Bulletin of the *Transilvania* University of Braşov • Vol 5(54), No. 2 - 2012
Series III: Mathematics, Informatics, Physics, 65-68

A GENERALIZATION OF KANTOROVICH OPERATORS AND A SHAPE-PRESERVING PROPERTY OF BERNSTEIN OPERATORS

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Abstract

We construct a generalization of the Kantorovich operators, depending on a parameter $b \geq 0$ and we prove that if a function $f \in C^1[0,1]$ with $f(0) = 0$, satisfies the differential inequality $f' + bf \geq 0$, then functions $B_n(f)$, $n \in \mathbb{N}$ satisfy the same inequality, where B_n are the Bernstein operators.

2000 *Mathematics Subject Classification:* 41A36, 41A35.
Key words: Kantorovich type operators, Bernstein operators, shape-preserving property.

1 Introduction

The Bernstein operators on the space $C[0,1]$ are defined by:

$$B_n(f,x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) p_{n,k}(x),\ f \in C[a,b],\ x \in [0,1],\ n \in \mathbb{N}, \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

The Kantorovich modification of the Bernstein operators are given by:

$$K_n(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,\ f \in C[0,1],\ x \in [0,1],\ n \in \mathbb{N}. \tag{2}$$

We note that, the Kantorovich operators K_n can be obtained by the following formula

$$K_n = D \circ B_{n+1} \circ I, \tag{3}$$

where D is the differentiation operator: $D(f) = f'$, $f \in C_1[0,1]$ and I is the antiderivative operator: $I(f,x) = \int_0^x f(t)dt$, $f \in C[0,1]$, $x \in [0,1]$. More general, if $L : C[0,1] \to C^r[0,1]$ is an arbitrary linear operator and $r \in \mathbb{N}$, if we denote by D^r and I^r, the iterates of operators D and I, then the operator $D^r \circ L \circ I^r$ is named the Kantorovich modification of operator L of order r. These operators play a crucial role in simultaneous approximation. Other types of generalizations or modifications of Kantorovich operators, partially included in References, are also known.

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2 Definition. Main results

We consider a generalization of the Kantorovich operators in the following sense.

Definition 2.1. Let a parameter $b \geq 0$. For any $n \in \mathbb{N}$ define the operator $K_n^b : C[0,1] \to C[0,1]$, defined by

$$\begin{aligned} K_n^b(f,x) \quad := \quad & (n+1+b)\sum_{k=0}^{n} p_{n,k}(x)e^{-b\frac{k+1}{n+1}} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} e^{bt} f(t)dt \\ & + \sum_{k=0}^{n} p_{n,k}(x)\left[(n+1+b) - (n+1-b)e^{\frac{b}{n+1}}\right] e^{-b\frac{k+1}{n+1}} \int_0^{\frac{k}{n+1}} e^{bt} f(t)dt, \quad (4) \end{aligned}$$

for $f \in C[0,1],\ x \in [0,1]$.

Remark 2.1. If we take $b = 0$ in (4) we obtain the Kantorovich operators given in (2).

Theorem 2.1. *Operators K_n^b are linear and positive, for any $n \in \mathbb{N}$ and $b \geq 0$.*

Proof. The linearity is clear. In order to prove the positivity it is enough to show that

$$(n+1+b) - (n+1-b)e^{\frac{b}{n+1}} \geq 0.$$

Consider function $\varphi(t) = 1 + t + (t-1)e^t$, $t \in \mathbb{R}$. If we denote $t = \frac{b}{n+1}$ it is sufficient to show that $\varphi(t) \geq 0$, for $t \geq 0$. We have $\varphi'(t) = 1 + te^t$. The minimum of function φ' is reached at point $t = -1$ and $\varphi'(-1) = 1 - e^{-1} > 0$. Hence $\varphi'(t) > 0$, $t \in \mathbb{R}$. Then function φ is increasing on $\mathbb{R}$. But $\varphi(0) = 0$ and hence $\varphi(t) \geq 0$, for $t \geq 0$. □

In order to give another description of operators K_n^b we consider operators $D_b : C^1[0,1] \to C[0,1]$ and $I_b : C[0,1] \to C^1[0,1]$, given by

$$\begin{aligned} D_b(f,x) &= f'(x) + bf(x),\ f \in C^1[0,1],\ x \in [0,1], \\ I_b(f,x) &= e^{-bx}\int_0^x e^{bt} f(t)dt,\ f \in C[0,1],\ x \in [0,1. \end{aligned}$$

Lemma 2.1. *Let $n \in \mathbb{N}$ and $b \geq 0$. We have*

i) $(D_b \circ I_b)(f) = f$, for all $f \in C[0,1]$,

ii) $(I_b \circ D_b)(f) = f$, for all $f \in C^1[0,1]$, such that $f(0) = 0$.

Proof. i) If $f \in C[0,1]$, then $I_b(f)$ is the solution of the Cauchy problem $y' + by = f$, $y(0) = 0$. Then $(D_b \circ I_b)(f) = f$.

ii) If $f \in C^1[0,1]$ and $f(0) = 0$, then integrating by parts we obtain, for $x \in [0,1]$:

$$\begin{aligned} (I_b \circ D_b)(f,x) &= e^{-bx}\int_0^x e^{bt}(f'(t) + bf(t))dt \\ &= e^{-bx}\left[e^{bx}f(x) - f(0) - b\int_0^x e^{bt}f(t)dt + b\int_0^x e^{bt}f(t)dt\right. \\ &= f(x). \end{aligned}$$

□

Theorem 2.2. *For any $n \in \mathbb{N}$ and $b \geq 0$ we have:*

$$K_n^b = D_b \circ B_{n+1} \circ I_b. \tag{5}$$

Proof. Let $f \in C[0,1]$ and $x \in [0,1]$. Using the convention $P_{n,k}(x) = 0$, for $k < 0$ or $k > n$, we have:

$$\begin{aligned}
(D_b \circ B_{n+1} \circ I_b)(f,x) &= (B_{n+1}(I_b(f),x))' + bB_{n+1}(I_b(f),x) \\
&= (n+1)\sum_{k=0}^{n+1}[p_{n,k-1}(x) - p_{n,k}(x)]I_b\Big(\frac{k}{n+1}\Big) \\
&\quad + b\sum_{k=0}^{n+1}[p_{n,k-1}(x) + p_{n,k}(x)]I_b\Big(\frac{k}{n+1}\Big) \\
&= \sum_{k=0}^{n+1}[(n+1+b)p_{n,k-1}(x) - (n+1-b)p_{n,k}(x)]I_b\Big(\frac{k}{n+1}\Big) \\
&= \sum_{k=0}^{n} p_{n,k}(x)\Big[(n+1+b)I_b\Big(\frac{k+1}{n+1}\Big) - (n+1-b)I_b\Big(\frac{k}{n+1}\Big)\Big].
\end{aligned}$$

From this it follows immediately (4). □

The results above allow us to derive a more general shape-preservation property for Bernstein operators. For this, let $b \geq 0$. Set

$$\mathcal{D}_b := \{f \in C^1[0,1] : \ D_b(f) \geq 0, \ f(0) = 0\}. \tag{6}$$

We have

Theorem 2.3. *For any $n \in \mathbb{N}$, $n \geq 2$ and $b \geq 0$, we have $B_n(\mathcal{D}_b) \subset \mathcal{D}_b$.*

Proof. Let $f \in \mathcal{D}_b$. We have $(D_b \circ B_n)(f) = (D_b \circ B_n \circ I_b)(D_b(f)) = K_{n-1}^b(D_b(f))$. Since $D_b(f) \geq 0$ and K_{n-1}^b is a positive operator it follows $K_{n-1}^b(D_b(f)) \geq 0$, i.e. $(D_b \circ B_n)(f) \geq 0$. Also $B_n(f,0) = f(0) = 0$. Hence $B_n(f) \in \mathcal{D}_b$. □

Theorem 2.4. *We have*

$$K_n^b(f) \rightrightarrows f \tag{7}$$

for all $f \in C[0,1]$.

(The symbol $\rightrightarrows$ means the uniform convergence on the interval $[0,1]$.)

Proof. Since operators K_n^b are positive it suffices to prove relation (7) for three test functions. Let us denote $e_k(t) = t^k$, $t \in [0,1]$, for $k = 0,1,2$. Then denote $g_k = I_b(e_k)$, $k = 0,1,2$. From the convergence properties of Bernstein operators we have $B_{n+1}(g_k) \rightrightarrows g_k$ and $(B_{n+1}(g_k))' \rightrightarrows g_k$, for $k = 0,1,2$. Hence, for the same indices k we have $(D_b \circ B_{n+1})(g_k) \rightrightarrows D_b(g_k)$. But $(D_b \circ B_{n+1})(g_k) = K_n^b(e_k)$ and $D_b(g_k) = e_k$. Hence $K_n^b(e_k) \rightrightarrows e_k$, for $k = 0,1,2$. Therefore we can apply the theorem of Popoviciu-Bohmann-Korovkin and we obtain (7). □

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Erratum

Theorem 2.2 contains an error of computation. Consequently the operators given in Definition 1 are not the real Kantorovich operators attached to Bernstein operators and the differential operator D_b. The correction is made in the paper: R. Păltănea, *A note on generalized Bernstein-Kantorovich operators*, Bull. Transilvania Univ Brasov, Ser III, **6(55)**, No. 2 (2013), 27-32.
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