (23) gives the possibility of obtaining a $V_{k+1}{}^{(1)}$. We apply the successive approximations method and we obtain:

$$\begin{aligned} V_{k+1}^{(n+1)} &= V_k + \frac{251}{720}hf\left(x_{k+1}, V_{k+1}^{(n)}\right) + \\ &+ \frac{h}{720}\left(646f_k - 264f_{k-1} + 106f_{k-2} - 19f_{k-3}\right) \end{aligned} \tag{24}$$

The process of approximation continues until $\left|V_{k+1}{}^{(n+1)} - V_{k+1}{}^{(n)} < 10^{-11}\right|$. System (9) will be intd under the following condition: for $x = 1, p = t = 0, q = 1$, $E$ chosen. We perform the variable $y = 1 - x$, we denote the variable by $x$ as well, and thus system (9) becomes:

$$\begin{aligned} \frac{dp}{dx} &= \frac{pq}{t\left(1-x\right)^2} \\ \frac{dq}{dx} &= -\frac{p\left(1-x\right)^2}{t} \\ \frac{dt}{dx} &= \frac{1}{2.5E}\frac{pq}{t^{2.5}\left(1-x\right)^2} \end{aligned} \tag{25}$$

It has a singularity in point $x = 0$ because of $t$.

For system (25) we propose Taylor's series:

$$\begin{aligned} p\left(x\right) &= a_1x + a_2x^2 + a_3x^3 + ... \\ q\left(x\right) &= 1 + b_1x + b_2x^2 + b_3x^3 + ... \\ t\left(x\right) &= c_1x + c_2x^2 + c_3x^3 + ... \end{aligned} \tag{26}$$

Returning to the old variable in the neighbourhood of x = 1, we have for system (9):

$$\begin{aligned} p\left(x\right) &= \frac{E}{2.5^{2.5}}\left(1-x\right)^{2.5} + ... \\ q\left(x\right) &= 1 - \frac{E}{2.5^{2.5}}\left(1-x\right)^{2.5} + ... \\ t\left(x\right) &= \frac{1}{2.5}\left(1-x\right) + \frac{14E}{4+25E}\left(1-x\right)^2 + ... \end{aligned} \tag{27}$$

We use (26) in calculating the value of the solutions in one single point contiguous to 1. In order to calculate the values of the solutions in the following three points, we use Runge-Kutta's method for non-autonomous systems :

$$
\begin{aligned}
V_{k+1} &= V_k + h(\frac{1}{6}l_1 + \frac{1}{3}l_2 + \frac{1}{3}l_3 + \frac{1}{6}l_4) \\
l_1 &= f\left(t_k, V_k\right) \text{ and } l_2 = f\left(x_k + \frac{h}{2}, V_k + \frac{h}{2}l_1\right) \\
l_3 &= f\left(t_k + \frac{h}{2}\right), V_k + \frac{h}{2}l_2 \\
l_4 &= f\left(t_k + h, V_k + hl_3\right) \\
h &= x_{k+1} - x_k = 10^{-3}
\end{aligned}
\tag{28}
$$

Thus, we obtain the values of the solutions in four points, which allow us to continue with the predictor-corrector method.

## 3 Results and conclusion

As we have already stated in the first chapter, we choose a $p_c^*$ and perform the interpolation considering different values of $E$ until we obtain a connection within the plan $(U, V)$, and with the help of values $C$ and $D$ resulting from the calculus, we determine a chemical composition. The whole calculus is repeated by choosing another $p_c^*$ and obtaining a new model until the corresponding chemical composition is as close as possible to the one obtained spectroscopically. The results obtained are presented in Table1.

In this table the pressure (P) is expressed in units of $10^{18}$ dyne/cm$^2$, the temperature (T) in units of $10^6$K, the density $\rho$ in gr/cm$^3$, $q$ is the reduced mass and $f$ is the reduced luminosity. After the connection of the solution of the radiative nucleus to the one of the convective zone the following values are obtained:

$$p_c^* = 0.6805133181685,\ E = 0.80,\ X = 0.7405,\ Y = 0.246,\ Z = 0.0135.$$

Solving system (13), using the boundary conditions in the centre of the star (20) indeterminacy appears under the form of 0/0. I have proposed the Taylor's series $\Sigma a_n x^n$ for the integration of this system:

$$
\begin{aligned}
p(x) &= p_0 + A_1x + A_2x_2 + A_3x^3 + ... \\
q(x) &= B_1x + B_2x^2 + B_3x^3 + ... \\
f(x) &= C_1x + C_2x^2 + C_3x^3 + ... \\
t(x) &= 1 + D_1x + D_2x^2 + D_3x^3 + ...
\end{aligned}
\tag{29}
$$

| x | P | q | f | T | ρ |
|---|---|---|---|---|---|
| 0.000 | 0.17799 | 0.000 | 0.000 | 13.7187 | 94.0188 |
| 0.0057 | 0.17778 | 0.129E-4 | 0.1727E-3 | 13.7133 | 93.9492 |
| 0.0231 | 0.17479 | 0.822E-3 | 0.789E-2 | 13.6327 | 92.9105 |
| 0.0403 | 0.16840 | 0.434E-2 | 0.0397 | 13.4475 | 90.6617 |
| 0.0807 | 0.14271 | 0.0325 | 0.2423 | 12.7285 | 81.2352 |
| 0.1038 | 0.12370 | 0.0654 | 0.4132 | 12.1496 | 73.7822 |
| 0.2018 | 0.04753 | 0.3293 | 0.9205 | 9.1723 | 37.5714 |
| 0.2537 | 0.02400 | 0.4990 | 0.9723 | 7.6797 | 22.6540 |
| 0.3546 | 0.531E-2 | 0.7586 | 0.9995 | 5.3750 | 7.1702 |
| 0.4066 | 0.235E-2 | 0.8436 | 0.9998 | 4.4781 | 3.8105 |
| 0.4527 | 0.112E-2 | 0.8964 | 0.9999 | 3.8123 | 2.1530 |
| 0.5046 | 0.494E-3 | 0.9366 | 0.9999 | 3.1861 | 1.1258 |
| 0.5450 | 0.258E-3 | 0.9576 | 0.9999 | 2.7590 | 0.6768 |
| 0.6055 | 0.965E-4 | 0.9777 | 0.9999 | 2.2442 | 0.3123 |
| 0.6546 | 0.426E-4 | 0.9872 | 0.9999 | 1.8837 | 0.1637 |
| 0.7008 | 0.192E-4 | 0.9928 | 0.9999 | 1.5909 | 0.0874 |
| 0.7555 | 0.709E-5 | 0.9966 | 0.9999 | 12879 | 0.0398 |
| 0.8045 | 0.272E-5 | 0.9983 | 0.9999 | 1.0528 | 0.0285 |
| 0.8507 | 0.102E-5 | 0.9992 | 0.9999 | 0.8619 | 0.0085 |
| 0.8952 | 0.375E-6 | 0.9997 | 0.9999 | 0.7231 | 0.0037 |

Figure 1:

where $p_0 = p_c^*$, and I take $x$ instead of $x^*$ for an easier use. I have assumed that the pressure $p(x)$, the temperature $t(x)$, the luminosity $f(x)$ and the mass $q(x)$ are continuous functions and using series (28) in (13) I obtained their expressions given by (21).

Then I have showed the classical methods of numeric integration used to solve such a system (formulae 22-24) using the successive approximations until

$$\left| V_{k+1}{}^{(n+1)} - V_{k+1}{}^{(n)} \right| < 10^{-11}.$$

When the boundary conditions at the surface of the star are used:

$$x = 1,\ f = 1,\ q = 1,\ t = 0,\ p = 0. \tag{30}$$

system (9), which corresponds to the convective cover, has also the indeterminacy under the form of 0/0. Using the series of powers, we obtained for the convective cover expression (27) and we have shown how formulae (28) are used to continue the integration of system (25). In conclusion this way of mathematical and numerical approaches permits the obtaining of any homogeneous stellar model which has a radiative nucleus and a convective cover. The papers quoted in the text were consulted at the writing of this paper. The other papers quoted in the References are recommended to be read for a better understanding of the studied theme.

The values of the constants which appear in the paper are
$G = 6.672 \cdot 10^{-8} cm^3 g^{-1} s^{-1}$
$R = 6.96 \cdot 10^{10} cm$
$H = 1.6725 \cdot 10^{-24} g$
$M = 1.9891 \cdot 10^{33} g$
$k = 1.3805 \cdot 10^{-16} erg/K$
$\mu = \frac{4}{3+5X-Z}$
$a = 7.564\ erg \cdot cm^{-2} deg^{-4}$
$c = 2.99792458 cm \cdot s^{-1}$
$L = 3.12 \cdot 10^{33} erg \cdot s^{-1}$

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