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QUOTIENT CI–ALGEBRAS

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Abstract

In this paper we introduce the notion of (regular) congruence relations on CI–algebras and we construct quotient algebra $(\frac{X}{\theta_F}; *, F_1)$ via a closed filter F of X. Moreover, we show that there exists a bijection from the set of all filters containing filter G to the set of all filters of $\frac{X}{G}$.

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1 Introduction

Y. Imai and K. Iseki [3] introduced two classes of abstract algebras: BCK–algebras and BCI–algebras. BCI–algebras as a class of logical algebras are the algebraic formulations of the set difference together with its properties in set theory and the implicational functor in logical systems. They are closely related to partially ordered commutative monoids as well as various logical algebras. Their names are originated form the combinators B, C, K and I in combinatory logic. It is known that the class of BCK–algebras is a proper subclass of the class of BCI–algebras[2].

Recently, H. S. Kim and Y. H. Kim defined a BE-algebra [4]. Biao Long Meng, defined notion of CI–algebra as a generation of a BE–algebra.[6]. BE-algebras and CI-algebras are studied in detail be some researchers [1, 5, 7, 8] and some fundamental properties of CI–algebra are discussed.

For better understanding this algebraic structure we need to study it in detail and one of important tool is congruence relation and quotient structure of an algebraic structure. In this paper, we introduce the concept of congruence relation on CI–algebras and introduce the notion of closed filter in CI–algebra and construct quotient algebra via closed filter and investigate related properties.

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2 Preliminaries

Definition 1. *[6] An algebra* $(X; *, 1)$ *of type* $(2, 0)$ *is called a* CI*–algebra if*

$(CI1)$ $x * x = 1$;

$(CI2)$ $1 * x = x$;

$(CI3)$ $x * (y * z) = y * (x * z)$,

for all $x, y, z \in X$.

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.

Example 1. [5] Let $X := \{1, a, b, c, d\}$ be a set with the following table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

Then $(X; *, 1)$ is a CI–algebra.

Definition 2. *[5] A subset* F *of* X *is said to be a filter when it satisfies the conditions:*

(F1) $1 \in F$;

(F2) $x,\ x * y \in F \Rightarrow y \in F$.

Example 2. In Example 1 $F_1 = \{1, a\}$ and $F_2 = \{1, b\}$ are filters of X but $F_3 = \{1, c\}$ is not a filter because $c * b = 1 \in F_3$ and $c \in F_3$ but $b \notin F_3$.

Definition 3. *[5] A* CI*–algebra* X *is said to be transitive if for any* $x, y, z \in X$,

$$y * z \leq (x * y) * (x * z).$$

Example 3. [5] Let $X := \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is a transitive CI–algebra.

Definition 4. *[5] A* CI*–algebra* X *is called commutative if*

$$(x * y) * y = (y * x) * x, \text{ for any } x, y \in X.$$

Example 4. Let $X := \{1, a, b\}$ be a set with the following table.

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	a	1

Then X is a commutative CI–algebra.

3 Congruences Relations in CI–algebras

Throughout this section X always means a transitive CI–algebra.

Definition 5. *A relation θ on X is called a congruence relation if*

(C1) θ is an equivalence relation on X;

(C2) θ satisfies the substitution property with respect to $*$, that is,

$$(x, y), (u, v) \in \theta \Rightarrow (x * u, y * v) \in \theta.$$

(R) A congruence relation θ is called regular when it satisfies

$$(1, x * y), (1, y * x) \in \theta \Rightarrow (x, y) \in \theta.$$

Let $Con(X)$ be the set of all congruence relations on CI–algebra X and $Con_R(X)$ be the set of all regular congruence relations on X.

Example 5. Let $X := \{1, a, b\}$. Define a binary operation $*$ on X by the following table:

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

then $(X; *, 1)$ is a CI–algebra. Consider $\theta = \{(1, 1), (a, a), (b, b)\}$, we can see that θ is a congruence relation on X, but it is not regular because $(1, a * b), (1, b * a) \in \theta$, while $(a, b) \notin \theta$.

Example 6. Let $X := \{1, a, b\}$. Define a binary operation $*$ on X by the following table:

$*$	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1

then $(X; *, 1)$ is a CI–algebra. Consider $\theta = \{(1, 1), (a, a), (b, b), (1, b), (b, 1), (1, a), (a, 1), (a, b), (b, a)\}$, we can see that θ is a regular congruence relation on X.

Example 7. In Example 1, $\theta = \{(1, 1), (a, a), (b, b), (c, c), (d, d), (1, b), (b, 1), (1, a), (a, 1), (1, c), (c, 1), (a, b), (b, a), (a, c), (c, a), (c, b), (b, c)\}$, we can see that θ is a regular congruence relation on X.

Definition 6. *A filter F of X is called closed if $x * 1 \in F$, whenever $x \in F$.*

Example 8. In Example 1, F_1 and F_2 are closed filters of X.

Let F be a filter of X and $\theta \in Con(X)$. Define a relation θ_F on X as follows:

$$\theta_F = \{(x, y) : x * y, y * x \in F\}.$$

and define F_θ as follows:

$$F_\theta = \{x * y : (x, y) \in \theta\}.$$

Proposition 1. *If F is a filter of X and $\theta \in Con(X)$, then*

(i) $\theta_F \in Con_R(X)$;

(ii) F_θ is a closed filter on X;

(iii) $F_\theta = \{x : (1, x) \in \theta\}$;

(iv) F_{θ_F} is the largest closed filter contained in F.

Proof. (i) It is evident that θ_F is an equivalence relation on X. We only show that θ_F satisfies the substitution property and the condition (R).

Suppose that $(x, y), (u, v) \in \theta_F$, then $x*y, y*x \in F$ and $u*v, v*u \in F$. By transitivity we have $(u * v) * ((x * u) * (x * v)) = 1$ and $(v * u) * ((x * v) * (x * u)) = 1$. Since F is a filter, we have $(x * u) * (x * v) \in F$ and $(x * v) * (x * u) \in F$. Hence $(x * u, x * v) \in \theta_F$ and similarly $(x * v, y * v) \in \theta_F$. Since θ_F is an equivalence relation, we have $(x * u, y * v) \in \theta_F$. Thus it is proved that θ_F satisfies the substitution property. Now, we show that θ_F is regular. Suppose that $(1, x * y), (1, y * x) \in \theta_F$. By $(CI2)$ and definition of θ_F, we have $x * y = 1 * (x * y) \in F$ and $y * x = 1 * (y * x) \in F$, this implies that $(x, y) \in \theta_F$. Therefore $\theta_F \in Con_R(X)$.

(ii) Since $(x, x) \in \theta$, $x*x = 1 \in F_\theta$. Suppose that $x*y, x \in F_\theta$. There are $(u, v), (p, q) \in \theta$ such that $x*y = u*v$ and $x = p*q$. Since $(u, v) \in \theta \in Con(X)$, we have $(u*v, v*v) = (u*v, 1) = (x*y, 1) \in \theta$ and similarly $(1, x) \in \theta$. By $(CI2)$ we have $(1*y, x*y) = (y, x*y) \in \theta$. Hence by transitivity we have $(1, y) \in \theta$ and therefore $1 * y = y \in F_\theta$. Thus F_θ is a filter of X.

If $x \in F_\theta$, then there exists $(p, q) \in \theta$ such that $x = p * q$. By $\theta \in Con(X)$, we have $(p * p, p * q) = (1, p * q) = (1, x) \in \theta$. It follows that $x * 1 \in F_\theta$ and hence F_θ is closed.

(iii) For brevity, we put $A = \{x : (1, x) \in \theta\}$. Suppose that $x \in F_\theta$. There exists $(u, v) \in \theta$ such that $x = u * v$. Since θ is a congruence relation, we have $(u * u, u * v) = (1, u * v) = (1, x) \in \theta$. That is $F_\theta \subseteq A$. It is easy to show the converse.

(iv) By (ii), F_{θ_F} is a filter and it is obvious that $F_{\theta_F} \subseteq F$. Let G be another closed filter contained in F. For any $x \in G$, since G is a closed filter, we have $x * 1 \in G \subseteq F$. This means that $x * 1$, $1 * x \in F$ and $(1, x) \in \theta_F$. By (iii), we obtain $x \in F_{\theta_F}$. This yields that $G \subseteq F_{\theta_F}$ and hence that F_{θ_F} is the largest closed filter contained in F.

Theorem 1. *Let F be a filter of X. If F is closed filter, then $F = F_{\theta_F}$.*

Theorem 2. *Let $\theta \in Con_R(X)$. Then $\theta = \theta_{F_\theta}$.*

Proof. Let $\theta \in Con_R(X)$. It is sufficient to show that $\theta_{F_\theta} \subseteq \theta$. We assume that $(x, y) \in \theta_{F_\theta}$. By definition, we have $x * y$, $y * x \in F_\theta$ and hence there exist $(p, q), (u, v) \in \theta$ such that $x * y = u * v, y * x = p * q$. Since $\theta \in Con(X)$, we have $(1, x * y) = (1, u * v) = (u * u, u * v) \in \theta$. Similarly $(1, y * x) \in \theta$. Since θ is regular we have $(x, y) \in \theta$. This means that $\theta_{F_\theta} \subseteq \theta$ and hence $\theta = \theta_{F_\theta}$.

Conversely, suppose that $\theta = \theta_{F_\theta}$. Since F_θ is the closed filter, the congruence $\theta = \theta_{F_\theta}$ is regular.

Note. For a nonempty subset F of X we define the binary relation $\sim_F$ in the following way:

$$x \sim_F y \text{ if and only if } x * y \in F \text{ and } y * x \in F.$$

The set $\{b : a \sim_F b\}$ will be denoted by $[a]_F$ and denote $\frac{X}{\sim_F} = \{[x]_F : x \in X\}$. For $\theta \in Con(X)$ we will denote $[x]_\theta = \{y \in X : x\theta y\}$, abbreviated by F_x and since $1 \in X$, then $[1]_\theta = F_1$. We will call $F_x(\theta)$ the θ-equivalence class containing x, and denote $\frac{X}{\theta} = \{F_x : x \in X\}$. For a congruence relation θ the operation $*$ on $\frac{X}{\theta}$ is defined by $F_x * F_y = F_{x*y}$. This binary operation is well-defined.

Lemma 1. *If F is a closed filter of X, then $F_1 = F$.*

Proof. If $x \in F$, then by $(CI2)$ $1 * x = x$, and since F is closed then $x * 1 \in F$. Hence $x \sim_F 1$. Therefore $x \in [1]_F = F_1$.

Conversely, if $x \in F_1$, then $1 \sim_F x$, i.e., $x = 1 * x \in F$ by definition of $\sim_F$. Thus $F_1 \subseteq F$. Therefore $F_1 = F$.

Proposition 2. *Let $\theta \in Con(X)$. If θ is regular, then θ is identical with the congruence relation derived from the closed filter F_1.*

Proof. Let $x * y \in F_1$ and $y * x \in F_1$. Then $F_{x*y} = F_{y*x} = F_1$. Since θ is regular thus $F_x = F_y$, and therefore $x\theta y$.

Conversely, if $x\theta y$, then $x * y\theta y * y = 1$. In the same way we have $y * x\theta 1$. This shows $x * y \in F_1$ and $y * x \in F_1$.

Theorem 3. *Let $\theta \in Con(X)$. Then $(\frac{X}{\theta}; *, F_1)$ is a CI–algebra.*

Proof. Let $F_x, F_y, F_z \in \frac{X}{\theta}$, for any $x, y, z \in X$. Then
(1) $F_x * F_x = F_{x*x} = F_1$,
(2) $F_1 * F_x = F_{1*x} = F_x$,
(3) $F_x * (F_y * F_z) = F_x * F_{y*z} = F_{x*(y*z)} = F_{y*(x*z)} = F_y * F_{x*z} = F_y * (F_x * F_z)$.
This proves that $(\frac{X}{\theta}; *, F_1)$ is a CI–algebra.

Example 9. In Example 6, $\theta = \{(1, 1), (a, a), (b, b), (1, a), (a, 1)\}$, is a regular congruence relation on X. Hence $[1] = F_1 = \{1, a\}$, $[a] = F_a = \{1, a\}$, and $[b] = F_b = \{b\}$. Therefore $\frac{X}{\theta_F} = \{F_1, F_b\}$ with the following table.

$*$	F_1	F_b
F_1	F_1	F_b
F_b	F_b	F_1

is a CI–algebra.

Theorem 4. *If F is a closed filter of X, then $(\frac{X}{\theta_F}, *, F_1)$ is a CI–algebra.*

Proof. By Proposition 1 and Theorem 3, is clear.

Theorem 5. *Let F be a filter of a commutative CI–algebra X. Then the quotient $(\frac{X}{\theta_F}; *, F_1)$ is a commutative CI–algebra.*

Proof. Suppose that $F_x,\ F_y \in \frac{X}{\theta_F}$. Then

$$(F_x * F_y) * F_y = F_{x*y} * F_y = F_{(x*y)*y} = F_{(y*x)*x} = F_{y*x} * F_x = (F_y * F_x) * F_x.$$

This shows that $(\frac{X}{\theta_F}; *, F_1)$ is commutative.

Example 10. In Example 4 $F = \{1, a\}$ is a closed filter of X and $\theta_F = \{(1, 1), (a, a), (b, b), (a, 1), (1, a)\}$. Hence $[1] = F_1 = \{1, a\}$, $[a] = F_a = \{1, a\}$, and $[b] = F_b = \{b\}$. Therefore $\frac{X}{\theta_F} = \{F_1, F_b\}$ with the following table.

$*$	F_1	F_b
F_1	F_1	F_b
F_b	F_1	F_1

is a commutative CI–algebra.

A mapping $f : X \to Y$ of CI–algebras is called a CI–homomorphism if $f(x*y) = f(x)*f(y)$, for all $x, y \in X$. Since $x*x = 1$ for all $x \in X$, then $f(1) = f(x*x) = f(x)*f(x) = 1$. Therefore $f(1) = 1$.

Proposition 3. *Let $f : X \to Y$ be a CI–homomorphism and Y is a commutative CI–algebra. If $\theta := \{(x, y) : f(x) = f(y)\}$, then θ is a regular congruence relation on X.*

Proof. It is obvious θ is an equivalence relation on X. We only show that θ satisfies the substitution property and the condition regularity. Suppose that (x, y) and $(u, v) \in \theta$. Then we have $f(x) = f(y)$ and $f(u) = f(v)$. Since f is a homomorphism this yields,

$$f(x * u) = f(x) * f(u) = f(y) * f(v) = f(y * v).$$

Then $(x * u, y * v) \in \theta$.

Now, let $(x * y, 1), (y * x, 1) \in \theta$. Then $f(x * y) = f(y * x) = f(1) = 1$. Since Y is commutative we have

$$f(x) = 1 * f(x) = (f(y) * f(x)) * f(x) = (f(x) * f(y)) * f(y) = 1 * f(y) = f(y).$$

Hence $f(x) = f(y)$, and therefore $(x, y) \in \theta$, hence θ is regular.

Theorem 6. *Let $f : X \to Y$ be a CI–homomorphism, Y is a commutative CI–algebra and $\theta = \{(x, y) : f(x) = f(y)\}$. Then $\frac{X}{\theta} \cong f(X)$.*

Proof. By Proposition 3 and Theorem 3, we get that $(\frac{X}{\theta}; *, F_1)$ is a CI–algebra. Let $v : \frac{X}{\theta} \to f(X)$ be such that $v(F_x) = f(x)$ for all $F_x \in \frac{X}{\theta}$. Then
(1) if $F_x = F_y$, then $(x, y) \in \theta$ therefore $f(x) = f(y)$. Hence $v(F_x) = v(F_y)$.
(2) Let $y \in f(X)$. Then there exists $x \in X$ such that $f(x) = y$. Then $F_x \in \frac{X}{\theta}$ and $v(F_x) = f(x) = y$. Hence v is onto.

(3) if $f(x) = f(y)$, then $(x, y) \in \theta$. This implies that $F_x = F_y$. Hence v is one to one.
(4) $v(F_x * F_y) = v(F_{x*y}) = f(x * y) = f(x) * f(y) = v(F_x) * v(F_y)$. Then v is CI–homomorphism.

Theorem 7. *If G and F are filters of X and $G \subseteq F$, then*

(a) G is also a filter of F.

(b) $\frac{F}{G}$ as the quotient of the filters F via the filter G is a filter of $\frac{X}{G}$.

Proof. (a) is immediate from Definition 2.3.
In order to prove (b), first we must show that each element $\frac{F}{G}$ is also an element of $\frac{X}{G}$. To avoid the ambiguity, we denote the element of $\frac{F}{G}$ containing x by $F_x(F)$.
Suppose $y \in X$ and $x \in F$. If $x \sim_G y$, then $x * y \in G$, and so $x * y \in F$ and $x \in F$. By definition $y \in F$. This says $F_x(F) \in \frac{X}{G}$ or each element of $\frac{F}{G}$ is also an element of $\frac{X}{G}$.
Next we prove that $\frac{F}{G}$ is a filter of $\frac{X}{G}$. Since $G \subset F$, $G_1 = G \in \frac{F}{G}$. Let $G_x * G_y \in \frac{F}{G}$ for all $G_x \in \frac{F}{G}$. Then $G_x * G_y = G_{x*y} \in \frac{F}{G}$. It follows $x * y \in F$ and $x \in F$. By definition of F we have $y \in F$, and so $G_y \in \frac{F}{G}$.

Theorem 8. *If F^* is a filter of $\frac{F}{G}$, then $F = \cup\{x : F_x \in F^*\}$ is a filter of X and $G \subseteq F$.*

Proof. Since $F = F_1 \in F^*$, $1 \in F$ and $G \subseteq F$. Let $x * y \in F$ and $x \in F$. Then $F_{x*y} = F_x * F_y \in F^*$ and $F_x \in F^*$. By definition we have $F_y \in F^*$ and so, $y \in F$. This shows that F is a filter of X.

Note. The set of all filters of X is denoted by $F(X)$, the set of all filters containing filter G of X is denoted by $F(X, G)$.

Theorem 9. *If G is a filter of X, then there is a bijection from $F(X, G)$ to $F(\frac{F}{G})$.*

Proof. Define $f : F(X, G) \to F(\frac{F}{G})$ by $f(F) = \frac{F}{G}$. By Theorem 7(b), f is well-defined, also Theorem 8 implies that f is onto. We can also prove that f is one-to-one. Let F_1, $F_2 \in F(X, G)$ and $F_1 \neq F_2$. Without loss of any generality, we may assume that there exists a $y \in F_2 - F_1$. If $f(F_1) = f(F_2)$, then $F_y \in f(F_2)$ and $F_y \in f(F_1)$. Thus there exists $x \in F_1$ such that $F_x = F_y$, so $x \sim_G y$, that $x * y \in G$ and $y * x \in G$. Since $G \subseteq F_1$, we have $x * y \in F_1$ and $x \in F_1$. Hence $y \in F_1$, which is a contradiction, so f is one-to-one.

Theorem 10. *Let F be a filter of X. Then there is a canonical surjective homomorphism $\varphi : X \to \frac{F}{G}$ by $\varphi(x) = F_x$, and $\ker \varphi = F$, where $\ker \varphi = \varphi^{-1}(F_1)$.*

Proof. It is clear that φ is well-defined. Let $x, y \in X$. Then $\varphi(x*y) = F_{x*y} = F_x*F_y = \varphi(x) * \varphi(y)$. Hence φ is homomorphism. Clearly φ is onto. We have $\ker \varphi = \{x \in X : \varphi(x) = F_1\} = \{x \in X : F_x = F_1\} = \{x \in X : x * 1, 1 * x \in F\} = \{x \in X : x \in F\} = F$.
Note. It is well-known that, for every set X, the set of equivalence relations on X, $Eq(X)$, with the inclusion ordering (in the powerset of $X \times X$) is a complete lattice in which the infimum is the meet and the supremum is the transitive closure of the join.

Theorem 11. *$Con(X)$ is a sublattice of $Eq(X)$.*

4 Conclusion

Quotient algebra play a central roll in universal algebra and their properties are used for better understanding the algebraic structure. In this note, we have introduced the concept of congruences relations and closed filter of CI–algebras and investigated some of their useful properties of this structure. We show that Quotient of a CI-algebra via a regular congruences and closed filter is a CI-algebra. We prove that there exists a bijection from the set of all filters containing filter G to the set of all filters of $\frac{X}{G}$. We hope that our result can used for classification of this algebraic structure and finding the relationship to the other algebraic structures.

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