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ON UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract

In this work, we define an integral operator $K_{\alpha,\beta}$, for analytic functions f in the open unit disk $\mathcal{U}$, α, β be complex numbers, and we obtain certain conditions of the univalence for this integral operator.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$.

We denote by $\mathcal{P}$ the class of functions p which are analytic in $\mathcal{U}$, $p(0) = 1$ and $Re\ p(z) > 0$, for all $z \in \mathcal{U}$.

For the functions $f,\ g \in \mathcal{A}$, the integral operator is defined by

$$K_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} \left(g'(u)\right)^{\beta} du, \tag{1.1}$$

for some complex numbers α and β.

For $\beta = 0$, α be a complex number, $f \in \mathcal{A}$, from (1.1) we have the integral operator Kim-Merkes [1],

$$H_{\alpha}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} du. \tag{1.2}$$

From (1.1), for $\alpha = 0$, β be a complex number, $g \in \mathcal{A}$, we obtain the integral operator Pfaltzgraff [4],

$$G_{\beta}(z) = \int_0^z \left(g'(u)\right)^{\beta} du. \tag{1.3}$$

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2 Preliminary results

We need the following lemmas.

Lemma 1. *[3]. Let γ be a complex number, $0 < Re\ \gamma \leq 1$ and $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$. If*

$$\frac{1 - |z|^{2Re\ \gamma}}{Re\ \gamma} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then for any complex number δ, $Re\ \delta \geq Re\ \gamma$, the function

$$F_\delta(z) = \left[\delta \int_0^z u^{\delta-1} f'(u) \right]^{\frac{1}{\delta}} \tag{2.2}$$

is in the class $\mathcal{S}$.

Lemma 2. *(Schwarz [2]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \ (z \in \mathcal{U}_R), \tag{2.3}$$

the equality (in the inequality (2.3) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

3 Main results

Theorem 1. *Let α, β, γ be complex numbers, $0 < Re\ \gamma \leq 1$, M, L positive real numbers and the functions f, g from $\mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$, $g(z) = z + b_2 z^2 + \ldots$. If*

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| \leq M, \quad (z \in \mathcal{U}), \tag{3.1}$$

$$\left| \frac{z g''(z)}{g'(z)} \right| \leq L, \quad (z \in \mathcal{U}) \tag{3.2}$$

and

$$|\alpha| M + |\beta| L \leq \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma + 1}{2Re\ \gamma}}}{2}, \tag{3.3}$$

then the integral operator $K_{\alpha,\beta}$ is in the class $\mathcal{S}$.

Proof. Function $K_{\alpha,\beta}(z)$ is regular in $\mathcal{U}$ and $K_{\alpha,\beta}(0) = K'_{\alpha,\beta}(0) - 1 = 0$. We have

$$\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)} = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta\frac{zg''(z)}{g'(z)}, \tag{3.4}$$

for all $z \in \mathcal{U}$.

From (3.4) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left[|\alpha|\left|\frac{zf'(z)}{f(z)} - 1\right| + |\beta|\left|\frac{zg''(z)}{g'(z)}\right|\right], \tag{3.5}$$

for all $z \in \mathcal{U}$.

Using (3.1), (3.2) and Lemma 2 we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le M|z|, \quad (z \in \mathcal{U}),$$

$$\left|\frac{zg''(z)}{g'(z)}\right| \le L|z|, \quad (z \in \mathcal{U})$$

and hence, by (3.5) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}|z|\left(|\alpha|M + |\beta|L\right), \quad (z \in \mathcal{U}). \tag{3.6}$$

Since

$$\max_{|z|\le 1}\left[\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}|z|\right] = \frac{2}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}},$$

from (3.3) and (3.6), we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le 1 \quad (z \in \mathcal{U}). \tag{3.7}$$

From (3.7) and by Lemma 1, for $\delta = 1$, it results that the integral operator $K_{\alpha,\beta}$ belongs to the class $\mathcal{S}$. □

Theorem 2. *Let* α, β, γ *be complex numbers,* $0 < Re\,\gamma \le 1$, L *positive real number and the functions* $f \in \mathcal{S}$, $g \in \mathcal{A}$, $f(z) = z + a_2z^2 + \ldots$, $g(z) = z + b_2z^2 + \ldots$.

If

$$\left|\frac{zg''(z)}{g'(z)}\right| \le L, \quad (z \in \mathcal{U}), \tag{3.8}$$

and

$$\frac{|\alpha|}{Re\,\gamma}+\frac{|\beta|L}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}\leq\frac{1}{2},\ for\ 0<Re\,\gamma<\frac{1}{2} \tag{3.9}$$

or

$$|\alpha|+\frac{|\beta|L}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}\leq\frac{1}{2},\ for\ \frac{1}{2}\leq Re\,\gamma\leq 1, \tag{3.10}$$

then the integral operator $K_{\alpha,\beta}\in\mathcal{S}$.

Proof. From (3.4) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right|\leq$$

$$\leq\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left[|\alpha|\left(\left|\frac{zf'(z)}{f(z)}\right|+1\right)+|\beta|\left|\frac{zg''(z)}{g'(z)}\right|\right], \tag{3.11}$$

for all $z\in\mathcal{U}$.

Since $f\in\mathcal{S}$, we have

$$\left|\frac{zf'(z)}{f(z)}\right|\leq\frac{1+|z|}{1-|z|},\quad (z\in\mathcal{U}). \tag{3.12}$$

and from (3.8), using Lemma 2, we get

$$\left|\frac{zg''(z)}{g'(z)}\right|\leq L|z|,\quad (z\in\mathcal{U}). \tag{3.13}$$

From (3.12), (3.13) and (3.11) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right|\leq\frac{1-|z|^{2Re\,\gamma}}{1-|z|}\,\frac{2|\alpha|}{Re\,\gamma}+\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}|z||\beta|L, \tag{3.14}$$

for all $z\in\mathcal{U}$.

We define the function $\psi:[0,1]\to\mathbb{R}$, $\Psi(x)=\frac{1-x^{2Re\,\gamma}}{1-x}$, $x=|z|$. We have

$$\max_{x\in[0,1]}\Psi(x)=1,\ for\ 0<Re\,\gamma<\frac{1}{2} \tag{3.15}$$

and

$$\max_{x\in[0,1]}\Psi(x)=2Re\,\gamma,\ for\ \frac{1}{2}\leq Re\,\gamma\leq 1. \tag{3.16}$$

Let us consider the function $Q : [0,1] \to \mathbb{R}$, $Q(x) = \frac{1-x^{2Re\,\gamma}}{Re\,\gamma} \cdot x$, $x = |z|$. We obtain

$$\max_{x\in[0,1]} Q(x) = \frac{2}{(2Re\,\gamma + 1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}. \tag{3.17}$$

For $0 < Re\,\gamma < \frac{1}{2}$, from (3.15), (3.17) and (3.14) we have

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le \frac{2|\alpha|}{Re\,\gamma} + \frac{2|\beta|L}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}, \tag{3.18}$$

for all $z \in \mathcal{U}$. Using (3.9) and (3.18) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le 1, \quad \left(z \in \mathcal{U}; 0 < Re\,\gamma < \frac{1}{2}\right). \tag{3.19}$$

For $\frac{1}{2} \le Re\,\gamma \le 1$, from (3.16), (3.17) and (3.14) we get

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le 2|\alpha| + \frac{2|\beta|L}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}, \tag{3.20}$$

for all $z \in \mathcal{U}$.

Using (3.10) and (3.20) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le 1, \quad \left(z \in \mathcal{U}; \frac{1}{2} \le Re\,\gamma \le 1\right). \tag{3.21}$$

From (3.19) and (3.21), by Lemma 1, for $\delta = 1$, it results that $K_{\alpha,\beta}$ belongs to class $\mathcal{S}$. □

4 Corollaries

Corollary 1. *Let α, γ be complex numbers, $0 < Re\,\gamma \le 1$, M positive real number and the function $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \dots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le M, \quad (z \in \mathcal{U}), \tag{4.1}$$

and

$$|\alpha| \le \frac{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}{2M}, \tag{4.2}$$

then the integral operator H_α, defined by (1.2), is in the class $\mathcal{S}$.

Proof. For $\beta = 0$ in Theorem 1, we obtain the Corollary 1. □

Corollary 2. *Let β, γ be complex numbers, $0 < Re\ \gamma \leq 1$, L positive real number and the function $g \in \mathcal{A}$, $g(z) = z + b_2 z^2 + \ldots$.*

If

$$\left| \frac{zg''(z)}{g'(z)} \right| \leq L, \quad (z \in \mathcal{U}), \tag{4.3}$$

and

$$|\beta| \leq \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma + 1}{2Re\ \gamma}}}{2L}, \tag{4.4}$$

then the integral operator G_β, defined by (1.3), is in the class $\mathcal{S}$.

Proof. We take $\alpha = 0$ in Theorem 1. □

Corollary 3. *Let α, β be complex numbers, $0 < Re\ \gamma \leq 1$ and the function $f \in \mathcal{S}$, $f(z) = z + a_2 z^2 + \ldots$.*

If

$$|\alpha| \leq \frac{Re\ \gamma}{2},\ for\ 0 < Re\ \gamma < \frac{1}{2} \tag{4.5}$$

or

$$|\alpha| \leq \frac{1}{2},\ for\ \frac{1}{2} \leq Re\ \gamma \leq 1, \tag{4.6}$$

then the integral operator $H_\alpha \in \mathcal{S}$.

Proof. For $\beta = 0$ in Theorem 2, we have the Corollary 3. □

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