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LOCAL AND GLOBAL STRUCTURES ON AFFINE HOLOMORPHIC BUNDLES

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Abstract

In this paper we give some cohomological obstructions to globalization of the holomorphic Liouville vector field, of the totally singular complex Lagrangians and of the locally complex Lagrange structures defined on a local chart of an affine holomorphic bundle endowed with the natural holomorphic vertical foliation. We also consider the transversal distributions, we find the main obstructions to globalization of a complex nonlinear connection and of the existence of an affine transversal distribution.

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1 Introduction

In the smooth category the cohomological obstructions for the globalization of some local structures as Liouville vector fields or locally Lagrangians on Lagrangian foliations were intensively studied in [14, 15, 16]. Also, in [4, 9] some extensions of this results on holomorphic Lagrangian fibrations and on affine complex foliated manifolds endowed with a complex tangent structure are given.

The aim of this paper is to obtain similar results in the complex-analytic category for some local structures on affine holomorphic bundles. Firstly, following some results concerning affine bundles from the real case [2, 10], we define the affine holomorphic bundle notion and we discuss the holomorphic vector pseudo-fields. Also, the holomorphic semi-basic 1-forms are defined and a holomorphic transverse Liouville 1-form is obtained. Next, with respect to the natural holomorphic vertical foliation, we find the cohomological obstructions to globalization of the holomorphic Liouville vector field, of the totally singular complex Lagrangians and of the locally complex Lagrange structures defined on a local chart of an affine holomorphic bundle. Finally, we consider transversal distributions, we find the cohomological obstruction to globalization of a complex nonlinear connection and we discuss about the integrability of the horizontal distribution and its holomorphy. We also obtain the main cohomological obstructions for the existence of an affine transversal distribution. The notions are introduced here by analogy with the real case intensively studied by I. Vaisman in several papers.

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2 Affine holomorphic bundles

2.1 Basic definitions and notations

A holomorphic bundle (fibration) is a triplet $\xi = (E, \pi, M)$, where E and M are complex manifold which are connected and paracompact and $\pi : E \to M$ is a holomorphic submersion. We say that E is the *total space*, M is the *base manifold* and π is the *canonical projection* of the holomorphic bundle ξ. In the sequel we identify the holomorphic bundle with the total space E. For every $z \in M$, the sets $E_z = \pi^{-1}(z)$ are closed submanifolds of E, which are supposed to be connected. Let us denote by n the complex dimension of M and by m the complex dimension of E_z, for any $z \in M$. Let $(U_\alpha, \varphi_\alpha)$ be a local chart on M with the complex coordinates (z^k), $k = 1, \ldots, n$ and (V_α, ψ_α) be a local chart on E with the complex coordinates $u = (z^k, \eta^a)$, $k = 1, \ldots, n$, $a = 1, \ldots, m$ such that $\pi(V_\alpha) = U_\alpha$. Then, at local change maps $(V_\alpha, \psi_\alpha) \to (V_\beta, \psi_\beta)$ on E the change rules of the local complex coordinates on E have the form

$$z^{'j} = z^{'j}(z^k) \ , \ \eta^{'b} = \eta^{'b}(z^k, \eta^a), \tag{2.1}$$

where $z^{'j}$ are holomorphic functions on z^k, and $\eta^{'b}$ are holomorphic functions on z^k and η^a and $\det \frac{\partial \eta^{'b}}{\partial \eta^a} \neq 0$.

A morphism of the holomorphic bundles $\pi^{'} : E^{'} \to M^{'}$ and $\pi : E \to M$ is a couple (f_0, f_1), where $f_0 : M^{'} \to M$ and $f_1 : E^{'} \to E$ such that $\pi \circ f_1 = f_0 \circ \pi^{'}$, i.e. f_1 sends fibers to fibers; we also say that f_1 is a f_0-morphism of holomorphic bundles.

Definition 1. *An affine holomorphic bundle is a holomorphic bundle $\pi : E \to M$ in which the change rules of the local complex coordinates on E have the form*

$$z^{'j} = z^{'j}(z^k) \ , \ \eta^{'b} = M_a^b(z^k)\eta^a + B^b(z^k), \tag{2.2}$$

where M_a^b and B^b are holomorphic functions on z^k and $\det M_a^b \neq 0$.

Throughout this paper, we assume that E is an affine holomorphic bundle.

Let J be the natural complex structure of manifold E. We also consider $T^{'}E$ and $T^{''}E = \overline{T^{'}E}$ to be its holomorphic and antiholomorphic tangent bundles and $T_{\mathbb{C}}E = T^{'}E \oplus T^{''}E$ the complexified tangent bundle of the real tangent bundle $T_{\mathbb{R}}E$. From (2.2) the following change rules for the natural local frames on $T_u^{'}E$ result:

$$\frac{\partial}{\partial z^k} = \frac{\partial z^{'j}}{\partial z^k}\frac{\partial}{\partial z^{'j}} + \left(\frac{\partial M_a^b}{\partial z^k}\eta^a + \frac{\partial B^b}{\partial z^k}\right)\frac{\partial}{\partial \eta^{'b}} \ , \ \frac{\partial}{\partial \eta^a} = M_a^b \frac{\partial}{\partial \eta^{'b}}. \tag{2.3}$$

By conjugation over all in (2.3) we get the change rules of the natural local frames on $T_u^{''}E$, and then the behaviour of the J complex structure

$$J\left(\frac{\partial}{\partial z^k}\right) = i\frac{\partial}{\partial z^k} \ , \ J\left(\frac{\partial}{\partial \overline{z}^k}\right) = -i\frac{\partial}{\partial \overline{z}^k} \ , \ J\left(\frac{\partial}{\partial \eta^a}\right) = i\frac{\partial}{\partial \eta^a} \ , \ J\left(\frac{\partial}{\partial \overline{\eta}^a}\right) = -i\frac{\partial}{\partial \overline{\eta}^a}. \tag{2.4}$$

The natural dual bases on $T_u^{'*}E$ change according to the rule

$$dz^{'j} = \frac{\partial z^{'j}}{\partial z^k} dz^k \ , \ d\eta^{'b} = \left(\frac{\partial M_a^b}{\partial z^k} \eta^a + \frac{\partial B^b}{\partial z^k} \right) dz^k + M_a^b d\eta^a \tag{2.5}$$

and by conjugation we obtain the change rules of the natural dual bases on $T_u^{''*}E$.

Thus, the coordinates of the vectors $Z = Z^k \frac{\partial}{\partial z^k} + Z^a \frac{\partial}{\partial \eta^a} \in \Gamma(T^{'}E)$ have the following change rules

$$Z^{'j} = \frac{\partial z^{'j}}{\partial z^k} Z^k \ , \ Z^{'b} = \left(\frac{\partial M_a^b}{\partial z^k} \eta^a + \frac{\partial B^b}{\partial z^k} \right) Z^k + M_a^b Z^a \tag{2.6}$$

and the coordinates of the co-vectors $\omega = \omega_k dz^k + \omega_a d\eta^a \in \Gamma(T^{'*}E)$ change according to the rules

$$\omega_j^{'} = \frac{\partial z^k}{\partial z^{'j}} \omega_k + \left(\frac{\partial M_b^a}{\partial z^{'j}} \eta^{'b} + \frac{\partial B^a}{\partial z^{'j}} \right) \omega_a \ , \ \omega_b^{'} = M_b^a \omega_a. \tag{2.7}$$

By conjugation over all in (2.6) and (2.7) we get the change rules of the coordinates of the vectors from $\Gamma(T^{''}E)$ and of the co-vectors from $\Gamma(T^{''*}E)$, respectively.

Definition 2. *An affine local section in the affine holomorphic bundle E is a holomorphic map $s : U_\alpha \to E$ such that $\pi \circ s = Id|_{U_\alpha}$ and its local components change according to the rule*

$$s^{'b}(z^{'j}) = M_a^b(z^k) s^a(z^k) + B^b(z^k). \tag{2.8}$$

The set of affine sections on E is denoted by $\Gamma(E)$ and it is an affine module over $\mathcal{F}(M)$, i.e. for every $f_1, \ldots, f_p \in \mathcal{F}(M)$ such that $f_1 + \ldots + f_p = 1$ and $s_1, \ldots, s_p \in \Gamma(E)$, then $f_1 s_1 + \ldots + f_p s_p \in \Gamma(E)$, where the affine combination is taken at every point $z \in M$. Using a partition of the unity on the base M, which can be smooth but not holomorphic, it can be easily proved that every affine holomorphic bundle allows an affine section.

We notice that a holomorphic vector bundle $\pi : \overline{E} \to M$ can be canonically associated with an affine holomorphic bundle $\pi : E \to M$. More precisely, using local coordinates, an affine holomorphic bundle reduces to a holomorphic vector bundle if in (2.2) we have $B^b = 0$. In this case, we say that E is of *holomorphic vector type* or according to the terminology from [10], we call E a *central affin holomorphic bundle*.

Let us consider $V_u^{'}E = \ker \pi_*|_u$ for every $u = (z, \eta) \in E$, then we obtain the *vertical distribution* or the *vertical sub-bundle* of $T^{'}E$, denoted by $V^{'}E$ which in view of (2.3) is an integrable and holomorphic one. This distribution is tangent to the holomorphic vertical foliation $\mathcal{V}$ (the foliation by fibers of π). Let $\Gamma(V^{'}E)$ be the module of the holomorphic vertical sections. The local complex coordinates on $V^{'}E$ have the form (z^k, η^a, ζ^a) and change by the rules

$$z^{'j} = z^{'j}(z^k) \ , \ \eta^{'b} = M_a^b(z^k)\eta^a + B^b(z^k) \ , \ \zeta^{'b} = M_a^b \zeta^a. \tag{2.9}$$

Definition 3. *A Liouville type section is a holomorphic vertical section $S \in \Gamma(V^{'}E)$ which has the local form*

$$S^a(z^k, \eta^a) = \eta^a + C^a(z^k), \tag{2.10}$$

where C^a are holomorphic functions on (z^k) variables.

Similarly to the real case [10], using (2.2), (2.9) and (2.10) we obtain

Proposition 1. *Every Liouville type section in* $\Gamma(V^{'}E)$ *defines an affine section in* $\Gamma(E)$ *and conversely.*

2.2 Holomorphic vector pseudo-fields

In this subsection, by analogy with the real case, [10], we define the holomorphic vector pseudo-field notion on a holomorphic bundle. We also obtain an inductive method in obtaining affine holomorphic bundles canonically associated with a holomorphic vector pseudo-field.

Let $U_\alpha = \pi(V_\alpha)$. A complex $(1,0)$-vector field (or more generally a $(p,0)$-vector field) on M is holomorphic if its local components are holomorphic functions on (z^k) variables.

Definition 4. *A holomorphic vector pseudo-field on E is an asssociation of a local holomorphic vector field* $\Gamma_\alpha \in \mathcal{X}_{\mathrm{hol}}(V_\alpha)$ *with every domain* V_α *of the given atlas on* E*, such that* $\Gamma_\alpha(\eta^a) = 0$ *and for every two domains* V_α *and* V_β *which have the complex coordinates* (z^k, η^a) *and* $(z^{'j}, \eta^{'b})$*, respectively, then on the intersection* $V_\alpha \cap V_\beta$ *we have* $\Gamma_\alpha(z^k) = \Gamma_\beta(z^k)$ *and* $\Gamma_\alpha(z^{'j}) = \Gamma_\beta(z^{'j})$*.*

From the above definition, we obtain the following change rule for Γ:

$$\Gamma_\beta = \Gamma_\alpha - \Gamma_\alpha(\eta^{'b})\frac{\partial}{\partial \eta^{'b}}. \tag{2.11}$$

Conversely, it can be proved that the association of a local holomorphic vector field $\Gamma_\alpha \in \mathcal{X}_{\mathrm{hol}}(V_\alpha)$ with the domain V_α, such that condition (2.11) holds on the intersection $V_\alpha \cap V_\beta$, then a holomorphic vector pseudo-field is obtained.

In the following we present some examples of holomorphic vector pseudo-fields:

1) Let $\pi : E \to M$ be a holomorphic bundle and $Z \in \mathcal{X}_{\mathrm{hol}}(M)$ be a holomorphic vector field on the base M. If the holomorphic vector field Z has the local form $Z = Z^k(z)\frac{\partial}{\partial z^k} \in \mathcal{X}_{\mathrm{hol}}(U_\alpha)$ then, $\Gamma_\alpha = Z^k(z)\frac{\partial}{\partial z^k}$ is a holomorphic vector pseudo-field on $V_\alpha = \pi^{-1}(U_\alpha)$.

2) Let $\pi : E \to M$ be a holomorphic bundle and $D : V^{'}(E) \to T^{'}M$ be an π-morphism of holomorphic vector bundles, where $T^{'}M$ is the holmorphic tangent bundle of M and $V \in \mathcal{X}_{\mathrm{hol}}(V^{'}(E))$ is a holmorphic vertical vector field on E.

   Using the local complex coordinates, if $V = V^a(z,\eta)\frac{\partial}{\partial \eta^a} \in \mathcal{X}_{\mathrm{hol}}(V^{'}E)$ and

$$V^a(z,\eta)\frac{\partial}{\partial \eta^a} \xrightarrow{D} D^k_a(z,\eta)V^a(z,\eta)\frac{\partial}{\partial z^k}$$

   ($Z^k = D^k_a V^a$ are holomorphic functions on $V^{'}(E)$) and $\{\frac{\partial}{\partial z^k}\}$ are holomorphic vector fields on M, then $\Gamma_\alpha = D^k_a(z,\eta)V^a(z,\eta)\frac{\partial}{\partial z^k} \in \mathcal{X}_{\mathrm{hol}}(V_\alpha)$ defines a holomorphic vector pseudo-field (here $\{\frac{\partial}{\partial z^k}\}$ are local vector fields on E).

Remark 1. *In the second example we can consider the particular case when* $V = \eta^a \frac{\partial}{\partial \eta^a}$ *is the holomorphic Liouville vector field. Then, the associated holomorphic pseudo-field has the local form* $\Gamma_\alpha = D^k_a(z,\eta)\eta^a \frac{\partial}{\partial z^k}$.

We have,

Proposition 2. *If* $\pi : E \to M$ *is an affine holomorphic bundle and* Γ *is a holomorphic vector pseudo-field on* E*, then there is an affine bundle* $\pi^{'} : E^{'} \to E$ *and a holomorphic vector pseudo-field* $\Gamma^{'}$ *on* $E^{'}$ *which is naturally induced by* Γ.

Proof. We assume that the local complex coordinates change on E according to the formulas (2.2) and we define the change rule of the complex coordinates on $E^{'}$ on $\pi^{'-1}(V_\alpha) \cap \pi^{'-1}(V_\beta)$ by

$$\zeta^{'b}(z^k, \eta^a, \zeta^a) = M^b_a(z^k)\zeta^a + \Gamma_\alpha(\eta^{'b}). \tag{2.12}$$

In the sequel we prove that $\pi^{'} : E^{'} \to E$ is an affine holomorphic bundle over E. Consider $(z^{''l}, \eta^{''c}, \zeta^{''c})$ the local complex coordinates in another local chart $\pi^{'-1}(V_\gamma)$ of $E^{'}$ which change according to the rules

$$z^{''l} = z^{''l}(z^k)\,,\; \eta^{''c} = M^c_b(z^k)\eta^{'b} + B^{''c}(z^k)\,,\; \zeta^{''c} = M^c_b(z^k)\zeta^{'b} + \Gamma_\beta(\eta^{''c}). \tag{2.13}$$

Let us prove that (2.13) is invariant at local charts changes on $E^{'}$. The link between the local complex coordinates (z^k, η^a, ζ^a) and $(z^{''l}, \eta^{''c}, \zeta^{''c})$ is

$$\begin{aligned} z^{''l} &= z^{''l}(z^k); \\ \eta^{''c} &= M^c_b(z^{'j})\eta^{'b} + B^{''c}(z^{'j}) \\ &= M^c_b(z^{'j})[M^b_a(z^k)\eta^a + B^{'b}(z^k)] + B^{''c}(z^{'j}) \\ &= M^c_a(z^k)\eta^a + B^{''c}(z^k), \end{aligned}$$

where $B^{''c}(z^k) = M^c_b(z^{'j})B^{'b}(z^k) + B^{''c}(z^{'j}(z^k))$ and

$$\begin{aligned} \zeta^{''c} &= M^c_b(z^{'j})\zeta^{'b} + \Gamma_\beta(\eta^{''c}) \\ &= M^c_b(z^{'j})[M^b_a(z^k)\zeta^a + \Gamma_\alpha(\eta^{'b})] + \Gamma_\beta(\eta^{''c}) \\ &= M^c_a(z^k)\zeta^a + M^c_b(z^{'j})\Gamma_\alpha(\eta^{'b}) + \Gamma_\beta(\eta^{''c}) \\ &= M^c_a(z^k)\zeta^a + M^c_b(z^{'j})\Gamma_\alpha(\eta^{'b}) + \Gamma_\alpha(\eta^{''c}) - \Gamma_\alpha(\eta^{'b})\frac{\partial \eta^{''c}}{\partial \eta^{'b}} \\ &= M^c_a(z^k)\zeta^a + \Gamma_\alpha(\eta^{''c}), \end{aligned}$$

where in the last equality we used (2.11). Thus $\pi^{'} : E^{'} \to E$ is an affine holomorphic bundle. We define now

$$\Gamma^{'}_\alpha = \Gamma_\alpha + \zeta^a \frac{\partial}{\partial \eta^a}$$

on $\pi^{'-1}(V_\alpha)$. We must prove that $\Gamma^{'}$ defines a holomorphic vector pseudo-field on $E^{'}$. Indeed, it is obvious that $\Gamma^{'}_\alpha(\zeta^a) = 0$ and on the intersection $\pi^{'-1}(V_\alpha) \cap \pi^{'-1}(V_\beta)$ of two domains of local charts on $E^{'}$, we have

$$\Gamma^{'}_\alpha(z^k) = \Gamma_\alpha(z^k) = \Gamma_\beta(z^k) = \Gamma^{'}_\beta(z^k)\,,\; \Gamma^{'}_\alpha(\eta^a) = \Gamma_\alpha(\eta^a) + \zeta^a\frac{\partial \eta^a}{\partial \eta^a} = \zeta^a,$$

$$\Gamma'_\alpha(z'^j) = \Gamma_\alpha(z'^j) + \zeta^a \frac{\partial z'^j}{\partial \eta^a} = \Gamma_\alpha(z'^j) = \Gamma_\beta(z'^j) = \Gamma'_\beta(z'^j),$$

and

$$\Gamma'_\beta(\eta^a) = \Gamma_\beta(\eta^a) + \zeta'^b \frac{\partial \eta^a}{\partial \eta'^b} = \Gamma_\beta(\eta^a) + \zeta'^b M^a_b = \zeta^a = \Gamma'_\alpha(\eta^a),$$

$$\Gamma'_\beta(\eta'^b) = \Gamma_\beta(\eta'^b) + \zeta'^b \frac{\partial \eta'^b}{\partial \eta'^b} = \zeta'^b = \Gamma_\alpha(\eta'^b) + M^b_a \zeta^a = \Gamma_\alpha(\eta'^b) + \zeta^a \frac{\partial \eta'^b}{\partial \eta^a} = \Gamma'_\alpha(\eta'^b)$$

which ends the proof. □

2.3 Holomorphic semi-basic 1-forms

Let us consider the quotient bundle $Q'E = T'E/V'E$. Then we obtain the following holomorphic vector bundles on E exact sequence

$$0 \to V'E \xrightarrow{i} T'E \xrightarrow{p} Q'E = E \times_M T'M \to 0, \tag{2.14}$$

where i and p are the canonical injection and the canonical projection, respectively. We have for $V'E$ and $Q'E$ the local bases $\left\{\frac{\partial}{\partial \eta^a}\right\}$, $a = 1, \dots, m$ and $\left\{[\frac{\partial}{\partial z^k}] = p(\frac{\partial}{\partial z^k})\right\}$, $k = 1, \dots, n$, respectively. If we put, for every $u \in E$,

$$V'^{\perp}_u E = \{\varphi \in \Gamma(T'^*_u E) \, : \, \varphi(Z) = 0 \, , \forall \, Z \in \Gamma(V'_u E)\},$$

we obtain a sub-bundle of T'^*E called the *orthogonal dual* of $V'_u E$. Now, if we consider $Q'^{\perp}E = T'^*E/V'^{\perp}E$ then we obtain a new exact sequence of vector bundles over E

$$0 \to E \times_M T'^*M = V'^{\perp}E \xrightarrow{j} T'^*E \xrightarrow{q} Q'^{\perp}E \to 0, \tag{2.15}$$

where j and q are the canonical injection and the canonical projection, respectively. For $V'^{\perp}E$ and $Q'^{\perp}E$ we have the local bases $\{dz^k\}$, $k = 1, \dots, n$ and $\{[d\eta^a] = q(d\eta^a)\}$, $a = 1, \dots, m$, respectively. If the change rule of the local coordinates is given by (2.1), then we have the following change rule for the local bases $\left\{[\frac{\partial}{\partial z^k}]\right\}$, $k = 1, \dots, n$ and $\{[d\eta^a]\}$, $a = 1, \dots, m$

$$\left[\frac{\partial}{\partial z^k}\right] = \frac{\partial z'^j}{\partial z^k} \left[\frac{\partial}{\partial z'^j}\right] \; , \; \left[d\eta'^b\right] = \frac{\partial \eta'^b}{\partial \eta^a} [d\eta^a] \, . \tag{2.16}$$

Thus, it follows that Q'^*E and V'^*E are canonically isomorphic with $V'^{\perp}E$ and $Q'^{\perp}E$, respectively. In a similar way $Q'E$ and Q'^*E are canonically isomorphic with $\pi^*(T'M)$ and $\pi^*(T'^*M)$, respectively.

A complex p-form φ on a complex manifold M is called *holomorphic* if it is of type $(p, 0)$ and its local coefficients are holomorphic functions on (z^k) variables, namely $\overline{\partial}\varphi = 0$.

Now, if γ is a holomorphic 1-form on the complex manifold E, then similarly to [5] Ch. II, we can prove that the following properties are equivalent:

i) For any vertical vector field $Z \in \Gamma(V'E)$, $i_Z \gamma = 0$.

ii) For any $u = (z, \eta) \in E$, there is a unique form $\varphi \in T'^{*}_{\pi(u)}M$ such that

$$\gamma_u = \pi_u^* \varphi. \tag{2.17}$$

iii) γ is a section of the holomorphic fibration

$$\mathrm{pr}_1 : E \times_M T'^{*}M \to E. \tag{2.18}$$

A holomorphic 1-form γ on E is said to be semi-basic if it has any one of these three properties. The form γ is basic, namely $i_Z\gamma = \mathcal{L}_Z\gamma = 0$ for any $Z \in \Gamma(V'E)$ if it is the pull-back $\pi^*\mu$ of a holomorphic 1-form μ on M. To any holomorphic semi-basic 1-form γ, there is a correspondent fiber morphism $f = \mathrm{pr}_2 \circ \gamma : E \to T'^{*}M$ (pr_2 being the projection $E \times_M T'^{*}M \to T'^{*}M$). Conversely, if a fiber morphism $f : E \to T'^{*}M$ is given, the form γ such that $\gamma_u = \pi_u^* f(u)$ is semi-basic. In particular for the holomorphic cotangent bundle $p : T'^{*}M \to M$, the semi-basic form corresponding to the identity mapping of $T'^{*}M$ is the natural holomorphic Liouville 1-form θ_M. Also, similarly to [5] Ch. II, we can prove in our case the relation between γ and f as follows:

Proposition 3. *For any holomorphic semi-basic* 1*-form* γ *on* E*, there is a fiber morphism* $f : E \to T'^{*}M$ *such that* $\gamma = f^*\theta_M$*. Conversely if* $f : E \to T'^{*}M$ *is a morphism then the form* $f^*\theta_M$ *is a holomorphic semi-basic* 1*-form on* E*.*

Proof. We use the calculus in local complex coordinates. Let $(z^1, \ldots, z^n, \eta^1, \ldots, \eta^m)$ be the local complex coordinates on $\pi^{-1}(U_\alpha)$ and $(z^1, \ldots, z^n, \zeta_1, \ldots, \zeta_n)$ be the local complex coordinates on $p^{-1}(U_\alpha)$, where $(U_\alpha, (z^k))$ is a local chart in M and $p : T'^{*}M \to M$. The forms γ and θ_M may be locally written by $\gamma = \gamma_i(z^k, \eta^\alpha)dz^i$, $\theta_M = \zeta_i dz^i$. Then the morphism f is defined by $\gamma_i = \zeta_i$. □

Proposition 4. *The* 1*-forms induced on the complex manifold* $E \times_M T'^{*}M$ *by the holomorphic Liouville* 1*-forms* θ_E *on* $T'^{*}E$ *and* θ_M *on* $T'^{*}M$ *coincide. In other words we have*

$$j^*\theta_E = \mathrm{pr}_2^*\theta_M. \tag{2.19}$$

Proof. This can be easily cheeked using local coordinates $(z^1, \ldots, z^n, \zeta_1, \ldots, \zeta_n)$ on $T'^{*}M$ and $(z^1, \ldots, z^n, \eta^1, \ldots, \eta^m, \zeta_1, \ldots, \zeta_n, t_1, \ldots, t_m)$ on $T'^{*}E$. The submanifold $E \times_M T'^{*}M$ of $T'^{*}E$ is defined locally by $t_1 = \ldots = t_m = 0$. We have $\theta_E = \zeta_i dz^i + t_\alpha d\eta^\alpha$, $\theta_M = \zeta_i dz^i$, hence $j^*\theta_E = \mathrm{pr}_2^*\theta_M = \zeta_i dz^i$. □

The form $\gamma = \mathrm{pr}_2^*\theta_M$ is by definition basic with respect to the holomorphic projection $E \times_M T'^{*}M \to T'^{*}M$. It could be called the holomorphic *transverse* Liouville 1-form.

3 Some global results

Let $\mathcal{V}$ be the holomorphic vertical foliation of E with the leaves characterized by $z^k = const$. As we have already seen, the holomorphic tangent vectors of the leaves define the structural subbundle $T^{'}\mathcal{V}=V^{'}E$ of $T^{'}E$ spanned by $\{\frac{\partial}{\partial\eta^a}\}$ and with the transition functions $M_a^b(z)$.

For the holomorphic vertical foliation $\mathcal{V}$, we denote by $\Omega^0_{pr}(E)$ the sheaf of germs of holomorphic projectable (foliated) fuctions on E and by $\mathcal{A}^0_{pr}(E,\mathcal{V})$ the sheaf of germs of leafwise holomorphic vertical functions, locally given by

$$f=\alpha_a(z^k)\eta^a+\beta(z^k), \tag{3.1}$$

where $\alpha_a(z)$, $\beta(z)\in\Omega^0_{pr}(E)$.

3.1 Holomorphic Liouville vector field

We can construct the following exact sequence

$$0\to\Omega^0_{pr}(E)\xrightarrow{i}\mathcal{A}^0_{pr}(E,\mathcal{V})\xrightarrow{p^{'}}\Omega^0_{pr}(E)\otimes V^{'*}E\to 0 \tag{3.2}$$

explicitly given by $\beta\xrightarrow{i}\alpha_a\eta^a+\beta\xrightarrow{p^{'}}\alpha_a d\eta^a$.

Now, let us consider the holomorphic Liouville vector field on E, locally given in the chart (V_α,ψ_α) by

$$\Gamma_\alpha=\eta^a\frac{\partial}{\partial\eta^a}. \tag{3.3}$$

Then, on the intesection $V_\alpha\cap V_\beta\neq\phi$ by (2.2) and (2.3) we have

$$\Gamma_\beta-\Gamma_\alpha=\eta^{'b}\frac{\partial}{\partial\eta^{'b}}-\eta^a\frac{\partial}{\partial\eta^a}=B^b\frac{\partial}{\partial\eta^{'b}} \tag{3.4}$$

and we see that the right-hand side of (3.4) defines a holomorphic vector field with coefficients in $\Omega^0_{pr}(E)$. Thus, the difference $\Gamma_{\alpha\beta}=\Gamma_\beta-\Gamma_\alpha$ yields a cocycle $(\delta\Gamma)_{\alpha\beta\gamma}=\Gamma_{\beta\gamma}-\Gamma_{\alpha\gamma}+\Gamma_{\alpha\beta}=0$. This cocycle defines a Cèch cohomology class

$$[\Gamma_\alpha]\in H^1(E,\Omega^0_{pr}(E)) \tag{3.5}$$

which will be called *linear obstruction* of $\mathcal{V}$, and its vanish yields Γ_α is globally defined. By the same considerations as in [14], we have

Proposition 5. *The affine holomorphic bundle $\pi:E\to M$ is of holomorphic vector type if and only if $[\Gamma_\alpha]=0$.*

Proof. The necessity is obvious. Conversely, if $[\Gamma_\alpha]=0$, then there is an adapted atlas where

$$B^{'b}\frac{\partial}{\partial\eta^{'b}}=\psi^{'b}(z^{'j})\frac{\partial}{\partial\eta^{'b}}-\psi^a(z^k)\frac{\partial}{\partial\eta^a} \tag{3.6}$$

with ψ^a holomorphic functions on (z^k) variables. Then, in the new coordinates $\widetilde{z}^k=z^k$ and $\widetilde{\eta}^a=\eta^a-\psi^a(z^k)$ we obtain $\widetilde{B}^b(\widetilde{z}^k)=0$. □

Remark 2. *We notice that the obstruction to globalization of the conjugated Liouville vector field $\overline{\Gamma}=\overline{\eta}^a\frac{\partial}{\partial\overline{\eta}^a}$ can be obtained in similar manner.*

3.2 Totally singular complex Lagrangians

In this subsection, as in the real case [11], we can consider the *totally singular complex Lagrangian* notion on an affine holomorphic bundle E, that is a real-valued function $\mathcal{L} : E \to \mathbb{R}$ which is affine in the fibers coordinates, or equivalently it has a null vertical complex hessian. Such a complex Lagrangian is locally given in the chart (V_α, ψ_α) by

$$\mathcal{L}_\alpha(z^k, \eta^a) = \alpha_a(z^k)(\eta^a + \overline{\eta}^a) + \beta(z^k), \tag{3.7}$$

where $\alpha = \alpha_a(z^k)d\eta^a \in \Gamma(V^{'*}E)$ and $\alpha_a(z^k)$, $\beta(z^k) \in \Omega^{\mathbb{R}}_{pr}(E)$, where $\Omega^{\mathbb{R}}_{pr}(E)$ is the sheaf of germs of real-valued projectable functions on E.

If we denote by $\mathcal{A}^{\mathbb{R}}_{pr}(E, \mathcal{V} \oplus \overline{\mathcal{V}})$ the sheaf of germs of functions locally given by (3.7), then similarly to (3.2) we can construct the following exact sequence

$$0 \to \Omega^{\mathbb{R}}_{pr}(E) \xrightarrow{i} \mathcal{A}^{\mathbb{R}}_{pr}(E, \mathcal{V} \oplus \overline{\mathcal{V}}) \xrightarrow{\widetilde{p}} \Omega^{\mathbb{R}}_{pr}(E) \otimes (V^{'*}E \oplus V^{''*}E) \to 0 \tag{3.8}$$

explicitly given by $\beta \xrightarrow{i} \alpha_a(\eta^a + \overline{\eta}^a) + \beta \xrightarrow{\widetilde{p}} \alpha_a(d\eta^a + d\overline{\eta}^a)$.

Then, on the intersection $V_\alpha \cap V_\beta \neq \phi$ from (2.2) and (3.7) we have

$$\mathcal{L}_{\alpha\beta} := \mathcal{L}_\beta - \mathcal{L}_\alpha = \alpha^{'}_b(B^b + \overline{B}^b) \tag{3.9}$$

which yields a cocycle $(\delta\mathcal{L})_{\alpha\beta\gamma} = \mathcal{L}_{\beta\gamma} - \mathcal{L}_{\alpha\gamma} + \mathcal{L}_{\alpha\beta} = 0$. This cocycle defines a Cèch cohomology class

$$[\mathcal{L}_\alpha] \in H^1(E, \Omega^{\mathbb{R}}_{pr}(E)). \tag{3.10}$$

Thus, we obtain

Proposition 6. $[\mathcal{L}_\alpha] = 0$ *yields* $\mathcal{L}_\alpha$ *is globally defined.*

3.3 Locally complex Lagrange structures

In this subsection we generalize in the case of affine holomorphic bundles some results from [9] concerning to globalization of some locally complex Lagrange structures on complex tangent manifolds.

Let $L_\alpha : E \to \mathbb{R}_+$ be a complex Lagrange function on the affine holomorphic bundle E, defined on $V_\alpha \subset E$, domain of local chart.

Definition 5. *We say that a family* $\{E, L_\alpha\}$ *is a locally complex Lagrange structure on* E*, if there is an atlas such that* $g_{a\bar{b}} = \partial^2 L_\alpha / \partial\eta^a \partial\overline{\eta}^b$ *glue up to a global hermitian metric on* $V^{'}E$*.*

If $\{E, L_\alpha\}$ defines a locally complex Lagrange structure on E, by integration of $g_{a\bar{b}}$, we obtain a complex Lagrangian $L : E \longrightarrow \mathbb{R}_+$ such that $L_\alpha = L|_{V_\alpha} + l_\alpha$, where l_α is an affine real valued form on E, i.e. there is $\overset{\alpha}{A}_a(z)$ and $\overset{\alpha}{B}(z) \in \mathbb{R}_+$ such that

$$L_\alpha = L|_{V_\alpha} + \overset{\alpha}{A}_a (\eta^a + \overline{\eta}^a) + \overset{\alpha}{B}\ . \tag{3.11}$$

On the intersection $V_\alpha \cap V_\beta$ we can define a cocycle $L_{\alpha\beta} := L_\beta - L_\alpha$, L being closed with respect to differential $(\delta L)_{\alpha\beta\gamma} = L_{\beta\gamma} - L_{\alpha\gamma} + L_{\alpha\beta} = 0$. Denoted by $[L_\alpha]$ the cohomology class defined by cocycle $L_{\alpha\beta}$. We have

Proposition 7. *$[L_\alpha] \in H^1(E, \mathcal{A}^0_R(E, \mathcal{V} \oplus \overline{\mathcal{V}}))$ and $[L_\alpha] = 0$ yields L is globally defined.*

Let us see when $[L_\alpha] = 0$. We can construct an exact sequence over affine real valued functions, without requesting their holomorphy,

$$0 \to \Phi^0_R(E) \xrightarrow{i} \mathcal{A}^0_R(E, \mathcal{V} \oplus \overline{\mathcal{V}}) \xrightarrow{\pi^{1,0}} \Phi^{(1,0)}_R(E) \to 0 \tag{3.12}$$

explicitly given by the corespondence $\overset{\alpha}{B} \to \overset{\alpha}{A}_a\ (\eta^a + \overline{\eta}^a) + \overset{\alpha}{B} \to \overset{\alpha}{A}_a\ d\eta^a$. This induces the following exact sequence of cohomology groups

$$0 \to H^1(E, \Phi^0_R(E)) \xrightarrow{i^*} H^1(E, \mathcal{A}^0_R(E, \mathcal{V} \oplus \overline{\mathcal{V}})) \overset{(\pi^{1,0})^*}{\to} H^1(E, \Phi^{(1,0)}_R(E)) \to \dots .$$

Let $[L_\alpha]_1 = (\pi^{1,0})^*[L_\alpha] \in H^1(E, \Phi^{(1,0)}_R(E))$. If $[L_\alpha]_1 = 0$ then $[L_\alpha]_1 \in \ker(\pi^{1,0})^* = \operatorname{Im} i^*$ and, therefore, there exists $[L_\alpha]_2 \in H^1(E, \Phi^0_R(E))$ such that $i^*[L_\alpha]_2 = [L_\alpha]$. Hence, we can state

Proposition 8. *$[L_\alpha] = 0$ if and only if $[L_\alpha]_1 = [L_\alpha]_2 = 0$.*

These are the main obstructions to globalization of locally complex Lagrange structure on an affine holomorphic bundle E.

4 Transversal distributions

As in the real case for vector (affine) bundles [2, 6] or the general case of holomorphic foliations [13], a normalization of the holomorphic vertical distribution $V'E$ is a distribution $H'E$ on E which is supplementary to $V'E$ in $T'E$. The distribution $H'E$ is called *horizontal distribution* (or complex nonlinear connection on E, briefly c.n.c.). Such a normalization can be defined by a right spliting of the exact sequence (2.14), i.e. by a map $\sigma : Q'E \to T'E$ which satisfies the conditions that σ is an E-morphism of holomorphic bundles and $p \circ \sigma = Id|_{Q'E}$.

Denoting as $H'E = \sigma(Q'E)$, it is a subbundle of $T'E$ which is supplementary to $V'E$, thus we obtain a normalization of $V'E$ with $H'E$ suitable horizontal bundle. In local coordinates, we can consider

$$\frac{\delta}{\delta z^k} = \sigma\left(p\left(\frac{\partial}{\partial z^k}\right)\right) = \frac{\partial}{\partial z^k} - N^a_k \frac{\partial}{\partial \eta^a},\ k = 1, \dots, n \tag{4.1}$$

and $\{\frac{\delta}{\delta z^k}\}$ is a local basis of the sections of $H'E$, called adapted for the c.n.c. The local functions $N^a_k(z, \eta)$ on E are called the coefficients of the c.n.c.

The change rule of the adapted basis is

$$\frac{\delta}{\delta z^k} = \frac{\partial z'^j}{\partial z^k} \frac{\delta}{\delta z'^j} \tag{4.2}$$

and consequently, the change rule for the coefficients N^a_k of the c.n.c. is

$$\frac{\partial z'^j}{\partial z^k} N'^b_j = M^b_a N^a_k - \left(\frac{\partial M^b_a}{\partial z^k} \eta^a + \frac{\partial B^b}{\partial z^k}\right). \tag{4.3}$$

Conversely, if we assume that on the domain of every local chart V_α on E, the local functions $N_k^a(z,\eta)$ are given such that the change rule (4.3) on the intersection of two domains hold, then the map σ locally given by (4.1) is a normalization of $V^{'}E$. The normalization σ gives an embedding of $Q^{'}E$ in $T^{'}E$ and a decomposition of $T^{'}E$ in the direct sum

$$T^{'}E = H^{'}E \oplus V^{'}E. \tag{4.4}$$

By conjugation over all in (4.4) we get a decomposition of the complexified tangent bundle of E, namely

$$T_{\mathbb{C}}E = H^{'}E \oplus V^{'}E \oplus H^{''}E \oplus V^{''}E, \tag{4.5}$$

where $H^{''}E$ is spanned by $\{\frac{\delta}{\delta\bar{z}^k}\}$ and $V^{''}E$ is spanned by $\{\frac{\partial}{\partial\bar{\eta}^a}\}$. The dual adapted bases are locally given by

$$\{dz^k\}\,,\ \{\delta\eta^a = d\eta^a + N_k^a dz^k\}\,,\ \{d\bar{z}^k\}\,,\ \{\delta\bar{\eta}^a = d\bar{\eta}^a + N_{\bar{k}}^{\bar{a}} d\bar{z}^k\}. \tag{4.6}$$

We notice that as in the case of holomorphic vector bundles [8], a normalization of $V^{'}E$ can be derived from a *regular complex Lagrangian* on E, that is a real valued function $L : E \to \mathbb{R}$ such that $g_{a\bar{b}} = \partial^2 L/\partial\eta^a\partial\bar{\eta}^b$ defines a hermitian metric tensor on the fibers of the vertical bundle $V^{'}E$. If we denote by $(g^{\bar{b}a})$ the inverse of $(g_{a\bar{b}})$, then by using (2.3), we obtain that the following local functions

$$\overset{CL}{N_k^a} = g^{\bar{b}a}\frac{\partial^2 L}{\partial z^k \partial\bar{\eta}^b} \tag{4.7}$$

verify the change rule (4.3) and we call this normalization *the Chern-Lagrange c.n.c.* on the affine holomorphic bundle E.

We can consider now the complex tensor field locally given in chart V_α by

$$N_\alpha = N_k^a \frac{\partial}{\partial\eta^a} \otimes dz^k \in \Gamma(V^{'}E \otimes Q^{'*}E).$$

Then, on the intersection $V_\alpha \cap V_\beta \neq \phi$ from (2.3), (2.5) and (4.3) we have

$$N_\beta - N_\alpha = -M_b^c\left(\frac{\partial M_a^b}{\partial z^k}\eta^a + \frac{\partial B^b}{\partial z^k}\right)\frac{\partial}{\partial\eta^c}\otimes dz^k \tag{4.8}$$

and we see that the right-hand side of (4.8) defines a complex tensor field with coefficients in $\mathcal{A}_{pr}^0(E,\mathcal{V})$. Thus, the difference $N_{\alpha\beta} := N_\beta - N_\alpha$ yields a cocycle $(\delta N)_{\alpha\beta\gamma} = N_{\beta\gamma} - N_{\alpha\gamma} + N_{\alpha\beta} = 0$. This cocycle defines a Cèch cohomology class

$$[N_\alpha] \in H^1(E, \mathcal{A}_{pr}^0(E,\mathcal{V})) \tag{4.9}$$

which will be called *obstruction to globalization of a c.n.c.* on an affine holomorphic bundle E. Thus, we obtain

Proposition 9. *$[N_\alpha] = 0$ yields N_α is globally defined.*

4.1 Integrability and holomorphy

However, $H^{'}E$ is smoothly isomorphic to $Q^{'}E$ which is holomorphic as $V^{'}E$, generally $H^{'}E$ is not a holomorphic subbundle of $T^{'}E$. The existence of a holomorphic supplementary distribution $H^{'}E$ is characterized in the general case of holomorphic foliations [13], by the vanishing of a certain cohomological obstruction, as follows:

By the change rule (4.3), the following 1-form:

$$\Phi_k^a = \overline{\partial} N_k^a \tag{4.10}$$

defines a global 1-form Φ on E with values in $Hom(Q^{'}E, V^{'}E)$ which is $d^{''}$-closed, hence it gives a cohomology class $[\Phi] \in H^1(E, \mathcal{O}(Hom(Q^{'}E, V^{'}E)))$ (in view of the Dolbeault-Serre theorem [12]). Thus, we have

Theorem 1. *([13]). The vertical distribution $V^{'}E$ admits a supplementary holomorphic distribution if and only if* $[\Phi] = 0$.

The horizontal distribution $H_{\mathbb{C}}E = H^{'}E \oplus H^{''}E$ is said to be *integrable* if it is closed under the Lie bracket operator, namely $[\Gamma(H_{\mathbb{C}}E), \Gamma(H_{\mathbb{C}}E)] \subset \Gamma(H_{\mathbb{C}}E)$.

The obstruction for the integrability of $H_{\mathbb{C}}E$ is given by the vanishing of the integrability tensors R_{jk}^a and $R_{j\overline{k}}^a$, locally given by

$$R_{jk}^a = \frac{\delta N_j^a}{\delta z^k} - \frac{\delta N_k^a}{\delta z^j}\ ,\ R_{j\overline{k}}^a = \frac{\delta N_j^a}{\delta \overline{z}^k}. \tag{4.11}$$

If the horizontal distribution is integrable, then its holomorphy can be studied with the help of *the partial Bott connection* on E, that is a connection D of $(1,0)$-type on $V^{'}E$ defined by

$$D_X Y = v^{'}[X, Y]\,,\ \forall\, X \in \Gamma(H^{'}E),\ \forall\, Y \in \Gamma(V^{'}E), \tag{4.12}$$

where $v^{'}$ is the natural vertical projection.

By similar calculations as in [1], we get that the curvature of the partial Bott connection D is given by

$$R = \Pi - \Lambda \wedge \overline{\Lambda}, \tag{4.13}$$

where $\Pi = (\Pi_b^a)$ and $\Lambda = (\Lambda_{\overline{b}}^a)$ are locally given by

$$\Pi_b^a = -\frac{1}{2}\frac{\partial R_{jk}^a}{\partial \eta^b} dz^j \wedge dz^k - \frac{\partial R_{j\overline{k}}^a}{\partial \eta^b} dz^j \wedge dz^{\overline{k}}\ ,\ \Lambda_{\overline{b}}^a = \frac{\partial N_k^a}{\partial \overline{\eta}^b} dz^k. \tag{4.14}$$

Now, from (4.13) and (4.14) we get

Proposition 10. *An integrable transversal distribution $H^{'}E$ is holomorphic if and only if the partial Bott connection D is flat.*

4.2 Affine transversal distributions

Definition 6. *We say that $H^{'}E$ is an affine transversal distribution of $V^{'}E$ if the local functions N_k^a are locally given by*

$$N_k^a(z,\eta) = \Gamma_{bk}^a(z)\eta^b + \beta_k^a(z), \tag{4.15}$$

where $\Gamma_{bk}^a(z)$ and $\beta_k^a(z)$ are projectable functions on E, not necessarily holomorphic.

Imposing the change rule (4.3) we get the change rules for Γ_{bk}^a and β_k^a, namely

$$\frac{\partial z^{'j}}{\partial z^k} M_c^d \Gamma_{dj}^{'b} = M_a^b \Gamma_{ck}^a - \frac{\partial M_c^b}{\partial z^k}\ ,\ \frac{\partial z^{'j}}{\partial z^k}\Gamma_{dj}^{'b} B^d + \frac{\partial z^{'j}}{\partial z^k}\beta_j^{'b} = M_a^b \beta_k^a - \frac{\partial B^b}{\partial z^k}. \tag{4.16}$$

The relations (4.3) and (2.2) show that $\theta = d_{\mathcal{V}}(\partial N_k^a/\partial\eta^b)$ glue up to a global $d_{\mathcal{V}}$-closed form which yields a cohomology class

$$[\theta] \in H^1(E, \underline{V^{'}E \otimes V^{'*}E \otimes H^{'*}E}), \tag{4.17}$$

where $\underline{E}$ denotes the sheaf of germs of foliated sections of a foliated holomorphic bundle and $d_{\mathcal{V}}$ is the exterior derivative along the leaves of holomorphic foliation $\mathcal{V}$.

By the same considerations as in the real case [14], we notice that $[\theta]$ does not depend on the choice of the affine transversal distribution from (4.15). Indeed, if we choose another affine transversal distribution $\widetilde{H}^{'}E$ with the local coefficients $\widetilde{N}_k^a$, then $P_k^a = \widetilde{N}_k^a - N_k^a$ defines a global section of $V^{'}E \otimes H^{'*}E$. Clearly, if an affine transversal distribution exists, then $[\theta] = 0$. Conversely, if $[\theta] = 0$, we have

$$d_{\mathcal{V}}(\partial N_k^a/\partial\eta^b) = -d_{\mathcal{V}}(\gamma_{bk}^a)\ ;\ \gamma_{bk}^a \in \Gamma(\underline{V^{'}E \otimes V^{'*}E \otimes H^{'*}E}). \tag{4.18}$$

The local forms $\gamma_{bk}^a \delta\eta^b$ are $d_{\mathcal{V}}$-closed, and provide some

$$[\gamma] \in H^1(E, \underline{V^{'}E \otimes H^{'*}E}) \tag{4.19}$$

which does not depend on the choice of γ_{bk}^a. Finally, if $[\gamma] = 0$, we shall obtain $P_k^a \in \Gamma(V^{'}E \otimes H^{'*}E)$ such that $\gamma_{bk}^a = \partial P_k^a/\partial\eta^b$, and

$$\widetilde{\delta}\eta^a = d\eta^a + (N_k^a + P_k^a)dz^k = 0$$

defines an affine transversal distribution $\widetilde{H}^{'}E$. Hence, we have

Proposition 11. *The holomorphic vertical distribution $V^{'}E$ has an affine transversal distribution if and only if $[\theta] = 0$ and $[\gamma] = 0$.*

Example 1. *Let $E = \overline{E}$ be a holomorphic vector bundle endowed with a complex Finsler structure F, purely hermitian [1, 7], with the fundamental tensor $h_{a\bar{b}}(z) = \partial^2 F^2/\partial\eta^a\partial\overline{\eta}^b$ and $\Gamma_{bk}^a(z) = h^{\bar{c}a}\partial h_{b\bar{c}}/\partial z^k$ the local coefficients of the Chern-Finsler linear connection on E. Then the Chern-Finsler c.n.c. $\overset{CF}{N_k^a} = \Gamma_{bk}^a(z)\eta^b = 0$ defines an affine transversal distribution on $\overline{E}$.*

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