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A STRONG LAW OF LARGE NUMBERS FOR A PROBABILISTIC CASH FLOW MODEL

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Abstract

In a previous paper, the first author introduced a probabilistic model for the cash flow in a (homogeneous) population. In the present paper we extend the model by considering the case of a non-homogeneous population, and we derive the properties of the model. We show that the random walk corresponding to the trajectory of a coin within the population is a recurrent and irreducible martingale, and we derive a corresponding Strong Law of Large numbers for it.

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1 Introduction

In [Pa], inspired by several papers in which the authors studied the effects of various moral beliefs on the interactive games among neighbors (see for example [Be1], [Be2], [Be3] and the references cited therein), the first author introduced a probabilistic model for the cash flow in a society in which the individuals decide to give (or not) money to their neighbors with a certain probability.

Continuing these ideas, in the present paper, we extend this model by considering the case when the population is non-homogeneous with respect to the decision of passing the coin to other members of the society. We still assume that the decisions of the individuals of the population are independent of each other, and also independent of previous decisions, but we no longer assume that the probability that an individual will give the coin to one of his neighbors is the same for all the members of the society.

The structure of the paper is the following. In Section 2 we introduce the extended probabilistic model for the cash flow and we set up the notation. In the next section, in Lemma 1 we show that the process V_n representing the number of transitions of the coin between different states up to time $n \in \mathbb{N}$ increases almost surely to infinity. Using

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this, and the classical Strong Law of Large Numbers, in Proposition 1 we show that the random walk S_n representing the position of the penny at time $n \in \mathbb{N}$ is a recurrent and irreducible martingale.

The result in Theorem 1 contains a Strong Law of Large Numbers for random walk S_n corresponding to the trajectory of the coin within the population, and the paper concludes with some remarks on the results obtained in Section 3.

2 The extended model

In [Pa], the first author introduced a probabilistic model for cash flow in a population which is homogenous, in the sense that the probability that an individual is willing to give the coin to one of his neighbors is the same for all individuals. Extending this model, we consider the case of a non-homogenous population, for which the probability that an individual is willing to give the coin to one of his neighbors depends on the individual. More precisely, we consider that the population occupies the integer positions on the real line, and we make the following assumptions: *the members of the population decide to keep or give the coin to one of their neighbors with a certain probability, constant in time but not necessarily the same for all the members of the population, the decision being independent of previous decisions, and also of the decisions of the rest of the population. If an individual decides to give the coin, he gives it to one of his adjacent neighbors, with equal probability.*

The mathematical model which describes the above assumptions can be formulated as follows. On a fixed probability space $(\Omega, \mathcal{F}, P)$, we consider two sequences of independent random variables, also independent of each other:

i) $(Y_i)_{i\in\mathbb{N}^*}$ - i.i.d. random variables taking ± 1 with equal probability. The random variable Y_i represents the increment of the position of the coin at time i, if the individual possessing the coin at this time is willing to give it to one of his neighbors.

ii) $(U_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}}$ - independent Bernoulli random variables with $P(U_{ij} = 1) = p_i \in (0, 1)$, $i \in \mathbb{Z}$, $j \in \mathbb{N}$. The random variable $U_{i,j}$ takes the value 1 if the individual i possesses the coin at time j and is willing to give it to one of the adjacent neighbors, and 0 otherwise.

Under the assumptions presented above, the position of the coin at time $n \in \mathbb{N}$ is given by the random walk $S_n(\mathbf{p})$, where $\mathbf{p} = (\ldots, p_{-1}, p_0, p_1, \ldots)$, and p_i represents the probability (constant in time) that if the individual at position $i \in \mathbb{Z}$ possesses the coin, he is willing to give it to one of his adjacent neighbors. In order to simplify the notation, we will drop the dependence on $\mathbf{p}$ and we will write S_n for $S_n(\mathbf{p})$.

Considering that the coin is initially located at the origin, the position of the coin in this extended model is given by the random walk $(S_n)_{n\in\mathbb{N}}$, where

$$S_n = \begin{cases} 0, & n = 0 \\ X_1 + \ldots + X_n, & n \geq 1 \end{cases}, \tag{1}$$

and

$$X_{n+1} = U_{S_n,n} Y_{n+1} = \begin{cases} Y_{n+1}, & \text{if } U_{S_n,n} = 1 \\ 0, & \text{otherwise} \end{cases}, \qquad n \in \mathbb{N}. \tag{2}$$

We consider the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$, where $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and

$$\mathcal{F}_n = \sigma\left(U_{i,j}, Y_k : i \in \mathbb{Z}, j < n, k \le n\right), \qquad n \ge 1,$$

represents the σ-algebra of events known up to time $n \in \mathbb{N}^*$. We note that with this definition, the random variable $U_{i,n}$ is $\mathcal{F}_{n+1}$-measurable and independent of $\mathcal{F}_n$, and the random variable Y_{n+1} is $\mathcal{F}_{n+1}$ measurable and independent of $\mathcal{F}_n$, for any $i \in \mathbb{Z}$ şi $n \in \mathbb{N}$.

We also consider the process V_n defined by

$$V_n = \sum_{j=1}^{n} 1_{\{S_j \ne S_{j-1}\}} = \sum_{j=0}^{n-1} U_{S_j,j}, \qquad n \ge 1, \tag{3}$$

representing the number of distinct transitions of the random walk S up to time n.

3 Main results

A first preliminary result is the following.

Lemma 1. *The process V_n defined by (3) satisfies $\lim_{n\to\infty} V_n = \infty$ a.s.*

Proof. From (3) it can be easily seen that V_n is a nondecreasing process, and therefore the limit $\lim_{n\to\infty} V_n$ exists. We have

$$\begin{aligned} \left\{\lim_{n\to\infty} V_n < \infty\right\} &\subset \bigcup_{j\in\mathbb{N}^*} \{V_j = V_{j+1} = \ldots\} \\ &= \bigcup_{j\in\mathbb{N}^*} \left\{0 = U_{S_j,j} = U_{S_{j+1},j+1} = \ldots\right\} \\ &= \bigcup_{j\in\mathbb{N}^*} \left\{0 = U_{S_j,j} = U_{S_j,j+1} = \ldots\right\} \\ &\subset \bigcup_{j\in\mathbb{N}^*} \bigcup_{i=-j}^{j} \{0 = U_{i,j} = U_{i,j+1} = \ldots\}, \end{aligned}$$

using that $U_{S_j,j} = 0$ implies $S_{j+1} = S_j$, and $S_j \in \{-j, \ldots, j\}$ for any $j \in \mathbb{N}^*$.

For an arbitrarily fixed $i \in \mathbb{Z}$, since $(U_{i,j})_{j\in\mathbb{N}}$ is an i.i.d. sequence of Bernoulli random variables with $P(U_{i,j} = 1) = 1 - P(U_{i,j} = 0) = p_i > 0$, we obtain

$$\begin{aligned} P(0 = U_{i,j} = U_{i,j+1} = \ldots) &= \lim_{n\to\infty} P(0 = U_{i,j} = U_{i,j+1} = \ldots = U_{i,n+j-1}) \\ &= \lim_{n\to\infty} (1 - p_i)^n \\ &= 0, \end{aligned}$$

and therefore

$$P\left(\lim_{n\to\infty} V_n < \infty\right) \leq \sum_{j\in\mathbb{N}^*} \sum_{i=-j}^{j} P\left(0 = U_{i,j} = U_{i,j+1} = \ldots\right) = 0,$$

or equivalent $P\left(\lim_{n\to\infty} V_n = \infty\right) = 1$, concluding the proof. □

The previous lemma shows that we can define the right inverse α_n of the process V_n by

$$\alpha_n = \min\left\{m \geq 0 : V_m \geq n\right\}, \qquad n \in \mathbb{N}. \tag{4}$$

With this preparation, we can show now that the random walk S_n corresponding to the trajectory of the coin in the extended (non-homogeneous) model has the same properties as in the original (homogeneous) model, namely that it is an irreducible and recurrent martingale (see for example [No] for the classical terminology).

Proposition 1. *The random walk $(S_n)_{n\in\mathbb{N}}$ defined by (1) – (2) is a recurrent and irreducible $(\mathcal{F}_n)_{n\in\mathbb{N}}$-martingale.*

Proof. The proof of the fact that S_n is recurrent is the same as the corresponding proof from [Pa]. For the sake of completeness, we present here the arguments.

From the definition of the processes S_n and α_n we obtain

$$S_{\alpha_n} = \ldots = S_{\alpha_{n+1}-1} \neq S_{\alpha_{n+1}}$$

and

$$P\left(S_{\alpha_{n+1}} - S_{\alpha_n} = \pm 1\right) = \frac{1}{2},$$

for any $n \in \mathbb{N}$, and therefore $(S_{\alpha_n})_{n\in\mathbb{N}}$ is a symmetric simple random walk on $\mathbb{Z}$.

According to a theorem of Pólya, the symmetric simple random walk on $\mathbb{Z}$ is recurrent (see for example [Bi], pp. 117 – 118). Using this result and Lemma 1 (which shows that V_n increases almost surely to infinity, and therefore its right inverse α_n has this property), it follows that the random walk $(S_n)_{n\in\mathbb{N}}$ is also recurrent.

Since S_n is recurrent, in order to show that it is also irreducible, it suffices to show that starting at the origin S_n will take any value $k \in \mathbb{Z}^*$ with positive probability. This follows however by using the independence of the random variables $U_{i,j}$ and Y_k:

$$\begin{aligned} P\left(\exists n \geq 1 : S_n = k\right) &\geq P\left(S_1 = 1, \ldots, S_k = k\right) \\ &= P\left(U_{0,0} = U_{1,1} = \ldots = U_{k-1,k-1} = 1, Y_1 = \ldots = Y_k = 1\right) \\ &= \frac{1}{2^k} \prod\nolimits_{i=0}^{k-1} p_i \\ &> 0, \end{aligned}$$

for any $k \in \mathbb{N}^*$. The proof being similar in the case $k < 0$, we omit it.

The proof of the fact that $(S_n)_{n\in\mathbb{N}}$ is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-martingale is similar to the corresponding proof from [Pa]. We consider the σ-algebra

$$\widetilde{\mathcal{F}}_n = \sigma\left(U_{i,j}, Y_j : i \in \mathbb{Z}, j \leq n\right) = \sigma\left(\mathcal{F}_n \cup \left\{U_{i,n} : i \in \mathbf{N}\right\}\right) \supset \mathcal{F}_n,$$

and we note that S_n is a $\mathcal{F}_n$-measurable random variable, $U_{S_n,n}$ is $\widetilde{\mathcal{F}}_n$-measurable, and Y_{n+1} is independent of $\widetilde{\mathcal{F}}_n$.

Using the properties of conditional expectation, we obtain

$$\begin{aligned}
E\left(S_{n+1}\middle|\,\mathcal{F}_n\right) &= S_n + E\left(X_{n+1}\middle|\,\mathcal{F}_n\right)\\
&= S_n + E\left(E\left(U_{S_n,n}Y_{n+1}\middle|\,\widetilde{\mathcal{F}}_n\right)\middle|\,\mathcal{F}_n\right)\\
&= S_n + E\left(U_{S_n,n}E\left(Y_{n+1}\middle|\,\widetilde{\mathcal{F}}_n\right)\middle|\,\mathcal{F}_n\right)\\
&= S_n + E\left(U_{S_n,n}E\left(Y_{n+1}\right)\middle|\,\mathcal{F}_n\right)\\
&= S_n + E\left(Y_{n+1}\right)E\left(U_{S_n,n}\middle|\,\mathcal{F}_n\right)\\
&= S_n + 0\cdot E\left(U_{S_n,n}\middle|\,\mathcal{F}_n\right)\\
&= S_n,
\end{aligned}$$

concluding the proof. □

The main result for the random walk S_n corresponding to the trajectory of the coin in the extended model is following theorem, which shows that the Strong Law of Large Numbers holds for S_n.

Theorem 1. *If* $\inf_{i\in\mathbb{N}} p_i = p > 0$ *and* $(S_n)_{n\in\mathbb{N}}$ *is the random walk defined by (1) – (2), then almost surely we have*

$$\lim_{n\to\infty}\frac{S_n}{n} = 0. \tag{5}$$

Proof. From Proposition 1 it follows that S_n is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-martingale with $S_0 = 0$. Since $ES_n^2; \leq n < \infty$ for any $n \in \mathbb{N}$, S_n^2 is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-submartingale.

Denoting by $A_n = \langle S\rangle_n$ the quadratic variation of S_n (the unique predictable and nondecreasing process with $A_0 = 0$ for which $S_n^2 - A_n$ is a martingale), we have

$$E\left(S_{n+1}^2 - A_{n+1}\middle|\,\mathcal{F}_n\right) = S_n^2 - A_n, \qquad n\in\mathbb{N},$$

which due to the fact that A_n is a predictable process (and therefore A_{n+1} is a $\mathcal{F}_n$-measurable random variable), can be written in the equivalent form

$$\begin{aligned}
A_{n+1} - A_n &= E\left(S_{n+1}^2 - S_n^2\middle|\,\mathcal{F}_n\right)\\
&= E\left(\left(S_{n+1} - S_n\right)^2\middle|\,\mathcal{F}_n\right)\\
&= E\left(X_{n+1}^2\middle|\,\mathcal{F}_n\right).
\end{aligned}$$

Since $X_{n+1}^2 = \left(U_{S_n,n}Y_{n+1}\right)^2 = U_{S_n,n}^2 = U_{S_n,n}$, and using the fact that $S_n \in \{-n,\ldots,n\}$ is a $\mathcal{F}_n$-measurable random variable, and $U_{i,n} \in \{0,1\}$ are random variables which are independent of $\mathcal{F}_n$ for any $n \in \mathbb{N}$, from the previous relation we obtain

$$\begin{aligned} A_{n+1} - A_n &= E\left(U_{S_n,n} \middle| \mathcal{F}_n\right) \\ &= E\left(\sum_{i=-n}^{n} U_{i,n} 1_{\{S_n=i\}} \middle| \mathcal{F}_n\right) \\ &= \sum_{i=-n}^{n} 1_{\{S_n=i\}} E\left(U_{i,n} \middle| \mathcal{F}_n\right) \\ &= \sum_{i=-n}^{n} 1_{\{S_n=i\}} E\left(U_{i,n}\right) \\ &= \sum_{i=-n}^{n} p_i 1_{\{S_n=i\}}. \end{aligned}$$

Adding the previous relations for $n = 0, 1, \ldots$, and using the fact that $A_0 = 0$, we obtain

$$A_{n+1} = \sum_{j=0}^{n} \sum_{i=-j}^{j} p_i 1_{\{S_j=i\}}, \qquad n \in \mathbb{N}. \tag{6}$$

Using the hypothesis $\inf_{i\in\mathbb{N}} p_i = p > 0$, we obtain

$$A_{n+1} = \sum_{j=0}^{n} \sum_{i=-j}^{j} p_i 1_{\{S_j=i\}} \geq \sum_{j=0}^{n} \sum_{i=-j}^{j} p 1_{\{S_j=i\}} = p \sum_{j=0}^{n} 1_{\{S_j=\{-j,\ldots,j\}\}} = p \sum_{j=0}^{n} 1 = (n+1)\, p, \tag{7}$$

which in particular shows that $\lim_{n\to\infty} A_n = \infty$ a.s.

Applying a martingale version of Strong Law of Large Numbers (see for example [Wi], pp. 123 – 124), we obtain the almost sure convergence

$$\lim_{n\to\infty} \frac{S_n}{A_n} = 0. \tag{8}$$

Using (6) we obtain

$$A_{n+1} = \sum_{j=0}^{n} \sum_{i=-j}^{j} p_i 1_{\{S_j=i\}} \leq \sum_{j=0}^{n} \sum_{i=-j}^{j} 1_{\{S_j=i\}} = \sum_{j=0}^{n} 1_{\{S_j=\{-i,\ldots,i\}\}} = \sum_{j=0}^{n} 1 = n+1,$$

and therefore

$$\left|\frac{S_n}{n}\right| = \left|\frac{S_n}{A_n}\right| \frac{A_n}{n} \leq \left|\frac{S_n}{A_n}\right|,$$

for all sufficiently large $n \in \mathbb{N}^*$.

Combining with (8) we obtain the almost sure convergence

$$\lim_{n\to\infty} \frac{S_n}{n} = 0,$$

concluding the proof of the theorem. □

Remark 1. *For the particular choice of the probabilities* $p_1 = p_2 = \ldots = p \in (0,1)$*, the non-homogeneous model presented above becomes the homogeneous model introduced in [Pa], and from the results presented in this section we can recover the corresponding results in [Pa].*

We conclude with the remark that, as pointed out in [Pa], the results on the random walk S_n corresponding to the trajectory of the coin within the population have, aside from the importance in their own right, Economic importance. The fact that by Proposition 1 the random walk S_n is recurrent shows that an individual possessing the coin at some time is sure to receive again the coin, infinitely often, in the future. From the Economic point of view this is reassuring, for it shows a good circulation of money within the society. The fact that the random walk described by the coin is irreducible shows that the model is fair, in the sense that all the individuals of the society will be in the possession of the coin at some time. Also, the fact that the random walk is a martingale shows again the fairness of the model, in the sense that there is no particular tendency of the coin to favor a certain region of the population.

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