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A REVIEW OF STOCHASTIC COMPARISON OF EXPONENTIAL AND GEOMETRIC CONVOLUTIONS

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Abstract

In this paper, we stochastically compare two sums of independent heterogeneous exponential random variables. By grouping a series of recent results in the literature, we obtain nice equivalent relations. We also formulate similar results for the convolutions of heterogeneous geometric random variables.

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1 Introduction

The exponential distribution plays a fundamental role in many problems in applied probability and statistics. For example, Markov models in reliability theory are connected to the exponential distribution. A problem of great interest, that arises naturally in various contexts, is the stochastic comparison of sums of independent heterogeneous exponential random variables. On the other hand, there are important models, especially in reliability, based of the convolutions of geometric random variables and their comparison. There are many classical and recent books discussing properties and applications of the convolutions of exponentials or geometrics; one may refer to Gnedenko et al. (1969), Barlow and Proschan (1975), Balakrishnan and Nevzorov (2003).

In this paper, we review in an unitary vision a series of recent results regarding the comparisons of the exponential convolutions and of the geometric convolutions, respectively. More precisely, we express (in terms of some weak majorization type orders) necessary and sufficient conditions for the comparisons of these convolutions in various classical senses: likelihood ratio order, hazard rate order, reversed hazard order, stochastic order, dispersive order, mean residual life order and increasing convex order. A surprising closeness between the exponential distribution and the geometric distribution is noticed.

Let us first recall the definition of the most usual stochastic orderings. Let X and Y be two positive, absolutely continuous, random variables with their distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, and densities f and g, respectively. Then:

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1. X is said to be larger than Y in the *likelihood ratio order* (denoted by $X \geq_{lr} Y$) if $f(t)/g(t)$ is increasing in $t \geq 0$;

2. X is said to be larger than Y in the *hazard rate order* (denoted by $X \geq_{hr} Y$) if $\overline{F}(t)/\overline{G}(t)$ is increasing in $t \geq 0$;

3. X is said to be larger than Y in the *reversed hazard rate order* (denoted by $X \geq_{rh} Y$) if $F(t)/G(t)$ is increasing in $t \geq 0$;

4. X is said to be larger than Y in the *dispersive order* (denoted by $X \geq_{disp} Y$) if $F^{-1}(v) - F^{-1}(u) \geq G^{-1}(v) - G^{-1}(u)$, for all $0 \leq u \leq v \leq 1$, where F^{-1} and G^{-1} are the right inverses of F and G, respectively;

5. X is said to be larger than Y in the *usual stochastic order* (denoted by $X \geq_{st} Y$) if $\overline{F}(t) \geq \overline{G}(t)$ for all $t \geq 0$;

6. X is said to be larger than Y in the *mean residual life order* (denoted by $X \geq_{mrl} Y$) if $\frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x)\,dx \geq \frac{1}{\overline{G}(t)} \int_t^\infty \overline{G}(x)\,dx,\ \forall\ t \geq 0.$

7. X is said to be larger than Y in the *increasing convex order* (denoted by $X \geq_{icx} Y$) if $\int_t^\infty \overline{F}(x)\,dx \geq \int_t^\infty \overline{G}(x)\,dx,\ \forall\ t \geq 0.$

Also, the distribution of a positive random variable is said to have an Increasing Failure Rate (IFR) if its hazard rate $f(t)/\overline{F}(t)$ is increasing in $t \geq 0$.

Let us also recall the discrete analog of above definitions. Let $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ be the set of non-negative integers. For two discrete random variables X and Y, with values in $\mathbb{N}_0$, denote their respective probability mass function f and g and their corresponding survival functions $\overline{F}$ (that is $\overline{F}(k) = P(X \geq k) = \sum_{i=k}^\infty f(i)$, for $k \in \mathbb{N}_0$) and $\overline{G}$. Then:

1. X is said to be larger than Y in the *likelihood ratio order* (denoted by $X \geq_{lr} Y$) if $f(k)/g(k)$ is increasing in $k \in \mathbb{N}_0$;

2. X is said to be larger than Y in the *hazard rate order* (denoted by $X \geq_{hr} Y$) if $\overline{F}(k)/\overline{G}(k)$ is increasing in $k \in \mathbb{N}_0$;

3. X is said to be larger than Y in the *reversed hazard rate order* (denoted by $X \geq_{rh} Y$) if $F(k)/G(k)$ is increasing in $k \in \mathbb{N}_0$;

4. X is said to be larger than Y in the *usual stochastic order* (denoted by $X \geq_{st} Y$) if $\overline{F}(k) \geq \overline{G}(k)$ for all $k \in \mathbb{N}_0$;

5. X is said to be larger than Y in the *mean residual life order* (denoted by $X \geq_{mrl} Y$) if
$$\frac{\sum_{i=k+1}^\infty \overline{F}(i)}{\overline{F}(k)} \geq \frac{\sum_{i=k+1}^\infty \overline{G}(i)}{\overline{G}(k)},\ \forall\ k \in \mathbb{N}_0.$$

6. X is said to be larger than Y in the *increasing convex order* (denoted by $X \geq_{icx} Y$) if

$$\sum_{i=k}^{\infty} \overline{F}(i) \geq \sum_{i=k}^{\infty} \overline{G}(i), \ \forall \ k \in \mathbb{N}_0.$$

A comprehensive treatment of the topic can be found in Shaked and Shanthikumar (2007).

2 Weak majorization type orders

The importance of the weak majorization order in statistical problems was emphasized by Nevius et al. (1974). The particular study of Schur-majorization properties of convolutions of exponential and geometric random variables has been initiated by Boland et al. (1994). A comprehensive treatment of the majorization and weak majorization orders and their applications can be found, for example, in Marshall and Olkin (1979) and Bhatia (1997). Remark that there are two versions of the weak majorization. The classical sense of the notion will be called the *weak submajorization*. But, a more proper ordering for comparing the convolutions is its complementary notion, viz., the *weak supermajorization*. In this context, Bon and Păltănea (1999) introduced *p-larger* order and, recently, Zhao and Balakrishnan (2009a) defined *reciprocal majorization* order.

The following definition presents these useful kinds of weak majorization order and related notions.

Definition 1. *Given a vector* $\boldsymbol{v}= (v_1, \cdots, v_n) \in \mathbb{R}^n$*, let* $v_{(1)} \leq \cdots \leq v_{(n)}$ *denote the increasing arrangement of its components. Then:*

(1) $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ *is said to majorize* $\boldsymbol{y}= (y_1, \cdots, y_n) \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$*) if*

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \quad \textit{for } k = 1, \cdots, n-1, \quad \textit{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)};$$

(2) $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ *is said to weakly supermajorize* $\boldsymbol{y}= (y_1, \cdots, y_n) \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \overset{w\uparrow}{\succeq} \boldsymbol{y}$*) if*

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \quad \textit{for } k = 1, \cdots, n;$$

(3) $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ *is said to weakly submajorize* $\boldsymbol{y}= (y_1, \cdots, y_n) \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \overset{w\downarrow}{\succeq} \boldsymbol{y}$*) if*

$$\sum_{i=1}^{k} x_{(n+1-i)} \geq \sum_{i=1}^{k} y_{(n+1-i)} \quad \textit{for } k = 1, \cdots, n;$$

(4) $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$ *is said to be p-larger than* $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$ *(written as* $\boldsymbol{x} \overset{p}{\succeq} \boldsymbol{y}$*) if*

$$\prod_{i=1}^{k} x_{(i)} \leq \prod_{i=1}^{k} y_{(i)} \quad \text{for } k = 1, \cdots, n;$$

(5) $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$ *is said to reciprocal majorize* $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$ *(written as* $\boldsymbol{x} \overset{rm}{\succeq} \boldsymbol{y}$*) if*

$$\sum_{i=1}^{k} \frac{1}{x_{(i)}} \geq \sum_{i=1}^{k} \frac{1}{y_{(i)}} \quad \text{for } k = 1, \cdots, n;$$

(6) $f : I^n \to \mathbb{R}$*, where* I *denotes a real interval, is said to be a Schur-convex function if* $f(\boldsymbol{x}) \geq f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in I^n$ *such that* $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$*.*

We observe that $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$ if and only if $\boldsymbol{x} \overset{w\uparrow}{\succeq} \boldsymbol{y}$ and $\boldsymbol{x} \overset{w\downarrow}{\succeq} \boldsymbol{y}$. Observe that, for two positive vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of the same size, we have $\boldsymbol{x} \overset{p}{\succeq} \boldsymbol{y} \Leftrightarrow \log \boldsymbol{x} \overset{w\uparrow}{\succeq} \mathbf{log}\ \boldsymbol{y}$ and $\boldsymbol{x} \overset{rm}{\succeq} \boldsymbol{y} \Leftrightarrow \boldsymbol{x}^{-1} \overset{w\downarrow}{\succeq} \boldsymbol{y}^{-1}$.

Now, let us recall some basic properties of Schur-convexity. Schur-convex functions are symmetric and convex symmetric functions are Schur-convex. In particular, if $g : \mathbb{R} \to \mathbb{R}$ is a convex function, then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, \cdots, x_n) = \sum_{i=1}^{n} g(x_i)$, for all $(x_1, \cdots, x_n) \in \mathbb{R}^n$, is Schur-convex; see Hardy et al. (1934). The functions which preserve the weak majorization are generally symmetric and monotone; see Nevius et al. (1974).

The hierarchy of above defined weak majorization type orders is established by the next elementary theorem; see Kochar and Xu (2010).

Theorem 1. *Suppose two vectors* $\boldsymbol{x} = (x_1, \cdots, x_n)$ *and* $\boldsymbol{y} = (y_1, \cdots, y_n)$ *are in* $\mathbb{R}^n_+$*. Then, the following implications hold:* $\boldsymbol{x} \overset{w\uparrow}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{p}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{rm}{\succeq} \boldsymbol{y}$*.*

There is a large number of classical results and recent refinements and extensions on the subject of stochastic comparison of sums of exponential and geometric random variables. The actual standard of the knowledge in the area permits us to formulate a series of best characterizations of classical stochastic orderings among these convolutions. In fact, we prove the equivalence between a system of functional inequalities (representing a set of stochastic inequalities) and a correspondent system of real inequalities involving positive numbers (i.e. the parameters of the associated random variables).

3 Comparison of exponential convolutions

Recall that an exponential random variable X with parameter (hazard rate) $a > 0$ has the distribution function $F_X(t) = 1 - e^{-at},\ t \geq 0$.

Theorem 2. *Let* $X_1, \cdots, X_n$ *and* $Y_1, \cdots, Y_n$ *be two sets of independent heterogeneous exponential random variables with the vectors of the associated parameters* $\boldsymbol{a} = (a_1, \cdots, a_n)$ *and* $\boldsymbol{b} = (b_1, \cdots, b_n)$*. Let* $a_{(1)} \leq \cdots \leq a_{(n)}$ *and* $b_{(1)} \leq \cdots \leq b_{(n)}$ *denote the increasing* *arrangement of the components of vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, respectively. For* $i \in \{1, \cdots, n\}$*, let us denote* $X_{(i)}$ *and* $Y_{(i)}$ *the random variables of the two sequences having the parameters* $a_{(i)}$ *and* $b_{(i)}$*, respectively.*

I The following statements are equivalent:

- *(a)* $\boldsymbol{a} \overset{w\uparrow}{\succeq} \boldsymbol{b}$*;*
- *(b)* $\sum_{i=1}^{k} X_{(i)} \geq_{lr} \sum_{i=1}^{k} Y_{(i)}$*, for* $k = 1, \cdots, n$*;*
- *(c)* $\sum_{i=1}^{k} X_{(i)} \geq_{rh} \sum_{i=1}^{k} Y_{(i)}$*, for* $k = 1, \cdots, n$*.*

II The following statements are equivalent:

- *(a)* $\boldsymbol{a} \overset{p}{\succeq} \boldsymbol{b}$*;*
- *(b)* $\sum_{i=1}^{k} X_{(i)} \geq_{hr} \sum_{i=1}^{k} Y_{(i)}$*, for* $k = 1, \cdots, n$*;*
- *(c)* $\sum_{i=1}^{k} X_{(i)} \geq_{disp} \sum_{i=1}^{k} Y_{(i)}$*, for* $k = 1, \cdots, n$*;*
- *(d)* $\sum_{i=1}^{k} X_{(i)} \geq_{st} \sum_{i=1}^{k} Y_{(i)}$*, for* $k = 1, \cdots, n$*.*

III The following statements are equivalent:

- *(a)* $\boldsymbol{a} \overset{rm}{\succeq} \boldsymbol{b}$*;*
- *(b)* $\sum_{i=1}^{k} X_{(i)} \geq_{mrl} \sum_{i=1}^{k} Y_{(i)}$*, for* $k = 1, \cdots, n$*;*
- *(c)* $\sum_{i=1}^{k} X_{(i)} \geq_{icx} \sum_{i=1}^{k} Y_{(i)}$*, for* $k = 1, \cdots, n$*.*

Proof.

Proof of I. Assume $\boldsymbol{a} \overset{w\uparrow}{\succeq} \boldsymbol{b}$ implies $(a_{(1)}, \cdots, a_{(k)}) \overset{w\uparrow}{\succeq} (b_{(1)}, \cdots, b_{(k)})$, for $k = 1, \cdots, n$. Then, using Theorem 3.3 in Zhao and Balakrishnan (2009b), we find $\sum_{i=1}^{k} X_{(i)} \geq_{lr} \sum_{i=1}^{k} Y_{(i)}$, for $k \in \{1, 2, \cdots, n\}$. Therefore $(a) \Rightarrow (b)$. Since the likelihood ratio order implies the reversed hazard rate order, we also obtain $\sum_{i=1}^{k} X_{(i)} \geq_{rh} \sum_{i=1}^{k} Y_{(i)}$, for $k = 1, 2, \cdots, n$, i.e. $(a) \Rightarrow (c)$. Conversely, suppose $\sum_{i=1}^{k} X_{(i)} \geq_{lr} \sum_{i=1}^{k} Y_{(i)}$, for $k \in \{1, 2, \cdots, n\}$. From Theorem 3.3 in Zhao and Balakrishnan (2009b) we have $a_{(1)} \leq b_{(1)}$ and $\sum_{i=1}^{k} a_{(i)} \leq \sum_{i=1}^{k} b_{(i)}$, for $2 \leq k \leq n$. Therefore $\boldsymbol{a} \overset{w\uparrow}{\succeq} \boldsymbol{b}$. Similarly, Theorem 3.7 in Mao et al. (2010) assures that, if $\sum_{i=1}^{k} X_{(i)} \geq_{rh} \sum_{i=1}^{k} Y_{(i)}$, for $k = 1, 2, \cdots, n$, then $\sum_{i=1}^{k} a_{(i)} \leq \sum_{i=1}^{k} b_{(i)}$, for $1 \leq k \leq n$. Thus, the implications $(b) \Rightarrow (a)$ and $(c) \Rightarrow (a)$ are proved.

Proof of II. Following the same reasoning as above, the equivalences
$\sum_{i=1}^{k} X_{(i)} \geq_{hr} \sum_{i=1}^{k} Y_{(i)},\ 1 \leq k \leq n,\ \Leftrightarrow\ \sum_{i=1}^{k} X_{(i)} \geq_{st} \sum_{i=1}^{k} Y_{(i)},\ 1 \leq k \leq n,\ \Leftrightarrow \boldsymbol{a} \overset{p}{\succeq} \boldsymbol{b}$
can be obtained from Theorem 1 in Bon and Păltănea (1999). Notice that an exponential convolution has the number 0 as the left-endpoint of its support. Then applying Theorem 3.B.13 in Shaked and Shanthikumar (2007), we have $\sum_{i=1}^{k} X_{(i)} \geq_{disp} \sum_{i=1}^{k} Y_{(i)},\ 1 \leq k \leq n,\quad \Rightarrow\quad \sum_{i=1}^{k} X_{(i)} \geq_{st} \sum_{i=1}^{k} Y_{(i)},\ 1 \leq k \leq n$. Finally, the implication $\boldsymbol{a} \overset{p}{\succeq} \boldsymbol{b}\ \Rightarrow\ \sum_{i=1}^{k} X_{(i)} \geq_{disp} \sum_{i=1}^{k} Y_{(i)},\ 1 \leq k \leq n$ follows from a more general result (involving gamma distribution) due to Khaledi and Kochar (2004).

Proof of III. Suppose $\sum_{i=1}^{k} X_{(i)} \geq_{mrl} \sum_{i=1}^{k} Y_{(i)}$ for $k=1,\cdots,n$. Theorem 4.1 in Zhao and Balakrishnan (2009a) ensures $\sum_{i=1}^{k}(a_{(i)})^{-1} \geq \sum_{i=1}^{k}(b_{(i)})^{-1},\ k=1,\cdots,n$. It follows that $\boldsymbol{a}\overset{rm}{\succeq}\boldsymbol{b}$. Since $\boldsymbol{a}\overset{rm}{\succeq}\boldsymbol{b}$ implies $(a_{(1)},\cdots,a_{(k)}) \overset{rm}{\succeq} (b_{(1)},\cdots,b_{(k)})$, for $k=1,\cdots,n$, we note that the converse implication is given by Theorem 4.1, item (1), in Zhao and Balakrishnan (2009a). Hence, $(a) \Leftrightarrow (b)$. Similarly, the equivalence $(a) \Leftrightarrow (c)$ follows from Theorem 3.3 in Mao et al. (2010) (see also Păltănea (2009)). □

Mention that Kochar and Xu (2010) obtained several similar properties of the comparison of exponential convolutions in the following stochastic orders: *Lorenz order*, *right spread order* and *NBUE order*. But, for these orderings, we have not yet established equivalent characterizations, as in the above theorem.

4 Comparison of geometric convolutions

Now, we formulate some correspondent results for the geometric distribution. A geometric random variable X with parameter $p \in (0,1)$ has the probability mass function

$$f_X(k) = P(X=k) = p(1-p)^k,\ k \in \mathbb{N}_0,$$

the survival function

$$\overline{F}_X(k) = P(X \geq k) = (1-p)^k,\ k \in \mathbb{N}_0,$$

and the constant mean residual life

$$m_X(k) = E[X-k|X \geq k] = \frac{\sum_{j=k+1}^{\infty} \overline{F}_X(j)}{\overline{F}_X(k)} = \frac{1-p}{p},\ k \in \mathbb{N}_0.$$

Remark the memoryless property of the geometric distribution. Also recall that a geometric random variable X with parameter p has a constant hazard rate function $h_X = p$ (so X is IFR). Then, we can say that the geometric distribution is the discrete analog of the exponential distribution. This fact can be taken as an explanation for the similarity between the ordering properties of the convolutions of the two discussed distributions.

The following theorem is the "geometric" correspondent of Theorem 2.

Theorem 3. *Let $X_1,\cdots,X_n$ and $Y_1,\cdots,Y_n$ be two sets of independent heterogeneous geometric random variables with the vectors of the associated parameters $\boldsymbol{p}=(p_1,\cdots,p_n)$ and $\boldsymbol{q}=(q_1,\cdots,q_n)$, respectively. Let $p_{(1)} \leq \cdots \leq p_{(n)}$ and $q_{(1)} \leq \cdots \leq q_{(n)}$ denote the increasing arrangement of the components of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively. Let us consider $X_{(i)}$ having the parameter $p_{(i)}$ and $Y_{(i)}$ having the parameter $q_{(i)}$, for $i=1,\cdots,n$.*

I The following statements are equivalent:

(a) $\boldsymbol{p}\overset{w\uparrow}{\succeq}\boldsymbol{q}$*;*

(b) $\sum_{i=1}^{k} X_{(i)} \geq_{lr} \sum_{i=1}^{k} Y_{(i)}$*, for* $k=1,\cdots,n$*;*

(c) $\sum_{i=1}^{k} X_{(i)} \geq_{rh} \sum_{i=1}^{k} Y_{(i)}$, *for* $k = 1, \cdots, n$.

II *The following statements are equivalent:*

(a) $\boldsymbol{p} \overset{p}{\succeq} \boldsymbol{q}$;

(b) $\sum_{i=1}^{k} X_{(i)} \geq_{hr} \sum_{i=1}^{k} Y_{(i)}$, *for* $k = 1, \cdots, n$;

(c) $\sum_{i=1}^{k} X_{(i)} \geq_{st} \sum_{i=1}^{k} Y_{(i)}$, *for* $k = 1, \cdots, n$.

III *The following statements are equivalent:*

(a) $\boldsymbol{p} \overset{rm}{\succeq} \boldsymbol{q}$;

(b) $\sum_{i=1}^{k} X_{(i)} \geq_{mrl} \sum_{i=1}^{k} Y_{(i)}$, *for* $k = 1, \cdots, n$;

(c) $\sum_{i=1}^{k} X_{(i)} \geq_{icx} \sum_{i=1}^{k} Y_{(i)}$, *for* $k = 1, \cdots, n$.

Proof.
We use the same idea as in the proof of Theorem 4. Thus, the equivalent statements of item I are direct consequences of Theorem 4.3 in Zhao and Balakrishnan (2009b) and of Theorem 4.4 in Mao et al. (2010). Then, the equivalences of item II arise from Theorem 4.1 in Zhao and Hu (2010) and from the well-known relation $\geq_{hr} \Rightarrow \geq_{st}$. The last equivalences (item III) result from Theorem 4.1 and Theorem 4.2 in Mao et al. (2010) (see also Păltănea (2010)). $\square$

References

[1] Balakrishnan, N., and Nevzorov, V. B. (2003). *A Primer on Statistical Distributions.* John Wiley & Sons, Hoboken, New Jersey.

[2] Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Reinhart and Winston, New York.

[3] Bhatia, R. (1997). *Matrix Analysis.* Springer-Verlag, New York.

[4] Boland, P. J. El-Neweihi, E. and Proschan, F. (1994). Schur properties of convolutions of exponential and geometric random variables. *Journal of Multivariate Analysis* **48**, 157-167.

[5] Bon, J.-L. and Păltănea, E. (1999). Ordering properties of convolutions of exponential random variables. *Lifetime Data Analysis* **5**, 185-192.

[6] Gnedenko, B. V., Belyayev, Y. K. and Solovyev, A. D. (1969). *Mathematical Methods of Reliability Theory.* Academic Press, London.

[7] Hardy, G.H., Littlewood, J.E. and Pólya, G. (1934). *Inequalities.* Cambridge University Press, Cambridge, England.

[8] Khaledi, B. and Kochar, S. C. (2004). Ordering convolutions of gamma random variables. *Sankhya* **66**, 466-473.

[9] Kochar, S. and Xu, M. (2010). On the right spread order of convolutions of heterogeneous exponential random variables. *Journal of Multivariate Analysis* **101** (1), 165-176.

[10] Mao, T., Hu, T. and Zhao, P. (2010). Ordering convolutions of heterogeneous exponential and geometric distributions revisited. *Probability in the Engineering and Informational Sciences*, **24** (3), 329-348.

[11] Marshall, A. W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and its Applications*. Academic Press, New York.

[12] Nevius, S.E., Proschan, F. and Sethuraman, J. (1974). Schur functions in statistics. III: A stochastic version of weak majorization, with applications. *FSU Statistics Report M307*, The Florida State University, Tallahassee, Florida.

[13] Păltănea, E. (2009). Stochastic ordering of sums of two independent exponential random variables. *Scientific Studies and Research - Series Mathematics and Informatics*, **19** (2), 351-356.

[14] Păltănea, E. (2010). On the comparison in MRL order of the convolutions of geometric random variables. *Bulletin of the Transilvania University of Braşov, Series III*, **3** (**52**), 81-92.

[15] Shaked, M. and Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer-Verlag, New York.

[16] Zhao, P. and Balakrishnan, N. (2009a). Mean residual life order of convolutions of heterogeneous exponential random variables. *Journal of Multivariate Analysis* **100**, 1792-1801.

[17] Zhao, P. and Balakrishnan, N. (2009b). Likelihood ratio ordering of convolutions of heterogeneous exponential and geometric random variables. *Statistics* & *Probability Letters* **79**, 1717-1723.

[18] Zhao, P. and Hu, T. (2010). On hazard rate ordering of the sums of heterogeneous geometric random variables. *Journal of Multivariate Analysis*, **101** (1), 44-51.