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ESTIMATES OF THE DEGREE OF APPROXIMATION FOR SMOOTH FUNCTIONS ON SIMPLEX

Maria TALPĂU DIMITRIU[1]

Abstract

We present general estimates with optimal constants of the degree of approximation by positive linear operators for smooth functions on simplex in $\mathbb{R}^d$ using weighted K-functionals of first order.

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1 Introduction

The quantitative estimate for the remainder in Taylor's formula using the least concave majorant of the modulus of continuity (via K-functional K_1^1) was established in [1]. In this paper we give the estimates for the remainder in Taylor's formula in several variables with the weighted K-functional $K_{1,\varphi}^{\infty}$ and we obtain general estimates for smooth functions on simplex in $\mathbb{R}^d$, $d \in \mathbb{N}$, defined by:

$$S = \left\{ \mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d | x_1, \cdots, x_d \geq 0,\, x_1 + \cdots + x_d \leq 1 \right\}.$$

We use the notation e_0 for function $e_0 : \mathbb{R}^d \to \mathbb{R}$, $e_0(\mathbf{x}) = 1$, e_1 for function $e_1 : \mathbb{R}^d \to \mathbb{R}^d$, $e_1(\mathbf{x}) = \mathbf{x}$ and $\pi_i : \mathbb{R}^d \to \mathbb{R}$ for the projection on component i, $i \in \{1, \ldots, d\}$.

2 Estimates with $K_{1,\varphi}^{\infty}$

Starting from the weight function used in [2], we consider the function

$$\varphi(\mathbf{x}) = \left[(x_1 + \cdots + x_d)(1 - x_1) \cdots (1 - x_d) \right]^{\alpha},\ \alpha \in (0, 1).$$

We denote by $\mathbf{C}_\varphi(S) = \left\{ f \in \mathbf{C}(S \setminus \left\{ \mathbf{v}_i,\, i = \overline{0,d} \right\}) \,|\, (\exists) \lim\limits_{\mathbf{x} \to \mathbf{v_i}} f(\mathbf{x})\varphi(\mathbf{x}) \in \mathbb{R},\, i = \overline{0,d} \right\}$ and

$$\mathbf{W}^1_{\mathbf{C}_\varphi}(S) = \left\{ f \in \mathbf{C}(\mathbf{S}) \,|\, \frac{\partial f}{\partial x_i} \in \mathbf{C}_\varphi(S),\, i = \overline{1,d} \right\},$$

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: mdimitriu@unitbv.ro

where $\mathbf{v_i}$, $i=\overline{0,d}$ are simplex vertices. We consider the K-functional

$$K_{1,\varphi}^{\infty}(f,t)=K^{\infty}\left(f,t;\mathbf{C}(S),\mathbf{W}^1_{\mathbf{C}_\varphi}(S)\right),\ t>0,\ 1\le s\le\infty$$

defined for the Banach space $(\mathbf{C}(S),\|\cdot\|)$ and the semi-Banach subspace $\left(\mathbf{W}^1_{\mathbf{C}_\varphi}(S),|\cdot|_{W^1_{C_\varphi}}\right)$, $|f|_{W^1_{C_\varphi}}=\|\varphi\nabla f\|_\infty=\left\|\left(\left\|\varphi\frac{\partial f}{\partial x_1}\right\|,\cdots,\left\|\varphi\frac{\partial f}{\partial x_d}\right\|\right)\right\|_\infty$ by

$$K_{1,\varphi}^{\infty}(f,t)=\inf_{g\in\mathbf{W}^1_{\mathbf{C}_\varphi}(S)}\max\left\{\|f-g\|,t\,\|\varphi\nabla g\|_\infty\right\}.$$

Lemma 1. *If $f\in\mathbf{C^r}(S)$, $\mathbf{x}\in S\setminus\left\{\mathbf{v}_i,\ i=\overline{0,d}\right\}$, $\mathbf{y}\in S$ then for the remainder in Taylor's formula of order r we have the following estimate*

$$|R_{r,f,\mathbf{x}}(\mathbf{y})|\le\frac{1}{r!}\left(2+\frac{\|\mathbf{y}-\mathbf{x}\|_1}{t(r-\alpha+1)\varphi(\mathbf{x})}\right)\cdot \tag{1}$$

$$\cdot\sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^{d}|y_i-x_i|^{r_i}\,K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)$$

from where

$$|R_{r,f,\mathbf{x}}(\mathbf{y})|\le\frac{\|\mathbf{y}-\mathbf{x}\|_1^r}{r!}\left(2+\frac{\|\mathbf{y}-\mathbf{x}\|_1}{t(r-\alpha+1)\varphi(\mathbf{x})}\right)\max_{r_1+\cdots+r_d=r}K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right). \tag{2}$$

Proof. Let $f\in\mathbf{C^r}(S)$ and $\mathbf{x}\in S\setminus\left\{\mathbf{v}_i,\ i=\overline{0,d}\right\}$, $\mathbf{y}\in S$, $\mathbf{x}\ne\mathbf{y}$. We consider the function $\psi(u)=(1-u)\mathbf{x}+u\mathbf{y}$, $u\in[0,1]$ and $h(u)=f(\psi(u))$.

Step 1: we prove that

$$|R_{r,f,\mathbf{x}}(\mathbf{y})|=|R_{r,h,0}(1)|\le\frac{1}{r!}\left(2+\frac{1}{t(r-\alpha+1)\varphi(\mathbf{x})}\right)K_{1,\varphi\circ\psi}^{\infty}\left(h^{(r)},t\right). \tag{3}$$

Let $g\in\mathbf{W}^{\mathbf{r+1}}_{\mathbf{C}_{\varphi\circ\psi}}[0,1]$. Using the integral form of the remainder and the fact that the function $u\mapsto\dfrac{1-u}{\varphi(\psi(u))^{\frac{1}{\alpha}}}$, $u\in(0,1)$ is decreasing [2] we have

$$\begin{aligned}|R_{r,g,0}(1)|&=\left|\frac{1}{r!}\int_0^1 g^{(r+1)}(u)(1-u)^r du\right|\\&\le\frac{1}{r!}\int_0^1\left|\varphi(\psi(u))g^{(r+1)}(u)\right|\frac{(1-u)^r}{\varphi(\psi(u))}du\\&\le\frac{\left\|(\varphi\circ\psi)g^{(r+1)}\right\|}{r!}\int_0^1(1-u)^r\frac{1}{\varphi(\psi(0))(1-u)^\alpha}du\\&=\frac{\left\|(\varphi\circ\psi)g^{(r+1)}\right\|}{r!(r-\alpha+1)\varphi(\mathbf{x})}.\end{aligned}$$

We have

$$
\begin{aligned}
|R_{r,h-g,0}(1)| &= \left|(h-g)(1) - \sum_{k=0}^{r} \frac{(h-g)^{(k)}(0)}{k!}\right| = \left|R_{r-1,h-g,0}(1) - \frac{(h-g)^{(r)}(0)}{r!}\right| \\
&\le |R_{r-1,h-g,0}(1)| + \frac{\left\|(h-g)^{(r)}\right\|}{r!} \le \frac{2}{r!} \cdot \left\|h^{(r)} - g^{(r)}\right\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
|R_{r,h,0}(1)| &\le |R_{r,h-g,0}(1)| + |R_{r,g,0}(1)| \\
&\le \frac{1}{r!}\left(2\left\|h^{(r)} - g^{(r)}\right\| + \frac{1}{(r-\alpha+1)\varphi(\mathbf{x})}\left\|(\varphi\circ\psi)g^{(r+1)}\right\|\right) \\
&\le \frac{1}{r!}\left(2 + \frac{1}{t(r-\alpha+1)\varphi(\mathbf{x})}\right)\max\left\{\left\|h^{(r)} - g^{(r)}\right\|, t\left\|(\varphi\circ\psi)g^{(r+1)}\right\|\right\}.
\end{aligned}
$$

Since g is arbitrary this implies (3).

Step 2: From (3) it results

$$
|R_{r,f,\mathbf{x}}(\mathbf{y})| \le \frac{1}{r!}\left(2 + \frac{\|\mathbf{y}-\mathbf{x}\|_1}{t(r-\alpha+1)\varphi(\mathbf{x})}\right) K^{\infty}_{1,\varphi\circ\psi}\left(h^{(r)}, \frac{t}{\|\mathbf{x}-\mathbf{y}\|_1}\right). \tag{4}
$$

Let $\varepsilon > 0$. We choose $g_{r_1,\cdots,r_d} \in \mathbf{C^1}(S)$, $r_i \in \mathbb{N}\cup\{0\}$, $i = \overline{1,d}:$ $r_1 + \cdots + r_d = r$, such that

$$
\begin{aligned}
&K^{\infty}_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t\right) + \varepsilon \\
&\qquad \ge \max\left\{\left\|\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d}\right\|, \, t\left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_{\infty}\right\}.
\end{aligned}
$$

We consider the function

$$
h_0(u) = \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} g_{r_1,\cdots,r_d}\left(\psi(u)\right)\prod_{i=1}^{d}(y_i - x_i)^{r_i}.
$$

We have

$$
K^{\infty}_{1,\varphi\circ\psi}\left(h^{(r)}, \frac{t}{\|\mathbf{x}-\mathbf{y}\|_1}\right) \le \max\left\{\left\|h^{(r)} - h_0\right\|, \frac{t}{\|\mathbf{x}-\mathbf{y}\|_1}\left\|(\varphi\circ\psi)h_o'\right\|\right\}.
$$

Since

$$
\begin{aligned}
&\left|h^{(r)}(u) - h_0(u)\right| = \left|d^r f\left(\psi(u)\right)(\mathbf{y}-\mathbf{x})^r - h_0(u)\right| \\
&= \left|\sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d}\right)(\psi(u))\prod_{i=1}^{d}(y_i - x_i)^{r_i}\right| \\
&\le \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!}\left\|\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d}\right\| \cdot \prod_{i=1}^{d}|y_i - x_i|^{r_i}.
\end{aligned}
$$

hold for u arbitrary this implies

$$\left\|h^{(r)}-h_0\right\| \le \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!}\left\|\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}-g_{r_1,\cdots,r_d}\right\|\cdot\prod_{i=1}^d |y_i-x_i|^{r_i}.$$

Also, since

$$\begin{aligned}\left|\varphi(\psi(u))h_0'(u)\right| &= \left|\sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\varphi(\psi(u))dg_{r_1,\cdots,r_d}\left(\psi(u)\right)(\mathbf{y}-\mathbf{x})\prod_{i=1}^d (y_i-x_i)^{r_i}\right| \\ &\le \sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_\infty\cdot\|\mathbf{y}-\mathbf{x}\|_1\cdot\prod_{i=1}^d|y_i-x_i|^{r_i}.\end{aligned}$$

hold for u arbitrary this implies

$$\left\|(\varphi\circ\psi)h_0'\right\| \le \sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_\infty\cdot\|\mathbf{y}-\mathbf{x}\|_1\cdot\prod_{i=1}^d|y_i-x_i|^{r_i}.$$

Then

$$\begin{aligned}&K_{1,\varphi\circ\psi}^\infty\left(h^{(r)},\frac{t}{\|\mathbf{x}-\mathbf{y}\|_1}\right)\le\max\left\{\left\|h^{(r)}-h_0\right\|,\frac{t}{\|\mathbf{x}-\mathbf{y}\|_1}\left\|(\varphi\circ\psi)h_o'\right\|\right\}\\ &\le\sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^d|y_i-x_i|^{r_i}\max\left\{\left\|\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}-g_{r_1,\cdots,r_d}\right\|,t\left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_\infty\right\}\\ &\le\sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^d|y_i-x_i|^{r_i}\left(K_{1,\varphi}^\infty\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)+\varepsilon\right).\end{aligned}$$

Since ε is arbitrary this implies

$$\begin{aligned}&K_{1,\varphi\circ\psi}^\infty\left(h^{(r)},\frac{t}{\|\mathbf{x}-\mathbf{y}\|_1}\right)\\ &\qquad\le\sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^d|y_i-x_i|^{r_i}K_{1,\varphi}^\infty\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right).\end{aligned}$$

Finally, with (4) it results (1). □

Theorem 1. *Let* $r\in\mathbb{N}$, $L:\mathbf{C}(S)\longrightarrow\mathbf{C}(S)$ *a positive linear operator and* $f\in\mathbf{C}^{\mathbf{r}}(S)$.

Then $(\forall)\mathbf{x} \in S \setminus \left\{\mathbf{v}_i,\, i=\overline{0,d}\right\}$, $(\forall)t>0$ *we have*

$$\begin{aligned}
&|L(f,\mathbf{x})-f(\mathbf{x})| \le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| \\
&+\sum_{k=1}^{r}\frac{1}{k!}\sum_{k_1+\cdots+k_d=k}\frac{k!}{k_1!\cdots k_d!}\cdot\left|\frac{\partial^k f}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(\mathbf{x})\right|\cdot\left|L\left(\bigotimes_{i=1}^{d}(\pi_i-x_ie_0)^{k_i},\mathbf{x}\right)\right| \\
&+\left(\frac{2}{r!}L\left(\|e_1-\mathbf{x}e_0\|_1^r,\mathbf{x}\right)+\frac{L\left(\|e_1-\mathbf{x}e_0\|_1^{r+1},\mathbf{x}\right)}{tr!(r-\alpha+1)\varphi(\mathbf{x})}\right)\cdot \\
&\cdot\max_{r_1+\cdots+r_d=r}K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right).
\end{aligned}\tag{5}$$

Conversely, if $(\exists)A_k,\, B,\, C\ge 0,\ k=\overline{0,r}$ *such that*

$$\begin{aligned}
&|L(f,\mathbf{x})-f(\mathbf{x})| \le A_0\cdot|f(\mathbf{x})|\,|L(e_0,\mathbf{x})-1| \\
&+\sum_{k=1}^{r}A_k\sum_{k_1+\cdots+k_d=k}\frac{k!}{k_1!\cdots k_d!}\cdot\left|\frac{\partial^k f}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(\mathbf{x})\right|\cdot\left|L\left(\bigotimes_{i=1}^{d}(\pi_i-x_ie_0)^{k_i},\mathbf{x}\right)\right| \\
&+\left(B\cdot L\left(\|e_1-\mathbf{x}e_0\|_1^r,\mathbf{x}\right)+C\frac{L\left(\|e_1-\mathbf{x}e_0\|_1^{r+1},\mathbf{x}\right)}{t\varphi(\mathbf{x})}\right)\cdot \\
&\cdot\max_{r_1+\cdots+r_d=r}K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)
\end{aligned}\tag{6}$$

holds for all positive linear operator $L:\mathbf{C}(S)\longrightarrow\mathbf{C}(S)$, *any* $f\in\mathbf{C}^{\mathbf{r}}(S)$, *any* $\mathbf{x}\in S$ *and any* $t>0$ *then* $A_0\ge 1$, $A_k\ge\frac{1}{k!}$, $k=\overline{1,r}$, $B\ge\frac{2}{r!}$ *and for* $A_k=\frac{1}{k!}$, $k=\overline{1,r}$ *we have* $C\ge\frac{1}{r!(r-\alpha+1)}$.

Proof. We have

$$\begin{aligned}
f(\mathbf{y})-f(\mathbf{x}) &= \sum_{k=1}^{r}\frac{1}{k!}d^kf(\mathbf{x})\,(\mathbf{y}-\mathbf{x})^k+R_{r,f,\mathbf{x}}(\mathbf{y}) \\
&=\sum_{k=1}^{r}\frac{1}{k!}\sum_{k_1+\cdots+k_d=k}\frac{k!}{k_1!\cdots k_d!}\cdot\frac{\partial^k f}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(\mathbf{x})\cdot\prod_{i=1}^{d}(y_i-x_i)^{k_i}+R_{r,f,\mathbf{x}}(\mathbf{y})
\end{aligned}$$

from which

$$\begin{aligned}
&L\left(f-f(\mathbf{x})e_0,\mathbf{x}\right) \\
&\quad=\sum_{k=1}^{r}\frac{1}{k!}\sum_{k_1+\cdots+k_d=k}\frac{k!}{k_1!\cdots k_d!}\cdot\frac{\partial^k f}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(\mathbf{x})\cdot L\left(\bigotimes_{i=1}^{d}(\pi_i-x_ie_0)^{k_i},\mathbf{x}\right) \\
&\quad+L\left(R_{r,f,\mathbf{x}},\mathbf{x}\right)
\end{aligned}$$

Then

$$\begin{aligned}
|L(f,\mathbf{x})-f(\mathbf{x})| &\le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1|+|L(f-f(\mathbf{x})e_0,\mathbf{x})| \\
&\le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| \\
&\quad+\sum_{k=1}^{r}\frac{1}{k!}\sum_{k_1+\cdots+k_d=k}\frac{k!}{k_1!\cdots k_d!}\cdot\left|\frac{\partial^k f}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(\mathbf{x})\right|\cdot\left|L\left(\bigotimes_{i=1}^{d}(\pi_i-x_ie_0)^{k_i},\mathbf{x}\right)\right| \\
&\quad+L(|R_{r,f,x}|,\mathbf{x}) \\
&\le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| \\
&\quad+\sum_{k=1}^{r}\frac{1}{k!}\sum_{k_1+\cdots+k_d=k}\frac{k!}{k_1!\cdots k_d!}\cdot\left|\frac{\partial^k f}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(\mathbf{x})\right|\cdot\left|L\left(\bigotimes_{i=1}^{d}(\pi_i-x_ie_0)^{k_i},\mathbf{x}\right)\right| \\
&\quad+\left(\frac{2}{r!}L\left(\|e_1-\mathbf{x}e_0\|_1^r,\mathbf{x}\right)+\frac{L\left(\|e_1-\mathbf{x}e_0\|_1^{r+1},\mathbf{x}\right)}{tr!(r-\alpha+1)\varphi(\mathbf{x})}\right)\cdot \\
&\quad\cdot\max_{r_1+\cdots+r_d=r}K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right),
\end{aligned}$$

which is (5).

If we choose $L(h,\mathbf{x})=0$ and $f=e_0$ and replace in (6) we obtain $A_0\ge 1$.

If we choose $L(h,\mathbf{x})=h(1,0,\ldots,0)$, $f(\mathbf{x})=x_1^k$, $k=\overline{1,r}$, $\mathbf{x}=\mathbf{0}$ and replace in (6) we obtain $A_k\ge\dfrac{1}{k!}$.

To show that $B\ge\dfrac{2}{r!}$ we choose $L(h,x)=h(1,0,\ldots,0)$ and $f(\mathbf{x})=2x_1^{r+a}$ with $a>0$. For $g=(r+a)\cdot(r+a-1)\cdots(a+1)\,e_0$ we have

$$\begin{aligned}
\max_{r_1+\cdots+r_d=r}K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)&=K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r}},t\right) \\
&\le\max\left\{\left\|\frac{\partial^r f}{\partial x_1^r}-g\right\|,t\,\|\nabla g\|_\infty\right\} \\
&=\left\|\frac{\partial^r f}{\partial x_1^r}-g\right\|=(r+a)\cdot(r+a-1)\cdots(a+1).
\end{aligned}$$

We replace in (6) and passing to limit $t\to\infty$, $\mathbf{x}\to\mathbf{0}$, $a\to 0$ (in this order) we obtain $B\ge\dfrac{2}{r!}$.

To show that $C\ge\dfrac{1}{r!(r-\alpha+1)}$ if $A_k=\dfrac{1}{k!}$, $k=\overline{1,r}$ we choose $L(h,x)=h(\mathbf{0})$ and

$$f(x)=\frac{\alpha}{1-\alpha}\int_{(x_1+\cdots+x_d)^{1-\alpha}}^{1}\left(u^{\frac{1}{1-\alpha}}-x_1-\cdots-x_d\right)^r du.$$

We have $f(\mathbf{0})=\dfrac{\alpha}{r-\alpha+1}$,

$$\frac{\partial^k f}{\partial x_1^{k_1}\cdots x_d^{k_d}} = \frac{\alpha}{1-\alpha}(-1)^k\frac{r!}{(r-k)!}\int\limits_{(x_1+\cdots+x_d)^{1-\alpha}}^{1}\left(u^{\frac{1}{1-\alpha}}-x_1-\cdots-x_d\right)^{r-k}du,\ k = \overline{1,r}$$

and

$$K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)\le t\left\|\varphi\frac{\partial^{r+1}f}{\partial x_1^{r+1}\partial x_2^{r_2}\cdots x_d^{r_d}}\right\| = tr!\alpha.$$

We replace in (6) and passing to limit $t\to 0$ we obtain

$$\frac{\alpha}{r-\alpha+1}-\frac{\alpha}{1-\alpha}\int\limits_{(x_1+\cdots+x_d)^{1-\alpha}}^{1}\left(u^{\frac{1}{1-\alpha}}-x_1-\cdots-x_d\right)^{r}du$$

$$\le\frac{\alpha}{1-\alpha}\int\limits_{(x_1+\cdots+x_d)^{1-\alpha}}^{1}\sum_{k=1}^{r}\binom{r}{k}\left(u^{\frac{1}{1-\alpha}}-x_1-\cdots-x_d\right)^{r-k}(x_1+\cdots+x_d)^k du$$

$$+C\cdot\frac{r!\alpha(x_1+\cdots+x_d)^{r+1-\alpha}}{(1-x_1)^{\alpha}\cdots(1-x_d)^{\alpha}}$$

ie

$$\frac{\alpha}{r-\alpha+1}-\frac{\alpha}{1-\alpha}\int\limits_{(x_1+\cdots+x_d)^{1-\alpha}}^{1}\left(u^{\frac{1}{1-\alpha}}-x_1-\cdots-x_d\right)^{r}du$$

$$\le\frac{\alpha}{1-\alpha}\int\limits_{(x_1+\cdots+x_d)^{1-\alpha}}^{1}\left(u^{\frac{r}{1-\alpha}}-\left(u^{\frac{1}{1-\alpha}}-x_1-\cdots-x_d\right)^{r}\right)du$$

$$+C\cdot\frac{r!\alpha(x_1+\cdots+x_d)^{r+1-\alpha}}{(1-x_1)^{\alpha}\cdots(1-x_d)^{\alpha}}$$

from where $\frac{1}{r-\alpha+1}\le C\cdot\frac{r!}{(1-x_1)^{\alpha}\cdots(1-x_d)^{\alpha}}$. Passing to limit $\mathbf{x}\to\mathbf{0}$ we obtain $C\ge\frac{1}{r!(r-\alpha+1)}$. □

Example 1. *The Bernstein operators are defined by*

$$B_n(f,\mathbf{x}) = \sum_{k_1+\cdots+k_d=0}^{n} f\left(\frac{k_1}{n},\ldots,\frac{k_d}{n}\right)b_{n,\mathbf{k}}(\mathbf{x}) \tag{7}$$

where

$$b_{n,\mathbf{k}}(\mathbf{x}) = \frac{n!}{k_1!\cdots k_d!\cdot\left(n-\sum\limits_{i=1}^{d}k_i\right)!}x_1^{k_1}\cdots x_d^{k_d}\left(1-\sum_{i=1}^{d}x_i\right)^{n-\sum\limits_{i=1}^{d}k_i}$$

with $n \in \mathbb{N}$ *and* $k_1, \ldots, k_d \in \mathbb{N} \cup \{0\}$*. We have*

$$\begin{aligned} B_n\left(\|e_1 - \mathbf{x}e_0\|_1, \mathbf{x}\right) &= \sum_{i=1}^{d} B_n\left(|\pi_i - x_i e_0|, \mathbf{x}\right) \\ &\leq \sum_{i=1}^{d} \sqrt{B_n\left((\pi_i - x_i e_0)^2, \mathbf{x}\right)} = \sum_{i=1}^{d} \sqrt{\frac{x_i(1-x_i)}{n}} \end{aligned}$$

and

$$\begin{aligned} & B_n\left(\|e_1 - \mathbf{x}e_0\|_1^2, \mathbf{x}\right) \\ &= \sum_{i=1}^{d} B_n\left((\pi_i - x_i e_0)^2, \mathbf{x}\right) + 2 \sum_{1 \leq i < j \leq d} B_n\left(|\pi_i - x_i e_0| \cdot |\pi_j - x_j e_0|, \mathbf{x}\right) \\ &\leq \sum_{i=1}^{d} B_n\left((\pi_i - x_i e_0)^2, \mathbf{x}\right) + 2 \sum_{1 \leq i < j \leq d} \sqrt{B_n\left((\pi_i - x_i e_0)^2, \mathbf{x}\right) \cdot B_n\left((\pi_j - x_j e_0)^2, \mathbf{x}\right)} \\ &= \left(\sum_{i=1}^{d} \sqrt{B_n\left((\pi_i - x_i e_0)^2, \mathbf{x}\right)}\right)^2 = \left(\sum_{i=1}^{d} \sqrt{\frac{x_i(1-x_i)}{n}}\right)^2. \end{aligned}$$

For $r = 1$ *and* $t = \sum_{i=1}^{d} \sqrt{\frac{x_i(1-x_i)}{n}}$*, from (5) result*

$$|B_n(f, \mathbf{x}) - f(\mathbf{x})| \leq \frac{5-\alpha}{2-\alpha} \left(\sum_{i=1}^{d} \sqrt{\frac{x_i(1-x_i)}{n}}\right) \max_{j=\overline{1,d}} K_{1,\varphi}^{\infty} \left(\frac{\partial f}{\partial x_j}, \frac{1}{\varphi(\mathbf{x})} \sum_{i=1}^{d} \sqrt{\frac{x_i(1-x_i)}{n}}\right).$$

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